Valuation of Vulnerable European Call Options

by

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Abstract

We develop a simple model for valuing vulnerable options subject to default risk on the part of the option writer, which we refer to as the "variable default boundary" model. This pricing model allows for the presence of other liabilities in the capital structure of the option writer while recognizing that the growth in the value of the option itself may be a major source of financial distress. This model, retains the attractive features of Johnson and Stulz (1987) and Klein (1996) by linking the payout ratio, in the event of default, to the value of the option writer's assets and by explicitly allowing for correlation between the options writer's assets and the asset underlying the option. It also corrects a defect that occurs in most fixed default boundary models where the pricing equations do not assure that the payment to claimants upon default is no greater than the assets of the option writer. Our model also incorporates stochastic interest rates, using a Vasicek (1979) term structure model.

An exact analytical solution to the variable default boundary model is not possible. However, we derive a simple approximate analytical solution, which depends on two parameters. We also develop an algorithm to estimate the optimal values of these parameters. Since Johnson and Stulz (1987) is a special case of our model the approximate analytical solution also provides an analytical solution to their model. Numerical examples compare the results of the approximate analytical solution to results derived from a Monte Carlo simulation. Comparisons to the results from the Black-Scholes-Merton model and the fixed default boundary model of Klein and Inglis (1999) are also given. Along with Klein and Inglis (1999), this thesis provides the only evidence of the importance of stochastic interest rates on the pricing and hedging of vulnerable European calls.

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... to Anne ...

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Chapter 1

Introduction

1.1 Definition of a Vulnerable Option:

Many financial institutions actively trade derivative contracts with their corporate clients as well as with other financial institutions in the over-the-counter (OTC) market. An investor or corporation who holds an over-the-counter (OTC) option is exposed to both market risk and default risk. Market risk arises primarily from movements in the value of the asset underlying the option, while default risk arises from the possibility of default on the part of the option writer. Unlike exchange traded options, there is no exchange or clearing house requiring OTC options to be marked to market on a daily basis or to ensure that the option writer posts sufficient margin to honor future obligations. Therefore, OTC option contracts are subject to default risk and have lower values then otherwise identical exchange traded options. Johnson and Stulz (1987) have labeled options subject to default risk as "vulnerable" options. It should be pointed out that another type of default risk may exist in some OTC option contracts. It is possible that the asset or security underlying the option may be affected by the default of the firm that issued that security, even if the counter-party that wrote the option is fully able to meet its obligations. For example, consider an OTC option written on a corporate bond. It is possible that the firm that issued the corporate bond may default on its obligations under terms of the bond covenants, even though the counter-party that wrote the option is able to meet its obligations. This thesis deals specifically with the pricing and hedging of vulnerable call options written by a counter-party that may default on its obligations under the terms of the option contract, and not with the issue of default in the underlying asset.

1.2 Brief Overview of the Literature:

A number of researchers have developed valuation frameworks that incorporate both market risk and credit risk. Three major categories of models can be distinguished. The first set are based on the seminal paper of Merton (1974), which allows shareholders to default on their obligations and surrender the firm's assets to the bondholders. Default occurs only if the value of the firm's assets is less than the face value of the firm's zero coupon debt at the maturity of the debt. An intuitively appealing aspect and strength of this approach is the link between the payout in the event of default and the value of the firm's assets.

An alternative to this approach proposed by Longstaff and Schwartz (1995) and Hull and White (1995a) allows default to occur at some random time prior to or at the maturity of the obligation. Default is triggered when the value of the firm's assets fall below some exogenously determined default boundary, usually related to the value of the firm's outstanding liabilities. However, these models assume an exogenous recovery rate in the event of default, thus breaking the intuitive link between payout in default and the value of the firm's assets.

The third approach to modeling default risk, developed by Duffie and Singleton (1994) and Jarrow and Turnbull (1995), models default as an exogenous event not directly related to the value of the firm's assets. As in the previous approach, default can occur at any time up to the maturity of the firm's obligations and the payout rate is again assumed to be exogenously determined.

1.3 Overview of Model Assumptions:

The traditional approach, in which default can only occur at the maturity of a firm's obligations, has been employed in two previous papers dealing with the valuation vulnerable options. Johnson and Stulz (1987) assume that the only liability of the firm is the written option and that the option holder receives all the assets of the writer in the event that option writer cannot make the required payment at the maturity of the option. Default risk can arise from both changes in the value of the option writer's assets and from changes in the value of the option itself. In fact, even if the assets of the writer have not decreased over the life of the option, the writer may still default if the value of the option out-grows the value of the option writer's assets. This approach has intuitive appeal because it links the amount recovered by the option holder to the value of the assets of the option writer. It should be noted, however, that Johnson and Stulz (1987) did not allow for the presence of other liabilities in the option writer's capital structure. This assumption is unrealistic in most business situations and may limit the applicability of their model.

Klein (1996) assumes that default will occur whenever the value of the option writer's assets drops below the value of some exogenously determined default boundary usually related to the option writer's other liabilities. This assumption ignores the potential liability created by the written option itself, but is reasonable provided the market value of the option is small relative to the value of the option writer's other liabilities. For example, if a large financial institution is the option writer, the probability is negligible that the option will increase sufficiently in value to materially change the total liabilities of the financial institution. However, the assumption is inappropriate if the value of the option represents a significant proportion of the writer's liabilities. For example, some commodity producers hedge a large proportion of their production by writing options. If the commodity price moves against these producers, the liability created by the option may be much larger than the producer's other liabilities. As a result, the increase in the value of the option itself may be the primary cause of financial distress. Therefore allowing the default barrier to depend on the value of the option should be an important consideration when valuing these types of vulnerable options.

It should also be pointed out that the model of Klein (1996) suffers from two potential defects[1]. It is possible that the option writer may still be solvent at maturity, but

[1] Hull and White (1995), Jarrow and Turnbull (1995), Longstaff and Schwartz (1995), Rich (1996) and Klein and Inglis (1999) may also suffer from these problems. Note that neither Merton (1974) or Johnson and Stulz (1987) suffer from these difficulties.

not have enough assets to meet all of the claims against it. It is also possible that the model will assume that the total recoveries of the claimants, in the event of bankruptcy, may be larger than the assets available to be paid out.

Both Johnson and Stulz (1987) and Klein (1996) explicitly allow for correlation between the process driving bankruptcy and the process driving the value of the asset underlying the option. The models developed by Hull and White (1995) and Jarrow and Turnbull (1995) dealing with vulnerable options impose the assumption of independence between the two processes to obtain analytical results. Hull and White (1995) argue that in the case of a large financial institution, the option is usually only a very small proportion of the institution's portfolio and that most institutions are reasonably well hedged against changes in the value of underlying assets. The assumption of independence is not unreasonable in these situations. However, for other realistic business situations this assumption could generate misleading results. For example, consider a call on oil written by a oil producer. In this case, the call would be exercised only when the price of oil is high and the credit worthiness of the writer is also likely to be high. Models assuming independence between the process driving financial distress and the underlying asset would therefore over-estimate the effect of credit risk in this situation. Explicitly allowing for a non-zero correlation also implies that option writers of identical credit ratings should rationally be offered different terms on the same derivative contract.

One of the objectives of this thesis is to combine the variable default boundary of Johnson and Stulz (1987) and the fixed default boundary of Klein (1996) into a single consistent model. We want to explicitly allow for other liabilities in the capital structure of the option writer, while still incorporating the fact that it may be the payoff on the option that drives the option writer into bankruptcy. We also want to ensure that the technical difficulties associated with the fixed default boundary type models are corrected. In addition, we want our model to maintain the intuitive link between the assets of the option writer and the proportion of the option holder's nominal claim that is recovered in the event of default. Finally we want to explicitly allow for correlation between the process driving bankruptcy and the process driving the value of the asset underlying the option.

1.4 Stochastic Interest Rates:

Longstaff and Schwartz (1995) extend the risky debt models of Merton (1974) and Black and Cox (1976) to allow not only for default risk but for interest rate risk as well. Longstaff and Schwartz (1995) identify the importance of correlation between the risk-free interest rate and the process driving bankruptcy when valuing risky corporate debt. In fact, they state that, "the variation in credit spreads due to changes in the level of interest rates is more important for investment grade bonds than the variation due to changes in the value of the firm". Using a calibrated version of their two-factor model, they demonstrate that a one standard deviation increase in the 30-year Treasury yield results in a 23.1 basis point per year decrease in average credit spreads for all Baa rated industrial bonds listed in Moody's Bond Record from 1977 to 1992. In contrast, a one standard deviation increase in the returns on an industrial index, a proxy for firm value, results in only a 7.3 basis point per year decrease in these average credit spreads. Thus, the correlation between changes in firm value and changes in interest rates has a significant effect on credit spreads. By extension it may also be important in the valuation of options vulnerable to financial distress on the part of the writer.

Valuation models for vulnerable options do not incorporate this issue. For example, Johnson and Stulz (1987) make the assumption that interest rates are fixed, while Jarrow and Turnbull (1995) assume " the bankruptcy process is independent of the spot interest rate process under the true probabilities." Making these assumptions greatly simplifies the valuation of vulnerable options, but ignores the potentially important influence of this correlation. Interest rate risk may have an important effect on option valuation in two ways, through the correlation between interest rates and the asset underlying the option as in Merton (1973) and the correlation between interest rates and the process driving financial distress as in Longstaff and Schwartz (1995). These effects, which have not been discussed in previous vulnerable option pricing papers, are explicitly incorporated in our model.

1.5 Approximate Analytical Solution:

Using standard risk-neutral valuation techniques we are able to obtain some analytical results to the valuation equations for the fixed default boundary model of Klein and Inglis (1999) and the variable default boundary model presented in this thesis. The model of Klein and Inglis (1999) has an exact analytical solution, however once we incorporate a variable component into the default boundary, we can no longer obtain an exact analytical solution. There are two problems. The first arises because the boundary condition becomes non-linear when the variable component is added. The second problem arises because of the hyperbolic functional form of the integrand of the valuation equation. We use two linear Taylor series expansions to simplify the valuation equation and develop an approximate analytical solution. This solution depends on two design parameters (i.e. the points of expansion of the Taylor series expansion). A simple algorithm, involving the solution to two uni-variate optimization problems, is developed to find these two parameters. For comparison purposes, a Monte Carlo simulation is also used to generate numerical solutions to the valuation equations.

As demonstrated with numerical simulations, once the two design parameters have been found the approximate analytical model will generate accurate results for the prices of vulnerable calls. We also perform a sensitivity analysis using both the fixed default boundary model of Klein and Inglis (1999) and the variable default boundary model developed in this thesis and demonstrate that the characteristics of the models can be very different from each other, as well as from standard default-free European calls.

A comparison with Rich's (1996) model, similar to the Hull and White (1995) model, using the estimated margin that would be required to remove the default risk from a vulnerable call was also performed. In all cases studied, the Klein and Inglis (1999) fixed default boundary model required the least amount of margin and the variable default boundary model required the greatest amount of margin. Rich's (1996) estimates always fell in the middle. Rich concludes that consumer margin requirements for exchange traded options are not market driven and are in fact set significantly above what the market would require. The variable default boundary model, while still predicting less margin then required by exchanges, is more in line with exchange determined margin requirements.

Finally, we look at the characteristics of the hedging parameters of both models. Results are again significantly different from those predicted by the standard Black-Scholes-Merton results. For example the delta of a vulnerable call can decrease with the moneyness of the call. Hedging experiments are also carried out on both the fixed and variable default boundary models.

1.6 Contributions of Thesis

This thesis makes the following contributions:

1. We develop a vulnerable option pricing model which allows for the presence of other liabilities in the capital structure of the option writer while recognizing that the growth in the value of the option itself may be a major source of financial distress. This model retains the attractive features of Johnson and Stulz (1987) and Klein (1996) by linking the payout ratio to the value of the option writer's assets and by explicitly allowing for correlation between the options writer's assets and the asset underlying the option. In addition, it incorporates stochastic interest rates in the modeling framework.
2. We derive an approximate analytical solution to the variable default boundary model developed above, which depends on two design parameters We develop a simple algorithm to estimate the optimal values of these two parameters. Since Johnson and Stulz (1987) is a special case of our model, the

approximate analytical solution also provides an analytical solution to their model. The analytical solution provides greater intuition about the nature of the solution and is considerably faster than numerical methods of solving the problem.

3. Along with Klein and Inglis (1999), this thesis provides the only evidence of the importance of stochastic interest rates on the pricing and hedging of vulnerable European calls. In particular, we demonstrate the importance of the correlation between 1) the assets of the option writer and the risk-free interest rate and 2) the asset underlying the option and the risk-free interest rate.

1.7 Organization of Thesis:

This thesis is organized as follows. Chapter 2 provides a more detailed review of the literature on models of default risk and vulnerable option models. It also briefly reviews some of the empirical work performed using various risky debt contracts. In chapter 3 we develop our valuation framework for vulnerable European calls, for the fixed default boundary model of Klein and Inglis (1999) and the variable default model mentioned above. Chapter 4 develops the exact pricing equation for the Klein and Inglis (1999) model and the approximate pricing equation for the variable default boundary model. It also develops the algorithm for choosing the two design parameters required for the approximate analytical solution. A brief review of the Monte Carlo procedure used to obtain a numerical value for the price of a European call is also provided. A number of numerical examples are provided in Chapter 5. We look at the sensitivity of the pricing equations to all of the model parameters and demonstrate the hedging effectiveness of each of the models. Concluding remarks are offered in Chapter 6.

Chapter 2

Literature Review

This chapter provides a brief overview of the literature dealing with models of default risk in general, followed by a more detailed look at vulnerable options papers. The majority of these papers look specifically at modeling risky debt and trying to explain or predict the default credit spread. More recently a number of papers have focused on the reduction in value and the properties of over-the-counter (OTC) options written by firms that may default on their obligations under these contracts. However, all of these papers must define the event of default and describe how and when the payout of any remaining assets will take place. That is they must develop some model of default risk. We look at default risk models in the next section.

2.1 Models of Default Risk

All existing models of default have two basic components. The first deals with how and when default will be defined to occur. The second deals with the payoff to be received in the event of default, either now or in the future. Although all models of firm default have these two components, the way in which each component is modeled can be quite different. All of the default models in the literature can be divided into two broad classes.

The first class of models are referred to as "structural" or "firm value" models, since they are based on modeling the firm's assets (the term "structural" model was first coined by Jarrow, Lando and Turnbull 1997). Structural models can be further sub divided into two subgroups. The first subgroup of models define the event of default as occurring when the assets of the firm fall below some default boundary, usually associated with the value of the firms outstanding debt, at the maturity of the debt under consideration. The payout in the event of default is determined endogenously within the model and typically depends on the value of the firms assets. The classic paper by Merton (1974) falls into this subgroup as do the papers by Johnson and Stulz (1987), Klein (1996) and Klein and Inglis (1999). The second subgroup of models allows greater flexability in the timing of default and greatly simplifies the payoff modeling. In these models, default can occur at any time prior to the maturity of the contingent claim and is defined to occur if the value of the firm's assets hit some exogenously specified default boundary. This boundary may be a function of any number of state variables, but is commonly assumed to be either a fixed constant related to the firm's outstanding debt obligations, or stochastic with the uncertainty being driven by stochastic interest rates. The payoff in the event of default is also given exogenously. A payout ratio is specified which determines the percentage of the nominal claim to be paid in the event of default. Note that these models lose the intuitive link between the payout in the event of default and the value of the assets of the firm available to be paid out. Papers using this approach include Black and Cox (1976), Longstaff and Schwartz (1995), Nielson, Saa-Requejo and Santa-Clara (1993), Madan and Unal (1994), and Rich (1996). Hull and White (1995) develop a structural model of the form described above. In the general form of their model, default occurs when some state variables (i.e. variables related to the value of the firm's assets or other variables that measure the financial health of the finn) hit some general boundary that determines default. The payout on default can be fixed or related to any state variables that are related to those that would determine the value of the nominal claim. To derive an analytical solution they assume that the state variables that determine default and the state variables that determine the nominal claim are uncorrelated. In this particular case the payout ratio is an exogenously determined constant which can be estimated by knowing the value of a bond with the same risk and maturity as the security under consideration.

In the framework outlined above, the nominal claim and the assets of the firm available to be paid out are equal at the time of default. The reduction in value of the claims against the firm comes from the exogenously determined payout ratio. This framework ignores the possibility that the firm may recover from financial distress and pay its claims in full if given sufficient time time to reorganize its affairs. In addition there is no provision in most standard over-the-counter (OTC) contracts that forces a holder of a long position in an option contract to calculate their nominal claim at the time of default. The option holder may well wait until the original maturity of the option to calculate their claim, if there is no immediate dissolution of the firm. Given that the average firm will take in excess of 18 months to emerge from financial distress (Wruck (1990)) these possibilities cannot be discounted.

Another difficulty with this approach is that there is no guarantee that the payment to claimants is no greater then firm value upon default. A different but similar problem can also occur. For example, in Longstaff and Schwartz (1995), it is possible for the firm to reach the maturity date of the bonds in a solvent position, but still not have sufficient assets to meet the required payments. We discuss these potential difficulties in Chapter 3. Briys and de Varenne (1997) also discuss these problems and develop a structural model for valuing risky debt that overcomes these problems. These problems are caused by the exogenously determined payout, in the event of default. (i.e. the payout is independent of the value of the firms assets).

Finally, a recent paper by Collin-Dufresne and Goldstein (2001), has pointed out that fixed default boundary models or stochastic default boundary models where the uncertainty is associated only with mean-reverting stochastic interest rates, always predict that credit spread's will decrease with the maturity of the debt. Since the value of firm assets are usually modeled as Brownian motion, the value of the assets will increase exponentially with time. However, if the debt remains constant, the probability of default decreases with time and therefore credit spreads decrease with bond maturity. This result is inconsistent with recent empirical findings that suggest that credit spreads increase with maturity Helwege and Turner (1999). Colline-Dufresne and Goldstein (2001) develop a structural model that allows for a dynamic capital struture. The firm continues to issue additional debt so as to maintain a constant leverage ratio. This structural model predicts increasing credit spreads which are consistent with the empirical results of Helwege and Turner (1999).

The second major class of default models are referred to a "reduced form" or "jump default" models. These models assume that default occurs at the first jump of an exogenously given point process, usually a Poisson process. The probability of default, conditional on no default having yet occurred is controlled by an intensity or hazard function which can be constant or be allowed to depend on time and various state variables. Examples or this approach include the papers by Litterman and Iben (1991), Duffie and Singleton (1994), Jarrow and Turnbull (1995), Lando (1994) and Madan and Unal (1995). A different approach in the same theme is to model the default process as a Markov chain, which allows the incorporation of not only the event of default but changes in credit ratings as well. Examples or this approach include Das and Tufano (1995) and Jarrow, Lando and Turnbull (1997). The payoff in the event of default is also determined exogenously by a payout ratio that can be either constant or stochastic. So again, the dependence of default risk on capital structure of the firm has been assumed away.

There has been a limited amount of empirical testing of default models and this work lags significantly behind theoretical developments. However, the empirical studies that have been published are all related to some form of debt obligation. For example, Jones, Mason and Rosenfeld (1984), study the ability of a Merton (1974) type model to predict corporate bond prices, Titman and Torous (1989) use a structural model to estimate rates on commercial mortgages and Wei (1995) compares Merton (1974) and Longstaff and Schwartz (1995) structural models with Eurodollar data. Madan and Unal (1995) calibrate a reduced form model to certificate of deposit rates and Monkkonen (1997) tests the one-step ahead prediction ability of six reduced form models with corporate bond data.

The general conclusions from these studies indicate that neither structural models or reduced form models predict security prices accurately across all credit ratings and different types of debt. For example tests using corporate bond data indicate that the issuing firm's capital structure seems to be important in valuing low rated bonds (Jones, Mason and Rosenfeld (1984)). Also, Shane (1994) presents empirical evidence that returns on low rated bonds have a higher correlation with an equity index, than investment grade bonds, which supports the idea of modeling the issuing firm's assets. However, reduced form models, which are easier to implement, tend to model the default probabilities of high rated bonds more accurately. Monkkonen (1997) compares six increasingly complex reduced form models using corporate bond data. Since reduced form models match the current bond prices exactly, the models are compared using one-step-ahead price forecasts computed with the implied default probabilities. He shows that prediction errors increase with decreasing credit ratings and increasing maturity, indicating that ignoring the capital structure of the issuing firm may be detrimental in certain circumstances.

There has been no tests of vulnerable option pricing models, since data is not readily available.

2.2 Vulnerable Option Models

There are a number of papers in the area of vulnerable option pricing. The first paper to consider vulnerable option pricing was by Johnson and Stulz (1987), who coined the term vulnerable options. They develop a structural model where default is only considered at the maturity of the option. Klein (1996) and Klein and Inglis (1999) also use this framework. Hull and White (1995) and Rich (1996) use variations of the second approach to structural modeling to account for default risk. The final paper we will look at is by Jarrow and Turnbull (1995). This is an example of a "reduced form" model.

Johnson and Stulz (1987) assume that a corporate entity, with no other outstanding liabilities, has written a European call option. Default occurs if the value of the option writer's assets, modeled as a geometric Brownian process, is less than value of the call at its maturity date. In the event of default the holder of the option receives all the assets of the option writer. In this model default can occur for two reasons: first the assets of the option writer decline sufficiently or second even if the writer's asset do not decline, the value of the asset underlying the call may increase at a rate sufficient to cause default. Thus, this model may be appropriate in those cases where the option's value is large in comparison to the option writer's assets and it is likely that default will be triggered by the exercise of the option. This can be an important consideration in certain cases. For example, the bankruptcy of Barrings' bank was caused to a large extent by the maturity of a large position in Nikkei Index futures contracts (Cao and Wei (1998). It is not hard to imagine a deep-in-the-money call causing the same havoc. Thus it seems prudent to include the possibility of the option itself causing default in any model of default risk.

However, the Johnson and Stulz (1987) model does not include any other liabilities in the capital structure of the option writer. This implies that default risk could be virtually non-existent if the value of the call was small relative to the option writer's assets, since it would be impossible for the option to grow enough in value to cause the writer to default. Therefore the assumption that the option is the only liability is likely to under estimate default risk It would be more realistic to assume that there are other claimants that will compete with the option holder for a share of the writer's remaining assets.

Klein (1996) uses the same approach as Johnson and Stulz (1987), but assumes that in addition to the written option there are other liabilities in the option writer's capital structure. The event of default is assumed to occur if the writer's assets are less then a fixed default boundary, related to the writer's debt, at the maturity of the option. This model allows for other liabilities, but loses the ability for the option itself to be the principal cause of default. This assumption is appropriate if the option's value is small compared to the value of the writer's total assets since it is unlikely that the option will be the cause of default.

Klein and Inglis (1999) is an extension of Klein (1996) that includes stochastic interest rates, which is an aspect of these models that is not addressed in any other vulnerable option pricing model. Klein and Inglis (1999) found that incorporating stochastic interest rates could be quite important, especially in long term options. In addition the correlation between the the writer's assets and the risk-free rate and the correlation between the asset underlying the option and the risk-free rate can have a significant impact on vulnerable option valuation.

Hull and White (1995) and Rich (1996) also use a structural model approach but assume that default can be triggered anytime prior to maturity of the option and occurs whenever the value of the option writer's assets equals some default barrier.

The payout in default in the Hull and White (1995) model is some predetermined portion of the nominal claim, which is the current value of the corresponding default free option. In general the payout ratio is allowed to be a function of a number of state variables, but is not directly related to the assets of the option writer. Hull and White are able to derive a analytic solution under the assumption of independence between the assets of the writer and the asset underlying the option. As Hull and White argue this assumption is reasonable if the option is written by a financial institution that writes and hedges a large number of option on a variety of underlying assets. In this case it is reasonable to assume that there will not be a high degree of correlation between the two assets. However, the independence assumption may not be appropriate in a number of other common business situations. For example natural resource producers writing options on their production.

Rich (1996) considers four recovery scenarios in the event of default, ranging from recovering a fixed proportion of margin posted by the option writer, up to recovering some fixed proportion of the moneyness of the option at the time of default. Note that this last recovery scenario, the most complicated he considers, ignores the time value of the option in setting the nominal claim. It certainly would not be optimal for a holder of an out-of-the-money option to walk away and not submit a claim against the writer, simply because the option has no intrinsic value. Also, the analytic formula Rich presents for this scenario is in the form of an integral equation that would require additional numerical techniques to solve. However, he does not need to impose the independence assumption made by Hull and White (1995).

The key point about this second subgroup of structural models is that they break the intuitive connection between the payout in the event of default and the value of the firms assets.

Jarrow and Turnbull (1996) use a reduced form approach to model default risk, where default is modeled as the first time to a jump in an exogenously given point process. The intensity rate, which determines the probability of default is assumed to be constant in Jarrow and Turnbull (1996) but can be allowed to follow a diffusion process (Duffie (1996)) or depend on various state variables (Lando (1994), Mandan and Unal (1995)). The payoff in the event of default is determined exogenously. It may be either constant as in the model developed by Jarrow and Turnbull (1996) or stochastic as in Das and Tufano (1995). These models are more mathematically tractable than Merton-type structural models, but they also fail to capture the dependence of default risk on the capital structure of the firm.

Jarrow and Turnbull (1996) make the further contribution of not only modeling default risk in the option writer (i.e. vulnerable options), but also model the impact of default risk in the asset underlying the option (e.g. corporate bonds). Another advantage of this approach is that the inputs of the model, which include the initial default free term structure, the term structure of credit yield spreads and the exogenous payout ratio are all directly observable. Some of the inputs of structural models are not directly observable. In particular the value of the option writer's assets and the volatility of the writer's assets need to be known to implement these models. Fortunately, it is relatively easy to estimate these values using the observable value of the writer's equity and the volatility of the equity.

Chapter 3

Valuation Framework

In this section we develop a simple continuous time framework for valuing European calls that allows for both default risk and interest rate risk. We present assumptions for two models: The first is the fixed default-boundary (FDB) model of Klein and Inglis (1999) and the second is a variable default-boundary (VDB) model, developed in this thesis. The basic assumptions of this framework follow those of Merton (1974), Johnson and Stulz (1987) and Klein (1996) and are outlined below.

3.1 The Valuation Framework

Assumption 1: Dynamics of the Option Writer's Assets:

Let V represent the total value of the assets of the option writer. The dynamics of V are given by

$$\frac{dV}{V} = \mu_V dt + \sigma_V dZ_V \tag{3.1.1}$$

where μ_V *is the instantaneous expected return on the assets of the option writer,* σ_V^2 *is the instantaneous variance of the return (assumed to be constant) and* Z_V *is a standard Wiener process.*

Assumption 2: Dynamics of the Asset Underlying the Written Option:

Let S denote the market value of the asset underlying the option. The dynamics of S are given by:

$$\frac{dS}{S} = \mu_S dt + \sigma_S dZ_S \tag{3.1.2}$$

where μ_S *is the instantaneous expected return on the asset underlying the option,* σ_S^2 *is the instantaneous variance of the return (again, assumed to be constant) and* Z_S *is a standard Wiener process. The instantaneous correlation between* Z_V *and* Z_S *is* ρ_{VS}.

We allow for stochastic interest rates by specifying the dynamics of risk-free discount bond prices as in Merton (1973). In section 4.5 of this paper we illustrate how our general valuation equation can be applied in the specific case when the Vasicek (1977) term structure is assumed. However, any term structure consistent with Merton's assumptions could be employed.

Assumption 3: Dynamics of the risk-free Bond:

Let $B(T)$ *denote the market value, at time zero, of a pure risk-free (in terms of default) bond which pays one dollar, at time* T, *where* T *corresponds to the exercise date of the option under consideration. The dynamics of* $B(T)$ *are given by:*

$$\frac{dB}{B} = \mu_B(T)dt + \sigma_B(T)dZ_B \tag{3.1.3}$$

where $\mu_B(T)$ *is the instantaneous expected return on the risk free bond,* $\sigma_B^2(T)$ *is the instantaneous variance of the return and* Z_B *is a standard Wiener process. The instantaneous correlation between* Z_V *and* Z_B *is* ρ_{VB}, *while the instantaneous correlation between* Z_S *and* Z_B *is* ρ_{SB}.

Following Merton (1973) $\sigma_B(T-t)$ is assumed to be non-stochastic and independent of the level of bond prices, however since $B(T-t)$ is the price of a risk-free discount bond, the value of the bond reverts to its face value at maturity (i.e. $B(0)=1$). Therefore, $\sigma_B(T-t)$ will depend on the time to maturity and will equal zero at the maturity of the bond (i.e. $\sigma_B(0)=0$). Note that the Vasicek (1977), Hull-White (1990) and Markov versions of Heath, Jarrow and Morton (1992) models of the term structure are consistent with the above formulation. Although, we derive the expressions for European calls in terms of B, we also consider the results in the context of a Vasicek (1977) term structure model.

Assumption 4: Perfect, frictionless markets:

We assume that there are no transactions costs or taxes and that securities trade in continuous time.

Assumption 5*:* Option Writer's Capital Structure:

The capital structure of the firm writing the option is composed of two parts: A zero coupon bond maturing at time T with a face value equal to a fixed amount D^* *and a written European call option also maturing at time T, with a payoff equal to* $\max(S_T - K, 0)$, *where* S_T *represents the price of the underlying asset at the maturity of the option and K represents the strike price of the option. Therefore the total nominal claims of the debtholder's and the option holder's, at* $t = T$, *are* D^* *and the intrinsic value of the option respectively.*

The next assumption defines the event of default in our model. In fact, we define the event of default in two different ways (i.e the FDB and VDB) and compare the results of this assumption in chapter 5.

Assumption 6a: Definition of Default: Fixed Default-Boundary:

Default occurs at the maturity of the option, T, only if the value of the option writer's assets V_T *is less the a fixed threshold value* $D = D^*$ *(i.e. the face value of the zero coupon bond). The threshold is fixed in the sense that* D^* *is known at time t.*

Assumption 6b: Definition of Default: Variable Default-Boundary:

Default occurs at the maturity of the option, T, *only if the value of the option writer's assets* V_T *is less than the variable threshold value* $D = D^* + \max(S_T - K, 0)$. *The threshold is variable in the sense that* S_T *is uncertain at time t.*

If we assume that the fixed default-boundary (i.e. $D = D^*$) is a reasonable description of reality than we are implicitly assuming that the value of the option writer's assets is large compared to the value of the option contract. Therefore the value of the option itself is assumed to not cause the firm to experience financial distress. This latter assumption is consistent with Longstaff and Schwartz (1995), as well as the vulnerable option models of Jarrow and Turnbull (1995), Hull and White (1995), Rich (1996) and Klein (1996). However, if the option is a significant component of the option writer's liabilities then there is a significant risk that the option itself may be the source of financial distress. In this case the "variable default-boundary" (i.e. $D = D^* + \max(S_T - K, 0)$) would be a better description of reality. This latter assumption is consistent with the framework of Johnson and Stulz (1987).

The next assumption deals with the reduction in value of the nominal claims if the default event should occur. We assume that both the debt-holders and the option holders rank equally in the event of default. Again, we differentiate between the fixed and variable default boundaries.

Assumption 7a: Write-down in the event of Default: Fixed Default-Boundary

The percentage write-down on the nominal claim of the option holder is

$w = 1 - \dfrac{(1-\alpha)V_T}{D^*}$ *where α represents the deadweight costs of the financial distress, expressed as a percentage of the value of the assets of the option writer. The ratio $\dfrac{V_T}{D^*}$ represents the value of the option writer's assets available to pay the claim expressed as a proportion of total claims at T.*

Assumption 7b: Write-down in the event of Default: Variable Default-Boundary

The percentage write-down on the nominal claim of the option holder is

$w = 1 - \dfrac{(1-\alpha)V_T}{D^* + \max(S_T - K)}$ *where α represents the deadweight costs of the financial distress, expressed as a percentage of the value of the assets of the option writer. The ratio $\dfrac{V_T}{D^* + \max(S_T - K)}$ represents the value of the option writer's assets available to pay the claim expressed as a proportion of total claims at T.*

We follow the approach of Longstaff and Schwartz (1995) when modeling the allocation of the option writer's assets upon resolution of the financial distress with appropriate modifications to account for the correlation between the returns on the assets of the option writer and the returns on the asset underlying the option. As in Longstaff and Schwartz (1995), " Rather than trying to model the complex bargaining process among corporate claimants during a restructuring or bankruptcy, we take the allocation of the firm's assets as exogenously given". We also assume all claims on the option writer are of equal priority, but note that our model could be extended to allow for multiple levels of seniority.

Longstaff and Schwartz (1995) also assume that both the deadweight bankruptcy costs and the value of the firm's assets available to be paid out to claimants are exogeneously determined. In their model, a holder of a vulnerable bond receives, in the event of financial distress, $(1-w)$ times the value of the non-vulnerable bond, where w represents the static percentage writedown due to financial distress. This approach is justified in their model since the nominal claim in bankruptcy is independent of the value of the firm's assets. As they point out, their model could be extended to link the payout ratio to the value of the assets of the firm. Provided the independence assumption is maintained w could simply be replaced with $\mathrm{E}[w]$ which could depend on the expected value of the assets at the time of the resolution of financial distress.

One of the goals of this paper is to analyze the effect on vulnerable options values when the percentage writedown (w) is not independent from the assets of the option writer. Therefore, we make an assumption similar to Longstaff and Schwartz concerning the deadweight costs of financial reorganization, but model directly the total amount of the firm's assets which are available to be paid out in the event of financial distress. In other words we divide the percentage writedown, w, into two components: α, which represents proportional deadweight costs, and, $\frac{V_T}{D^*}$ or $\frac{V_T}{D^* + \max(S_T - K)}$ which are linked to the assets of the option writer and have been explained above.

3.2 Criticism of Fixed Default-Boundary Models

A large number of models of financial distress use some variation of a fixed default-boundary to trigger financial distress. In the risky bond literature, both Longstaff and Schwartz (1995) and Kim, Ramaswamy, and Sundaresan (1993) use a fixed default-boundary in which default is triggered when the assets of the issuing firm drop below some exogenously determined level. In the vulnerable option literature, the models of, Klein (1996), Rich (1996) and Klein and Inglis (1999)[1] trigger default whenever the assets of the option writer fall below a pre-specified default-boundary. There are two major problems with this framework. The first major defect with this approach is that there is nothing in these models to assure that the actual payments to the bondholders and option holders are not greater than the value of the firm's assets. This is obviously not possible. This occurs weather default is allowed to occur at anytime or just at maturity of the option. The problem arises because the payoff in the event of default is exogenously specified and is independent of the value of the firm's assets. In these models the default-boundary is often chosen to be less than the promised repayment. This allows the firm to operate in a small negative net worth position and prevents default being triggered the instant the assets of the firm fall below the promised repayment. To illustrate the defect consider the following simple example. Assume that in the event of default the firm will pay 80% of the nominal claim of $100 or $80. If the default-boundary is set to $70, then in the event of default the value of the firm's assets will be worth $70, while the assumed repayment is $80. This example also, illustrates another problem that can occur at the maturity date. The firm could be solvent and yet not be able to make the promised payment. For example the firm's assets could always stay above the default-boundary and be worth say $75 at maturity. The firm would have always been solvent and yet not be able to make the promised $100 payment. Again, the problem is that the payment made in the event of default is not linked to the assets of the firm at that time. The fixed default-boundary model presented in this paper corrects this particular defect by linking the payout to the available assets of the firm.

[1] Hull and White (1995) may also suffer from these potential difficulties. If, as in the example presented in their paper, default is defined as dropping below a fixed default-boundary and the payout ratio is predefined constant.

In the fixed default-boundary vulnerable option pricing models of Hull and White (1995), Klein (1996) and Klein and Inglis (1999) a similar but different problem can occur. None of these models incorporate the fact that the firm may not have enough assets to payoff the debt plus the potentially large liability generated by the written option. That is, the potential payout on the written option is assumed to be relatively small compared to the remaining liabilities and is therefore ignored. This may be a reasonable assumption to make for large firms with few written options, but could lead to large pricing errors for other firms, where written options represent a large fraction of the firm's total liabilities. This defect is corrected in the variable default-boundary model presented in this paper, where the potential payout on the written option is directly considered to be liability. Consider an example where the assets of the firm turn out to be worth $65 at maturity and it is assumed that the firm will pay 80% of the nominal claim. Assume the promised payoff on the debt is $70 and the intrinsic value of the written option is $30 for a total nominal claim of $100. In the Klein and Inglis (1999) fixed default-boundary model, the option writer would be in default since total assets ($65) are less then the promised payoff of the debt (i.e. $70). The actual payoff would be 0.8*(65/70)*70 = $52 to the debt holders and 0.8*(65/70)*30 = $22.29 to the option holders for a total of $74.29 which is greater then the actual assets of $65 (even ignoring the cost of bankruptcy). Note that, when we use the fixed default-boundary models we are effectively assuming that the $22.29 payment is small relative to the total liabilities, an assumption which is clearly wrong is this example. Using the variable default-boundary model presented in this paper, the firm would again be in default at maturity since total assets ($65) are less than the total nominal claim ($100). In this case the debt-holders would receive 0.8*(65/100)*70= $36.40 and the option holder's 0.8*(65/100)*30 = $15.60 for a total payout of $52, which is exactly 80% of the total assets (i.e. is equal to the amount of assets available to be paid out. As a result the fixed default-boundary model will overprice all vulnerable options to some extent. The variable default-boundary model presented in this paper over-comes the problem of assuming higher payments to stakeholders than will actually occur. The key is to allow the actual payouts to be a function of the value of the firm's assets and to correctly specify all of the firm's liabilities.

The second problem is that the fixed boundary does not account for the possibility that the option's payoff can increase the possibility of financial distress on the part of the option writer. Using the fixed default-boundary model of Klein (1996) and Klein and Inglis (1999) we can illustrate this problem with the same example used above. Assume that in the event of default the firm will pay 80% of the total nominal claim of $100 ($70 to bondholders and $30 to option holders). If the default-boundary is set to $70 and the value of the firm's assets turns out to be $75, then the firm will not be in default, according to the FDB model. However, it is also assumed that the firm will be able to payout a total of $100, which is clearly not possible. Using the variable default model presented in this thesis the firm would be in default (since $75 < $100) and the payouts would be 0.8*(75/100)*70 = $42 to bondholders and 0.8*(75/100)*30 = $18 to option holders, which is exactly equal to the 0.8*75 = $60 of assets available to be paid out to claimants. Again, fixed default-boundary models, which ignore the option's potential to cause financial distress, will overprice vulnerable options. This problem is overcome in the variable default-boundary model presented here.

3.3 Justification of Only Testing for Default at the Maturity of the Option

Assumptions 5, 6 and 7, which allow default to occur only at the maturity of the option, are not as restrictive as they may first appear because of an important difference between the treatment of European options and other securities when financial distress occurs. In the case of debt instruments, for example, indentures typically provide that the repayment of the principal amount is accelerated. As a result, the exact timing of the initiation of financial distress is an important consideration when modeling the effect of credit risk on debt instruments, and justifies the focus on this issue in the recent literature such as in Longstaff and Schwartz (1995). In contrast, most over the counter options are governed by a standardized contract recommended by the International Swaps and Derivatives Association (ISDA). The standard ISDA agreement does not require acceleration of the exercise date of European options when the writer experiences financial distress. Instead, the option holder can terminate the contract at any subsequent time if financial distress "has occurred and is then continuing". This allows holders of options to wait until the original maturity date T when determining their nominal claim, which is consistent with Assumption 5. Since there is usually a substantial delay in the resolution of financial distress, option holders may often wait until the maturity date of the option before calculating their nominal claim.

If the option holder decides to terminate before T, the nominal claim of the option holder is determined by the "Market Quotation" provision in the ISDA agreement. This amount is defined as the amount that would have to be paid by "Reference Market-makers" to enter into a transaction "that would have the effect of preserving for such party the economic equivalent of any payment or delivery" that would have been required after that date, had the financial distress not occurred. In other words, the option holder's nominal claim is the non-vulnerable value of the option at the time the Market Quotation provision is applied. This non-vulnerable value depends in turn on the expected value, at the time the option holder elects to terminate the contract, of the underlying asset at the maturity date T of the option. Assumption 6 is considered to be a reasonable approximation of this non-vulnerable value because the intrinsic value of the option at time T also depends on the value of the underlying asset at the maturity of the option.

Assumption 7 also uses time T when determining the percentage writedown on the nominal claim of the option holder. We believe this choice is appropriate for a number of reasons. First, it should be noted that using the time of the start of financial distress (i.e., some point in time $\tau < T$) is not useful because at that time $V_\tau = D^*$ by definition and thus the percentage write-down becomes fixed. Second, there is usually a substantial delay in the resolution of financial distress as noted above. Since the assets of the firm are not distributed until sometime after the event of default, measuring the option writer's assets at τ ignores the possibility that the option writer may recover from financial distress by time T. In the absence of theoretical guidance on the exact timing of the resolution of financial distress, we claim that measuring the option writer's assets at time T represents a reasonable proxy for the amount of assets available to the option writer's creditors. This choice is also computationally convenient because it corresponds to the time at which financial distress is assumed to be able to occur in Assumption 5.[2] Checking for default only at the maturity of the option is obviously an unrealistically simple assumption and yet this simple assumption may be its greatest strength. Both academics and practitioners like the Black-Scholes model because its unrealistically simple assumptions are easily understood. The assumptions of this valuation framework preserve the intuitive appeal of the traditional Black-Scholes model and allow a relatively simple and tractable result that provides some broad insights into how default risk affects vulnerable option valuation.

3.4 Payoff Tables for Both the FDB and VDB Models

Although the assumptions presented above deal specifically with a call written by a firm that may default on its future obligations, we can also write down the payoffs that could be expected for the debt, call, total liabilities and equity of this firm. Tables 1a and 1b present the payoffs on each of the securities issued by the firm in each of the four states of the world considered in this framework

3.5 Partial Differential Equation

We use the standard no arbitrage results to derive the fundamental partial differential equation defining the price F(V,S,B,T) of any derivative security with payoff at time, T, contingent upon the value of the writer's assets and value of the asset underlying the option and the price of a risk-free discount bond. The value, in units of the discount bond, of any derivative, F, must satisfy the following partial differential equation:

$$F_t + \frac{1}{2}\sigma_V^2 V^2 F_{VV} + \frac{1}{2}\sigma_S^2 S^2 F_{SS} + \frac{1}{2}\sigma_B^2 B^2 F_{BB} + \rho_{VS}\sigma_V\sigma_B VSF_{VS} + \rho_{VB}\sigma_V\sigma_B VBF_{VB} + \rho_{SB}\sigma_S\sigma_B SBF_{SB} = 0 \tag{3.6.1}$$

[2] As noted by a reviewer, option buyers may require the writer of the option to post collateral, thus the full amount of the writer's obligation may not be at risk. This thesis assumes no such collateralization occurs.

Table 1a: Payoff Table for Fixed Default-Boundary Model

Money-ness of Call Option	$S_T \le K$ **(out-of-the-money)**		$S_T > K$ **(in-the-money)**	
Default Status of Option Writer	$V_T \le D^*$ (default)	$V_T > D^*$ (no-default)	$V_T \le D^*$ (default)	$V_T > D^*$ (no-default)
Debt	$(1-\alpha)V_T$	D^*	$(1-\alpha)\frac{V_T}{D^*}D^*$	D^*
Call	0	0	$(1-\alpha)\frac{V_T}{D^*}(S_T-K)$	S_T-K
Liabilities	$(1-\alpha)V_T$	D^*	$(1-\alpha)\frac{V_T}{D^*}(D^*+S_T-K)$	D^*+S_T-K
Equity	0	V_T-D^*	0	$V_T-D^*+S_T-K$
Total Assets (net of direct default costs)	$(1-\alpha)V_T$	V_T	$(1-\alpha)V_T$ *	V_T

*In the case where the option writer is in default on an in-the money call option, the payoff on the liabilities in this model is larger than the assets of the firm. This is clearly inconsistent with reality. This situation is corrected in the variable default-boundary model.

Table 1b: Payoff Table for Variable Default-Boundary Model

Moneyness of Call Option	$S_T \le K$ **(out-of-the-money)**		$S_T > K$ **(in-the-money)**	
Default Status of Option Writer	$V_T \le D^*$ **(default)**	$V_T > D^*$ **(no-default)**	$V_T \le D^*+S_T-K$ **(default)**	$V_T > D^*+S_T-$ **(no-default)**
Debt	$(1-\alpha)V_T$	D^*	$(1-\alpha)\frac{V_T}{D^*+S_T-K}D^*$	D^*
Call	0	0	$(1-\alpha)\frac{V_T}{D^*+S_T-K}(S_T-K)$	S_T-K
Liabilities	$(1-\alpha)V_T$	D^*	$(1-\alpha)V_T$	D^*+S_T-K
Equity	0	V_T-D^*	0	$V_T-D^*+S_T-K$
Total Assets (net of default costs)	$(1-\alpha)V_T$	V_T	$(1-\alpha)V_T$	V_T

The derivation of this PDE is given in Appendix A. The value of a vulnerable call under the FDB and VDB models can be obtained by solving the PDE subject to the boundary conditions given in Tables 1a and 1b. Note that neither equation 3.6.1 nor the boundary conditions contain any terms that involve investor's risk preferences.

Chapter 4

Valuation of Vulnerable European Calls

In this chapter we present the valuation equations used to find the value of vulnerable European call options. We start by presenting the valuation equation for the fixed default boundary model (FDB) of Klein and Inglis (1999). This model is analytically tractable and we present the valuation formula for this model. Next we present the valuation equation for the variable default boundary model (VDB). This equation is not analytically tractable, however by making some simplifying assumptions we derive an approximate analytical solution. We also show how Vasicek's (1977) term structure model can be incorporated into both of these models. Since, no exact valuation formula exists for the VDB model we also employ a Monte Carlo simulation to generate numerical results. The results of the two different solution techniques for the VDB model are compared in the next chapter.

4.1 Fixed Default Boundary: Analytical Solution

We use the risk-neutral pricing approach first proposed by Cox and Ross (1976) and formalized by Harrison and Pliska (1981). The valuation equation for a vulnerable European call, for the FDB model of Klein and Inglis (1999) is given by:

$$c = E\left[B\left\{\max(S_T - K,0) \;\mid\; V_T \geq D^*\right\}\right] + E\left[B(1-\alpha)\frac{V_T}{D^*}\max(S_T - K,0) \;\mid\; V_T < D^*\right] \tag{4.1.1}$$

where E denotes risk neutral expectations. This expression shows that the expected future payout on the nominal claim of amount $\max(S_T - K, 0)$ is comprised of two components, which are conditional on the terminal value of the option writer's assets. The nominal claim is paid out in full if the assets of the option writer are greater than the default boundary. If the assets are below the default boundary, the payout is only a proportion, $(1-\alpha)\frac{V_T}{D^*}$, of the nominal claim.

To derive the pricing equation for a vulnerable European call, assuming a fixed default boundary at maturity of the option we must evaluate to following integral:

$$c = \int_{K}^{\infty}\int_{D^*}^{\infty} BP(V_T, S_T)(S_T - K)\,dV_T\,dS_T + \int_{K}^{\infty}\int_{-\infty}^{D^*} BP(V_T, S_T)(1-\alpha)\frac{V_T}{D^*}(S_T - K)\,dV_T\,dS_T \tag{4.1.2}$$

where $P(V_T, S_T)$ represents a bi-variate lognormal distribution and the other variables are as defined above. This expression can be considerably simplified by writing the value of the call in units of the risk-free bond, B, and applying the standard log transformation. The valuation expression can then be expressed as:

$$c = \int_{\ln K}^{\infty}\int_{\ln D^*}^{\infty} n_2(X_T, Y_T)(e^{Y_T} - K)\,dX_T\,dY_T + \int_{\ln K}^{\infty}\int_{-\infty}^{\ln D^*} n_2(X_T, Y_T)(1-\alpha)\frac{e^{X_T}}{D^*}(e^{Y_T} - K)\,dX_T\,dY_T \tag{4.1.3}$$

where $n_2(X_T, Y_T)$ represents a bi-variate normal distribution. Appendix B demonstrates the simple mapping of the three random variables B, V_T and S_T into the two random variables X_T and Y_T. It also demonstrates that X_T and Y_T are bi-variate normally distributed.

Proposition 1: The value of a European call, assuming the fixed default boundary can be expressed as: (See appendix C for a proof of Proposition 1)

$$c = SN_2(a_1, a_2, \bar{\rho}_{VS}) - KBN_2(b_1, b_2, \bar{\rho}_{VS}) + \frac{S(1-\alpha)V}{DB}\exp(s_S s_V \bar{\rho}_{VS}) N_2(c_1, c_2, -\bar{\rho}_{VS}) - \frac{K(1-\alpha)V}{D} N_2(d_1, d_2, -\bar{\rho}_{VS}) \tag{4.1.4}$$

where N_2 represents the cumulative bi-variate normal distribution function and the arguments of N_2 are given by:

$$a_1 = \frac{\ln\left(\frac{S}{BK}\right) + \frac{s_S^2}{2}}{s_S} \qquad a_2 = \frac{\ln\left(\frac{V}{BD^*}\right) - \frac{s_V^2}{2} + s_V s_S \bar{\rho}_{VS}}{s_V}$$

$$b_1 = \frac{\ln\left(\frac{S}{BK}\right) - \frac{s_S^2}{2}}{s_S} \qquad b_2 = \frac{\ln\left(\frac{V}{BD^*}\right) - \frac{s_V^2}{2}}{s_V}$$

$$c_1 = \frac{\ln\left(\frac{S}{BK}\right) + \frac{s_S^2}{2} + s_V s_S \bar{\rho}_{VS}}{s_S} \qquad c_2 = -\frac{\ln\left(\frac{V}{BD^*}\right) + \frac{s_V^2}{2} + s_V s_S \bar{\rho}_{VS}}{s_V}$$

$$d_1 = \frac{\ln\left(\frac{S}{BK}\right) - \frac{s_S^2}{2} + s_V s_S \bar{\rho}_{VS}}{s_S} \qquad d_2 = -\frac{\ln\left(\frac{V}{BD^*}\right) + \frac{s_V^2}{2}}{s_V}$$

The additional parameters used in our model are given by:

$$s_V^2 = \int_0^T \hat{\sigma}_V^2(T-\tau)d\tau$$

$$s_S^2 = \int_0^T \hat{\sigma}_S^2(T-\tau)d\tau$$

$$s_{VS}(T) = \int_0^T \hat{\rho}_{VS}(T-\tau)\hat{\sigma}_V(T-\tau)\hat{\sigma}_S(T-\tau)d\tau$$

$$\hat{\sigma}_V^2(T) = \sigma_V^2 + \sigma_B^2(T) - 2\sigma_V\sigma_B(T)\rho_{VB}$$

$$\hat{\sigma}_S^2(T) = \sigma_S^2 + \sigma_B^2(T) - 2\sigma_S\sigma_B(T)\rho_{SB}$$

$$\hat{\rho}_{VS}(T) = \frac{\sigma_V\sigma_S\rho_{VS} - \sigma_V\sigma_B(T)\rho_{VB} - \sigma_S\sigma_B(T)\rho_{SB} + \sigma_B^2(T)}{\hat{\sigma}_V(T)\hat{\sigma}_S(T)}$$

$$\bar{\rho}_{VS} = \frac{s_{VS}}{s_V s_S}$$

4.2 Variable Default Boundary Model: Approximate Analytical Solution

The valuation equation for a vulnerable European call, assuming the VDB model proposed in this paper is given by:

$$\begin{aligned} c = {} & E\left[B\left\{\max(S_T - K, 0) \;\middle|\; V_T \geq D^* + S_T - K\right\}\right] + \\ & E\left[B\left\{(1-\alpha)\frac{V_T}{D^* + S_T - K}\max(S_T - K, 0) \;\middle|\; V_T < D^* + S_T - K\right\}\right] \end{aligned}$$

(4.2.1)

where E denotes risk neutral expectations. This expression shows that the expected future payout on the nominal claim of amount $\max(S_T - K, 0)$ is again comprised of two components. The nominal claim is paid out in full if the assets of the option writer are greater than the default boundary of $D^* + S_T - K$. If the assets are below the default boundary, a proportion, $(1-\alpha)\dfrac{V_T}{D^* + S_T - K}$, of the nominal claim is paid out. There are two major differences between this model and the FDB model. First the default boundary is split into two components: a fixed component, which is the result of the other liabilities (i.e. debt) of the firm and a stochastic component due to the potential payout on the option. The second difference is in the proportion of the nominal claim that is paid out in the event of default. The VDB model accounts directly for the potential payout created by the option and therefore guarantees that the total payout in the event of default is equal to the assets available to be paid out. This is not true of the FDB model.

To derive the pricing equation for a vulnerable European call, assuming a variable default boundary at maturity we must evaluate to following integral:

$$c = \int_{K}^{\infty} \int_{D^* + S_T - K}^{\infty} BP(V_T, S_T)(S_T - K)\,dV_T\,dS_T + \int_{K}^{\infty} \int_{-\infty}^{D^* + S_T - K} BP(V_T, S_T)(1-\alpha)\frac{V_T}{D^* + S_T - K}(S_T - K)\,dV_T\,dS_T \tag{4.2.2}$$

where $P(V_T, S_T)$ represents a bi-variate lognormal distribution and the other variables are as defined above. This expression can also be simplified by writing the value of the call in units of the risk-free bond, B, applying the standard log transformation as in the FDB model. The valuation expression for this model then becomes:

$$c = \int_{\ln K}^{\infty} \int_{\ln(D^* + e^{Y_T} - K)}^{\infty} n_2(X_T, Y_T)(e^{Y_T} - K)\,dX_T\,dY_T + \int_{\ln K}^{\infty} \int_{-\infty}^{\ln(D^* + e^{Y_T} - K)} n_2(X_T, Y_T)(1-\alpha)\frac{e^{X_T}}{D^* + e^{Y_T} - K}(e^{Y_T} - K)\,dX_T\,dY_T \tag{4.2.3}$$

where $n_2(X_T, Y_T)$ represents a bi-variate normal distribution with means, standard deviations and correlation as given in appendix B. We can standardize this distribution using a simple transformation of variables. The resulting valuation equation is given as:

$$
\begin{aligned}
c = \Bigg[& \int_{-a}^{\infty}\int_{f(\tilde{u})}^{\infty} \frac{S}{B}\exp\left(-\frac{s_S^2}{2}+s_S\tilde{u}\right)\frac{1}{2\pi\sqrt{1-\bar{\rho}_{VS}^{\,2}}}\exp\left\{-\frac{1}{2(1-\bar{\rho}_{VS}^{\,2})}\left[\tilde{u}^2-2\bar{\rho}_{VS}\tilde{u}\tilde{v}+\tilde{v}^2\right]\right\}d\tilde{v}\,d\tilde{u} \\
& -\int_{-a}^{\infty}\int_{f(\tilde{u})}^{\infty} \frac{K}{2\pi\sqrt{1-\bar{\rho}_{VS}^{\,2}}}\exp\left\{-\frac{1}{2(1-\bar{\rho}_{VS}^{\,2})}\left[\tilde{u}^2-2\bar{\rho}_{VS}\tilde{u}\tilde{v}+\tilde{v}^2\right]\right\}d\tilde{v}\,d\tilde{u} \\
& +\int_{-a}^{\infty}\int_{-\infty}^{f(\tilde{u})} \frac{(1-\alpha)\frac{V}{B}\frac{S}{B}\exp\left(-\frac{s_S^2}{2}+s_S\tilde{u}-\frac{s_V^2}{2}+s_V\tilde{v}\right)}{D^*-K+\frac{S}{B}\exp\left(-\frac{s_S^2}{2}+s_S\tilde{u}\right)}\frac{1}{2\pi\sqrt{1-\bar{\rho}_{VS}^{\,2}}}\exp\left\{-\frac{1}{2(1-\bar{\rho}_{VS}^{\,2})}\left[\tilde{u}^2-2\bar{\rho}_{VS}\tilde{u}\tilde{v}+\tilde{v}^2\right]\right\}d\tilde{v}\,d\tilde{u} \\
& -\int_{-a}^{\infty}\int_{-\infty}^{f(\tilde{u})} \frac{(1-\alpha)K\frac{V}{B}\exp\left(-\frac{s_V^2}{2}+s_V\tilde{v}\right)}{D^*-K+\frac{S}{B}\exp\left(-\frac{s_S^2}{2}+s_S\tilde{u}\right)}\frac{1}{2\pi\sqrt{1-\bar{\rho}_{VS}^{\,2}}}\exp\left\{-\frac{1}{2(1-\bar{\rho}_{VS}^{\,2})}\left[\tilde{u}^2-2\bar{\rho}_{VS}\tilde{u}\tilde{v}+\tilde{v}^2\right]\right\}d\tilde{v}\,d\tilde{u}\Bigg]
\end{aligned}
\tag{4.2.4}
$$

where

$$
a = \left[\frac{\ln\left(\frac{S}{KB}\right)-\frac{s_S^2}{2}}{s_S}\right] \quad \text{and} \quad f(\tilde{u}) = \frac{\ln\left(\dfrac{D^*-K+\frac{S}{B}\exp\left(-\frac{s_S^2}{2}+s_S\tilde{u}\right)}{\frac{V}{B}}\right)+\frac{s_V^2}{2}}{s_V}
$$

and $\tilde{u}$ and $\tilde{v}$ are standard normal variates with zero mean and standard deviation equal to one.

This valuation equation is not analytically tractable, however we can find an approximate analytical solution. The first problem occurs because of the non-linear boundary condition $f(\tilde{u})$ in second integral of all of the terms in equation 4.2.4. We employ a first order Taylor series expansion of the boundary about the point "p" to simplify the integrals in each of the above terms. The second problem occurs because of the hyperbolic form of the integrand in both the third and fourth terms on equation 4.2.4.

This function has to be transformed into an exponential form so that it can be combined with the bi-variate normal distribution. We employ a second first order Taylor series expansion about the point "q" to overcome this problem. Therefore, the approximate analytical solution becomes a function of two additional parameters, "p" and "q". We develop a simple procedure for estimating these additional parameters in a later section of this chapter.

Proposition 2: The approximate value of a vulnerable European call, assuming a variable default boundary at the option's maturity can be expressed as: (See appendix D for a proof of Proposition 2)

$$c = SN(a_1, a_2, \delta) - KBN(b_1, b_2, \delta)$$

$$+ \frac{(1-\alpha)V\frac{S}{B}\exp\left(-\frac{s_V^2}{2}-\frac{s_S^2}{2}\right)\exp(-gq)}{D^* - K + \frac{S}{B}\exp\left(-\frac{s_S^2}{2}+s_S q\right)}$$

$$\exp\left(\frac{(g+s_S+ms_V)^2 + 2(g+s_S+ms_V)(\bar{\rho}_{VS}-m)s_V + \left(\sqrt{1-2\bar{\rho}_{VS}m+m^2}\,s_V\right)^2}{2}\right)N_2(c_1, c_2, -\delta)$$

$$- \frac{(1-\alpha)KV\exp\left(-\frac{s_V^2}{2}\right)\exp(-gq)}{D^* - K + \frac{S}{B}\exp\left(-\frac{s_S^2}{2}+s_S q\right)}$$

$$\exp\left(\frac{(g+s_S+ms_V)^2 + 2(g+s_S+ms_V)(\bar{\rho}_{VS}-m)s_V + \left(\sqrt{1-2\bar{\rho}_{VS}m+m^2}\,s_V\right)^2}{2}\right)N_2(d_1, d_2, -\delta)$$

(4.2.5)

where N_2 represents the cumulative bivariate normal distribution function. The arguments of N_2 are given by:

$$a_1 = \frac{\ln\left(\frac{S}{BK}\right) + \frac{s_S^2}{2}}{s_S} \qquad a_2 = -\frac{b - mp}{\sqrt{1 - 2\bar{\rho}_{VS} m + m^2}} + \delta s_S.$$

$$b_1 = \frac{\ln\left(\frac{S}{BK}\right) + \frac{s_S^2}{2}}{s_S} \qquad b_2 = -\frac{b - mp}{\sqrt{1 - 2\bar{\rho}_{VS} m + m^2}}$$

$$c_1 = a + (g + s_S + ms_V) + \delta\left(\sqrt{1 - 2\bar{\rho}_{VS} m + m^2}\, s_V\right.$$

$$c_2 = \frac{b - mp}{\sqrt{1 - 2\bar{\rho}_{VS} m + m^2}} - \delta(g + s_S + ms_V) - \left(\sqrt{1 - 2\bar{\rho}_{VS} m + m^2}\, s_V\right)$$

$$d_1 = a + (g + ms_V) + \delta\left(\sqrt{1 - 2\bar{\rho}_{VS} m + m^2}\, s_V\right)$$

$$d_4 = \frac{b - mp}{\sqrt{1 - 2\bar{\rho}_{VS} m + m^2}} - \delta(g + ms_V) - \left(\sqrt{1 - 2\bar{\rho}_{VS} m + m^2}\, s_V\right)$$

The additional parameters used in this solution are given by:

$$a = \frac{\ln(\frac{S}{KB}) - \frac{s_S^2}{2}}{s_S}$$

$$b = \frac{\ln\left(\frac{D^{*} - K + \frac{S}{B}\exp\left(-\frac{s_S^2}{2} + s_S p\right)}{\frac{V}{B}}\right) - \frac{s_V^2}{2}}{s_V}$$

$$g = -\frac{s_S \frac{S}{B}\exp\left(-\frac{s_S^2}{2} + s_S q\right)}{D^* - K + \frac{S}{B}\exp\left(-\frac{s_S^2}{2} + s_S q\right)}$$

$$m = \frac{\sigma_S}{\sigma_V}\left[\frac{\frac{S}{B}\exp\left(-\frac{s_S^2}{2} + s_S p\right)}{D^* - K + \frac{S}{B}\exp\left(-\frac{s_S^2}{2} + s_S p\right)}\right]$$

$$\delta = \frac{\bar{\rho}_{VS} - m}{\sqrt{1 - 2\bar{\rho}_{VS} m + m^2}}$$

4.3 Observations about the Approximate Analytical Solution

Our objective in deriving the approximate analytical solution to the VDB model is to develop a simple analytical approximation that will capture the majority, but not all of the behaviour caused by the variable default boundary. Within this framework, we have allowed for a variable default condition at the maturity of the option as well as linking the nominal claim in the event of financial distress to the value of the assets of the option writer. We also, allow for existence of other liabilities in the capital structure of the option writer.

Although the terms of equation 4.2.5 are quite complicated, an intuitive explanation is relatively straightforward. The first two terms of this equation give the expected value of the option conditional on no financial distress occurring and are similar to the terms in Merton's (1973) derivation of the Black-Scholes valuation formula. The arguments a_1 and b_1 in the bi-variate cumculative normal distribution function correspond to the arguments in the uni-variate cumulative normal distribution function in the Black-Scholes formula. The arguments a_2 and b_2 relate to the probability of financial distress. The third and fourth terms in equation 4.2.5 provide the expected value of the call conditional upon financial distress having occurred. These terms are analogous to the first two terms but also incorporate the expected loss to the option holder because of the financial distress. Note that if there is no probability of financial distress (e.g. $V >> D^{*} + S - K$) the third and fourth terms disappear entirely. It can easily be shown that equation 4.2.5 reduces to the Black-Scholes formula if we assume no default risk and constant interest rates.

It is interesting to note that the VDB model converges to the FDB model of Klein and Inglis (1999) as D^{*} increases in value as compared to $S - K$. As D^{*} grows sufficiently large the variable default boundary becomes closer to the fixed default boundary. Also, note that as D^{*} approaches zero, our VDB valuation formula provides an approximation to the valuation equation of Jonhson and Stulz (1987).

4.4 Determination of the Design Parameters "p" and "q"

The approximate analytical solution of the variable default model depends on two design parameters, "p" and "q". These parameters represent the points of expansion in each of the two Taylor series expansions used to simplify the valuation integral. In this section we present a simple procedure for estimating reasonable values for these parameters independently of each other. Table 1b shows the payoff on the debt, written call, total liabilities and the equity of the firm at the maturity of the call assuming the VDB model. Note that the valuation equation for the total liabilities of the firm will depend on the parameter "p", but will not depend on the parameter "q". This is because the non-linear boundary condition will still impact the value of the liabilities, but the hyperbolic integrand does not appear in any of the payoffs, in any of the four states of the world. We can exploit this fact to estimate the parameter "p" independently of "q". First we need to develop the valuation equation for the total liabilities of the option writer.

4.4.1 Valuation of the Option Writer's Total Liabilities

The valuation equation for the total liabilities of the option writer, assuming the VDB model proposed in this paper is given by:

$$
\begin{aligned}
L = {} & E\left[B\left\{(1-\alpha)V_T \quad | \quad V_T \le D^*, \quad S_T \le K\right\}\right] + \\
& E\left[B\left\{D^* \mid V_T > D^*, \quad S_T \le K\right\}\right] + \\
& E\left[B\left\{(1-\alpha)V_T \quad | \quad V_T \le D^* + S_T - K, \quad S_T > K\right\}\right] + \\
& E\left[B\left\{(1-\alpha)\left(D^* + S_T - K\right) \quad | \quad V_T > D^* + S_T - K, \quad S_T > K\right\}\right]
\end{aligned}
$$

(4.4.1.1)

where E denotes risk neutral expectations. This valuation equation can be expressed as:

$$
\begin{aligned}
L = {} & \int_{-\infty}^{K}\int_{-\infty}^{D^*} BP(V_T,S_T)(1-\alpha)V_T dV_T dS_T + \int_{-\infty}^{K}\int_{D^*}^{\infty} BP(V_T,S_T)D^* dV_T dS_T + \\
& \int_{K}^{\infty}\int_{-\infty}^{D^*+S_T-K} BP(V_T,S_T)(1-\alpha)V_T dV_T dS_T + \int_{K}^{\infty}\int_{D^*+S_T-K}^{\infty} BP(V_T,S_T)(1-\alpha)\left(D^*+S_T-K\right)dV_T dS_T
\end{aligned}
$$

where $P(V_T, S_T)$ represents a bi-variate lognormal distribution. This expression can also be simplified by writing the value of the liabilities in units of the risk-free bond *B(T-t)* and applying the standard log transformation as in the FDB and VDB models. The valuation expression for this model then becomes:

$$
\begin{aligned}
L = {} & \int_{-\infty}^{\ln(K)}\int_{-\infty}^{\ln(D^*)} n_2(X_T,Y_T)(1-\alpha)e^{X_T} dX_T dY_T + \int_{-\infty}^{\ln(K)}\int_{\ln(D^*)}^{\infty} n_2(X_T,Y_T)D^* dX_T dY + \\
& \int_{\ln(K)}^{\infty}\int_{-\infty}^{\ln(D^*+e^{Y_T}-K)} n_2(X_T,Y_T)(1-\alpha)e^{X_T} dX_T dY + \int_{\ln(K)}^{\infty}\int_{\ln(D^*+e^{Y_T}-K)}^{\infty} n_2(X_T,Y_T)\left(D^*+e^{Y_T}-K\right)dX_T dY
\end{aligned}
$$

(4.4.1.2)

where $n_2(X_T, Y_T)$ represents a bi-variate normal distribution with means, standard deviations and correlation as given in appendix B. We again employ a simple transformation of variables to standardize the normal distribution. The valuation equation can then be expressed as:

$$L=\left[\int_{-\varepsilon}^{-a}\int_{-\varepsilon}^{-bb}(1-\alpha)\frac{V}{B}\exp\left(-\frac{s_V^2}{2}+s_V\tilde{v}\right)\frac{1}{2\pi\sqrt{1-\bar{\rho}_{VS}^{\;2}}}\exp\left\{-\frac{1}{2(1-\bar{\rho}_{VS}^{\;2})}\left[\tilde{u}^2-2\bar{\rho}_{VS}\tilde{u}\tilde{v}+\tilde{v}^2\right]\right\}d\tilde{v}d\tilde{u}\right.$$

$$+\int_{-\varepsilon}^{-a}\int_{-bb}^{\varepsilon}\frac{D^*}{2\pi\sqrt{1-\bar{\rho}_{VS}^{\;2}}}\exp\left\{-\frac{1}{2(1-\bar{\rho}_{VS}^{\;2})}\left[\tilde{u}^2-2\bar{\rho}_{VS}\tilde{u}\tilde{v}+\tilde{v}^2\right]\right\}d\tilde{v}d\tilde{u}$$

$$+\int_{-a}^{\varepsilon}\int_{-\varepsilon}^{f(\tilde{u})}(1-\alpha)\frac{V}{B}\exp\left(-\frac{s_V^2}{2}+s_V\tilde{v}\right)\frac{1}{2\pi\sqrt{1-\bar{\rho}_{VS}^{\;2}}}\exp\left\{-\frac{1}{2(1-\bar{\rho}_{VS}^{\;2})}\left[\tilde{u}^2-2\bar{\rho}_{VS}\tilde{u}\tilde{v}+\tilde{v}^2\right]\right\}d\tilde{v}d\tilde{u}$$

$$+\int_{-a}^{\varepsilon}\int_{f(\tilde{u})}^{\varepsilon}\frac{S}{B}\exp\left(-\frac{s_S^2}{2}+s_S\tilde{u}\right)\frac{1}{2\pi\sqrt{1-\bar{\rho}_{VS}^{\;2}}}\exp\left\{-\frac{1}{2(1-\bar{\rho}_{VS}^{\;2})}\left[\tilde{u}^2-2\bar{\rho}_{VS}\tilde{u}\tilde{v}+\tilde{v}^2\right]\right\}d\tilde{v}d\tilde{u}$$

$$\left.+\int_{-a}^{\varepsilon}\int_{f(\tilde{u})}^{\varepsilon}\frac{D^*-K}{2\pi\sqrt{1-\bar{\rho}_{VS}^{\;2}}}\exp\left\{-\frac{1}{2(1-\bar{\rho}_{VS}^{\;2})}\left[\tilde{u}^2-2\bar{\rho}_{VS}\tilde{u}\tilde{v}+\tilde{v}^2\right]\right\}d\tilde{v}d\tilde{u}\right] \tag{4.4.1.3}$$

where

$$a=\left[\frac{\ln(\frac{S}{BK})-\frac{s_S^2}{2}}{s_S}\right]\quad;\quad bb=\left[\frac{\ln(\frac{V}{BD^*})-\frac{s_V^2}{2}}{s_V}\right]\quad\text{and}\quad f(\tilde{u})=\frac{\ln\left(\dfrac{D^*-K+\frac{S}{B}\exp\left(-\frac{s_S^2}{2}+s_S\tilde{u}\right)}{\frac{V}{B}}\right)+\frac{s_V^2}{2}}{s_V}$$

and $\tilde{u}$ and $\tilde{v}$ are standard normal variates with zero mean and standard deviation equal to one.

This valuation equation is again not analytically tractable, however we can find an approximate analytical solution. The only problem, in this case, occurs because of the non-linear boundary condition in the second integral in the third, fourth and fifth terms of equation 4.4.1.3. We again employ a first order Taylor series expansion of the non-linear boundary $f(\tilde{u})$, about the point "p" to simplify the integrals.

Proposition 3: The value of the total liabilities of the option writer, assuming a variable default boundary can be expressed as: (See appendix E for a proof of Proposition 3)

$$L = (1-\alpha)VN_2(a_1, a_2, \bar{\rho}_{VS}) + D^* BN_2(b_1, b_2, -\bar{\rho}_{VS})$$

$$+ (1-\alpha)V\ N_2(c_1, c_2, -\delta) + SN_2(d_1, d_2, \delta) + (D^* - K)BN_2(e_1, e_2, \delta)$$

(4.4.1.4)

where N_2 represents the cumulative bivariate normal distribution function. The arguments of N_2 are given by:

$a_1 = -a - \bar{\rho}_{VS} s_V$ $\qquad$ $a_2 = -bb - s_V$.

$b_1 = -a$ $\qquad$ $b_2 = \mathrm{bb}$

$c_1 = a + \bar{\rho}_{VS} s_V$ $\qquad$ $c_2 = \dfrac{b - mp}{\sqrt{1 - 2\bar{\rho}_{VS} m + m^2}} - \delta m s_V - \sqrt{1 - 2\bar{\rho}_{VS} m + m^2}\, s_V$

$d_1 = a + s_S$ $\qquad$ $d_4 = -\dfrac{b - mp}{\sqrt{1 - 2\bar{\rho}_{VS} m + m^2}} + \delta s_S$

$e_1 = a$ $\qquad$ $e_2 = -\dfrac{b - mp}{\sqrt{1 - 2\bar{\rho}_{VS} m + m^2}}$

All of the additional parameters are the same as those defined for the VDB vulnerable call. Note that the key is that this valuation formula depends only on design parameter "p" and not on "q". The objective is to choose "p" so as to accurately price the liabilities of the firm.

We need to make one additional point before determining the parameter "p". In general, the valuation equations for both the vulnerable call and the total liabilities are approximations, however there is one particular case where the solutions are exact.

4.4.2 $D^* = K$: An Exact Analytical Solution

There is one specific case where an exact analytical solution exists for the VDB vulnerable call and total liabilities of the option writer. If the face value of the debt at maturity, D^*, is equal to the strike price, K, of the call option then the valuation formulas given in equations 4.2.5 and 4.4.1.4 are exact and independent of the choice of the design parameters "p" and "q". The easiest way to see this result is to look at the valuation equations 4.2.4 and 4.4.1.3. First note that the non-linear boundary, $f(\tilde{u})$, reduces to $\left[\ln\left(\frac{S}{V}\right) - \frac{1}{2s_V}\left(s_S^2 - s_V^2\right)\right] + \left[\frac{s_S}{s_V}\right]\tilde{u}$, which is a linear function of $\tilde{u}$. Therefore, there is no need to use a first order Taylor series expansion to linearize the boundary condition. Second, note that the hyperbolic functional form of the integrand in the second term of equation 4.2.4 disappears if $D^* = K$. Again, the Taylor series expansion used in the general case is not required. Neither of the design parameters enters the valuation formulas. Alternatively, we could impose the condition $D^* = K$ on the final valuation formulas (i.e. equations 4.2.5 and 4.4.1.4). It is relatively easy to show that any dependence of either "p" or "q" disappears.

In addition we can show that the general non-linear boundary in equations 4.2.4 and 4.4.1.3 is always less than or equal to its linear approximation if $D^* < K$ and always greater than or equal to its approximation if $D^* > K$. For a proof see appendix F. This result is the key to the determination of the design parameter "p".

4.4.3 Estimation of Design Parameter "p"

The objective of this section is to demonstrate one simple technique for estimating a reasonable value for the design parameter "p". The idea is to pick "p" to accurately value the total liabilities of the option writer.

To understand the technique for determining the parameter "p", look at Figures 1 2 and 3. The parameter values used to generate these three examples are specified on each of the respective figures. Each of these figures shows the boundaries of integration

Figure 1

Integration Region for Vulnerable European Calls: ($D^* > K$)

Calculations of vulnerable call option prices are based on the following parameter values: $S = 50$, $K = 40$, $V = 100$, $D^* = 90$, $T = 3, \alpha = 0$, $\sigma_V = 0.1$, $\sigma_s = 0.3$, $\rho_{VS} = 0$, $r = 0.05$, $a = 0.5, b = 0.08$, $\sigma_r = 0.03, \rho_{Vr} = 0$, $\rho_{Sr} = 0.0$. Figure shows the boundaries of the integration region superimposed on the contour lines of a bi-variate normal probability distribution. The actual non-linear default boundary of the variable default boundary model (VDB) is based on the function $f(\tilde{u})$ shown in equation 4.2.4 and 4.4.1.3. The linear approximation is based on the function given in Appendix D. The default boundary for the fixed default boundary model (FDB) is based on the boundary function shown in Appendix C. Note that:

$$\tilde{u} = \frac{\ln(S_T) - \ln\left(\frac{S}{B}\right) + \frac{s_S^2}{2}}{s_S} \qquad \tilde{v} = \frac{\ln(V_T) - \ln\left(\frac{V}{B}\right) + \frac{s_V^2}{2}}{s_V}$$

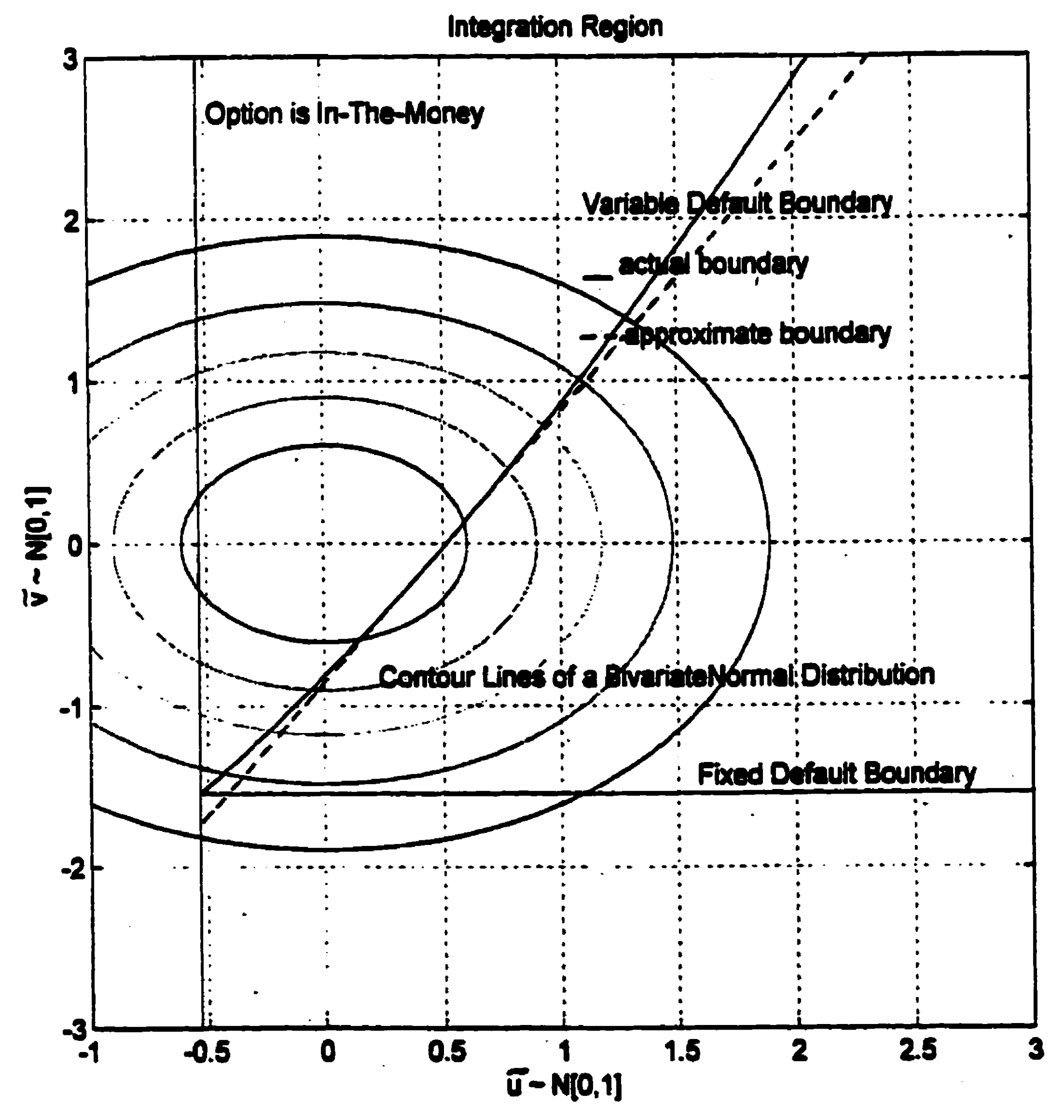


Figure 2

Integration Region for Vulnerable European Calls: ($D^* = K$)

Calculations of vulnerable call option prices are based on the following parameter values: $S = 30$, $K = 40$, $V = 50$, $D^* = 40$, $T = 1, \alpha = 0$, $\sigma_V = 0.1$, $\sigma_s = 0.2$, $\rho_{VS} = 0.5$, $r = 0.05$, $a = 0.5, b = 0.08, \sigma_r = 0.03$ $\rho_{Vr} = 0$, $\rho_{Sr} = 0.0$. Figure shows the boundaries of the integration region superimposed on the contour lines of a bi-variate normal probability distribution. The actual non-linear default boundary of the variable default boundary model (VDB) is based on the function $f(\tilde{u})$ shown in equation 4.2.4 and 4.4.1.3. The linear approximation is based on the function given in Appendix D. The default boundary for the fixed default boundary model (FDB) is based on the boundary function shown in Appendix C. Note that:

$$\tilde{u} = \frac{\ln(S_T) - \ln\left(\frac{S}{B}\right) + \frac{s_S^2}{2}}{s_S} \qquad \tilde{v} = \frac{\ln(V_T) - \ln\left(\frac{V}{B}\right) + \frac{s_V^2}{2}}{s_V}$$

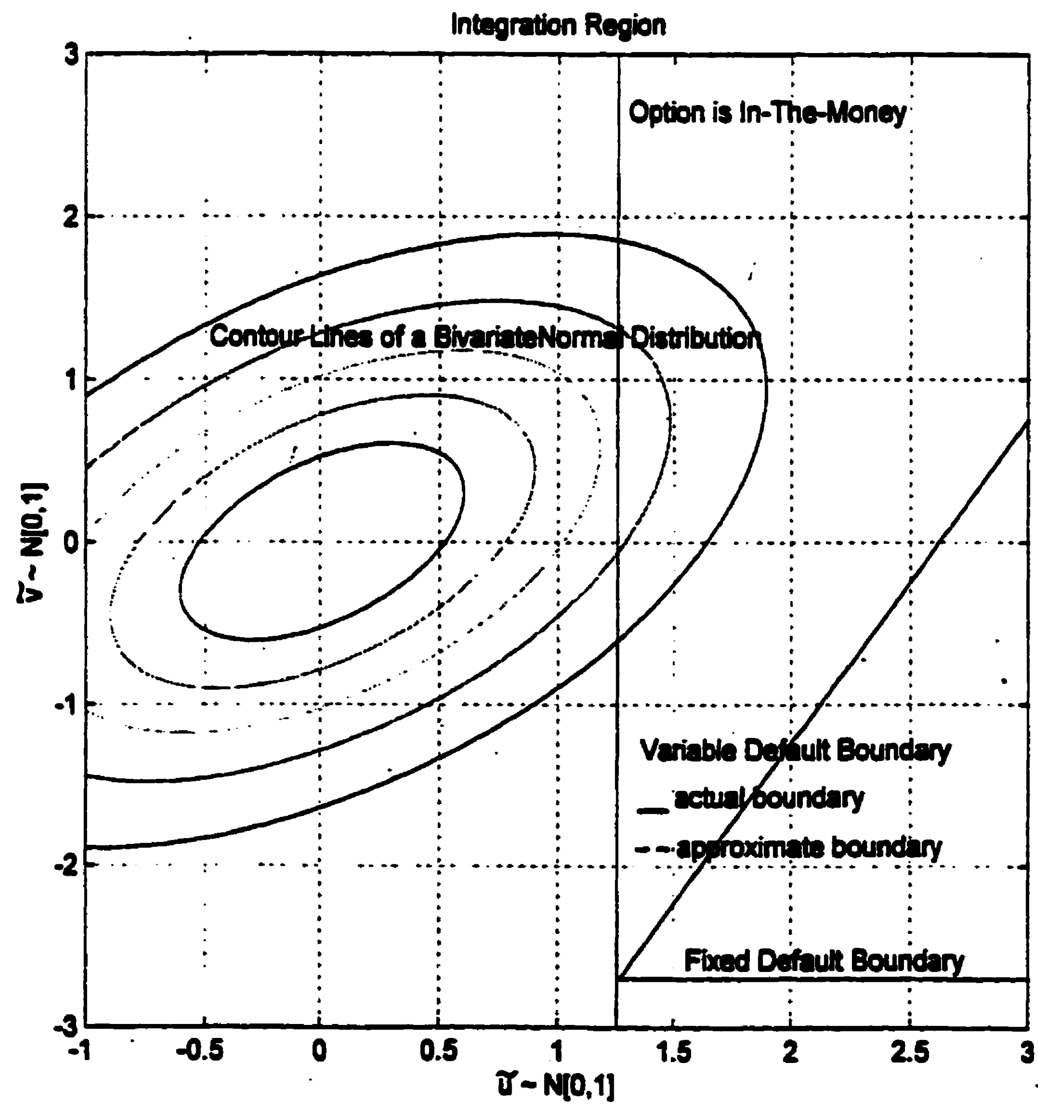
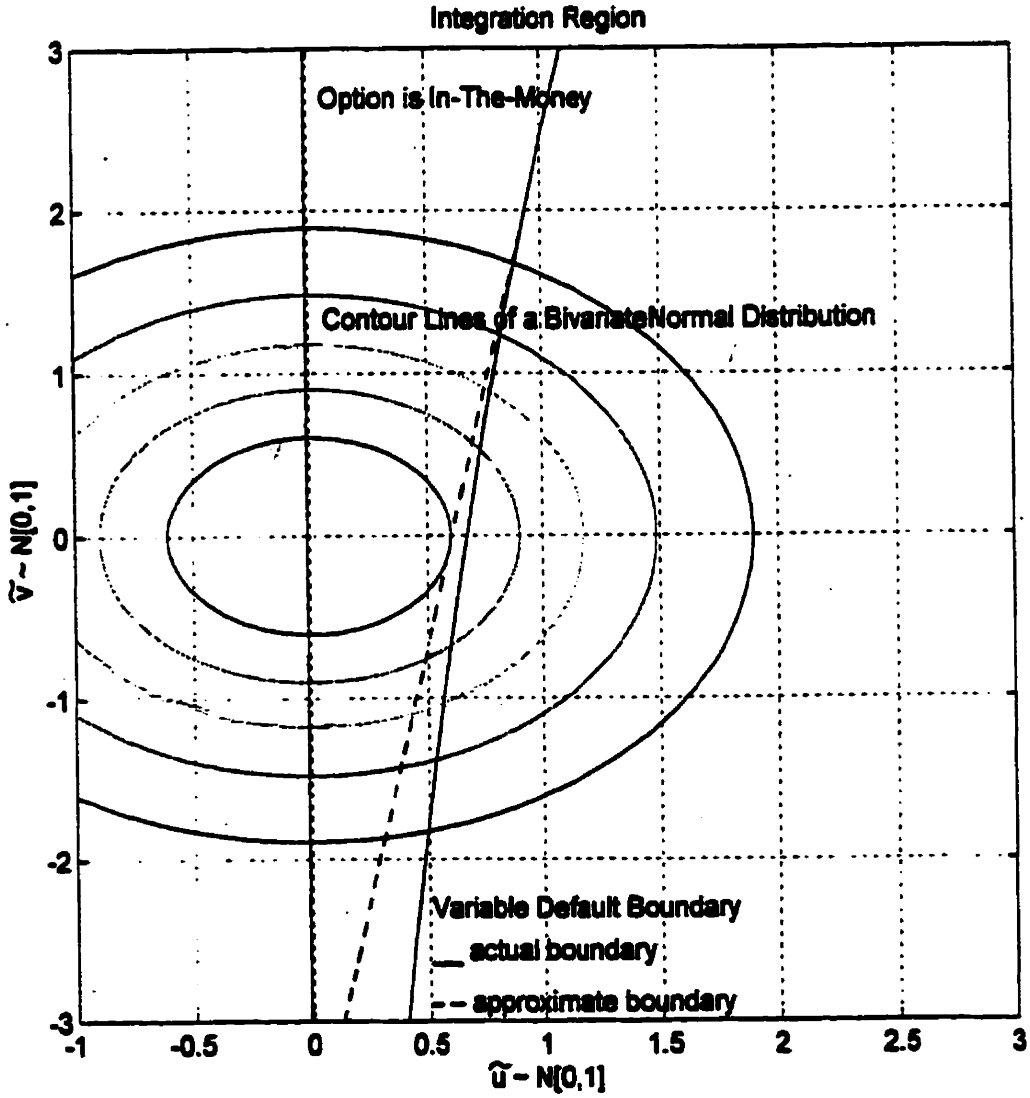


Figure 3

Integration Region for Vulnerable European Calls: ($D^{*} < K$)

Calculations of vulnerable call option prices are based on the following parameter values: $S = 40$, $K = 40$, $V = 5$, $D^{*} = 0$, $T = 0.3333, \alpha = 0$, $\sigma_{V} = 0.3$, $\sigma_{s} = 0.3$, $\rho_{VS} = 0.0$; $r = 0.05$, $a = 0.1, b = 0.05$, $\sigma_{r} = 0, \rho_{Vr} = 0$, $\rho_{Sr} = 0.0$. The actual non-linear default boundary of the variable default boundary model (VDB) is based on the function $f(\tilde{u})$ shown in equation 4.2.4 and 4.4.1.3. The linear approximation is based on the function given in Appendix D. The default boundary for the fixed default boundary model (FDB) is based on the boundary function shown in Appendix C. Note that:

$$\tilde{u} = \frac{\ln(S_T) - \ln\left(\frac{S}{B}\right) + \frac{s_S^2}{2}}{s_S} \qquad \tilde{v} = \frac{\ln(V_T) - \ln\left(\frac{V}{B}\right) + \frac{s_V^2}{2}}{s_V}$$



for a vulnerable call for both the FDB and VDB models in terms of the random variables $\tilde{v}$ and $\tilde{u}$. These variables are standard normal random variates with zero mean and standard deviations equal to one. For the VDB model the non-linear boundary condition, $f(\tilde{u})$, is shown along with a representative linear approximation that is used to simplify the valuation equations. Contour lines for the bi-variate normal distribution are also shown in the figures. The key difference between the three examples is the relative size of D^* and K. Figure 1 corresponds to a case in which $D^* > K$, Figure 2 to the case in which $D^* = K$ and Figure 3 to the case in which $D^* < K$. Note that as stated above and proved in Appendix F if $D^* > K$ then the non-linear boundary is always greater than or equal to its linear approximation. If $D^* < K$ then the non-linear boundary is always below its linear approximation. Finally, if $D^* = K$ then the boundary condition is already linear and no approximation is required. This boundary is the same boundary in the valuation equations for both the call option and the total liabilities and represents the dividing line between the firm defaulting and remaining solvent. The area above the boundary represents no-default and the area below the line represents default.

Assume that we are trying to value the total liabilities of the option writer and that $D^* > K$. If we implement the linear boundary condition as an approximation to the non-linear default boundary then the approximate default region is smaller then the actual default region and the probability of default will be underestimated. Therefore, the approximate valuation formulas will always overvalue the total liabilities. Since the value of the liabilities will always be overvalued, we want the choose "p" that minimizes the value of the liabilities and therefore minimizes the error in the approximate solution. If $D^* < K$ then the non-linear default boundary is always below its linear approximation. Therefore the approximate default region will be larger then the actual default region and the approximate solution will always undervalue the liabilities. To minimize the error in the approximate solution we want to choose the value of "p" that maximizes the value of the liabilities.

Due to the complex nature of the approximate solution we perform the minimization and maximization problems numerically using a BFGS Quasi-Newton method, with a numerically estimated first derivatives. The algorithm, which employs a standard hill-climbing technique with adjustable step size, is robust to the initial guess, which was chosen to be +1 in all of the cases presented in the paper.

If $D^* = K$ then, as shown above, the valuation formula is exact and any value of "p" will give the same value for the total liabilities.

We assume that the value of the parameter "p" estimated using this technique is a good estimate for valuing both the total liabilities and the vulnerable call. The next step is to estimate the parameter "q" for the vulnerable call.

4.4.4 Estimation of Design Parameter "q"

The objective of this section is to estimate the value of the design parameter "q" to accurately value the vulnerable call in the VDB framework. The parameter "q" arose because of the hyperbolic functional form in the integrand of the third and fourth terms of equation 4.2.3. We need to transform this hyperbolic functional form, $F(\tilde{u})$, into an exponential functional form to proceed with the integration. In appendix G we show how this objective can be achieved using a first order Taylor series expansion. The actual and approximate functions are shown below.

The actual hyperbolic function is given by:

$$F(\tilde{u}) = \frac{1}{D^* - K + \frac{S}{B}\exp\left(-\frac{s_S^2}{2} + s_S\tilde{u}\right)} \tag{4.4.4.1}$$

The approximate exponential function is given by:

$$F(\tilde{u}) \approx \frac{\exp[g(\tilde{u} - q)]}{D^* - K + \frac{S}{B}\exp\left(-\frac{s_S^2}{2} + s_S q\right)} \tag{4.4.4.2}$$

where:

$$g = \frac{-s_S \dfrac{S}{B}\exp\left(-\dfrac{s_S^2}{2} + s_S q\right)}{D^* - K + \dfrac{S}{B}\exp\left(-\dfrac{s_S^2}{2} + s_S q\right)}$$

The relationship between the actual and approximate function is illustrated in Figures 4, 5 and 6. The specific parameters for each example are given in the respective figures. Again, the key difference between the three examples is the relative size of D^* and K. Figure 4 corresponds to a case in which $D^* > K$, Figure 5 to the case in which $D^* = K$ and Figure 6 to the case in which $D^* < K$. Notice that when $D^* > K$ that the actual function is always less than or equal to the approximate function. See appendix G for a proof of this result. Therefore, the value of the call given by the exponential approximation will always be greater than it would have been using the actual hyperbolic function. To achieve the best estimate of the call we should chose the design parameter "q" to minimize the value of the call given by the approximate valuation formula. As shown in Figure 6, the reverse argument holds if $D^* < K$ and we should chose "q" to maximize the value of the call. So the choice of the parameter "q" reduces to solving a simple optimization problem. Again, given the complex nature of the approximate valuation formula for the vulnerable call, these optimization problems are solved numerically. Finally, if $D^* = K$ then the actual and approximate values of the function, $F(\tilde{u})$, are identical and the valuation formulas are exact and independent of the choice of "q".

4.5 Vasicek Term Structure Model

The valuation formulas for both the FDB and VDB models incorporate stochastic interest rates in the same fashion as Merton (1973) in his alternative derivation of the Black and Scholes option pricing model. The time dependent parameter $\hat{\sigma}_V^2(T-t)$ represents the instantaneous variance of the return of the writer's assets in terms of the value of the risk-free bond (i.e. V / B). $\hat{\sigma}_S^2(T-t)$ represents the instantaneous variance of the return on the assets underlying the option normalized by the value of the risk-free

Figure 4

Comparison of the Actual Hyperbolic Integrand and the Approximate Exponential Integrand Functions for the VDB Model: ($D^{*} > K$)

Calculations of vulnerable call option prices are based on the following parameter values: $S = 50$, $K = 40$, $V = 100$, $D^{*} = 90$, $T = 3$, $\alpha = 0$, $\sigma_{V} = 0.1$, $\sigma_{s} = 0.3$, $\rho_{VS} = 0$, $r = 0.05$, $a = 0.5$, $b = 0.08$, $\sigma_{r} = 0.03$, $\rho_{Vr} = 0$, $\rho_{Sr} = 0.0$. The actual hyperbolic integrand function in the variable default boundary model (VDB) is based on 4.4.4.1. The approximate exponential integrand function is based on equation 4.4.4.2.
The actual hyperbolic function and approximate exponential function are given by:

$$F(\tilde{u}) = \frac{1}{D^{*} - K + \frac{S}{B}\exp\left(-\frac{s_S^2}{2} + s_S\tilde{u}\right)} \qquad f(\tilde{u}) \approx \frac{\exp[g(\tilde{u} - q)]}{D^{*} - K + \frac{S}{B}\exp\left(-\frac{s_S^2}{2} + s_S q\right)}$$

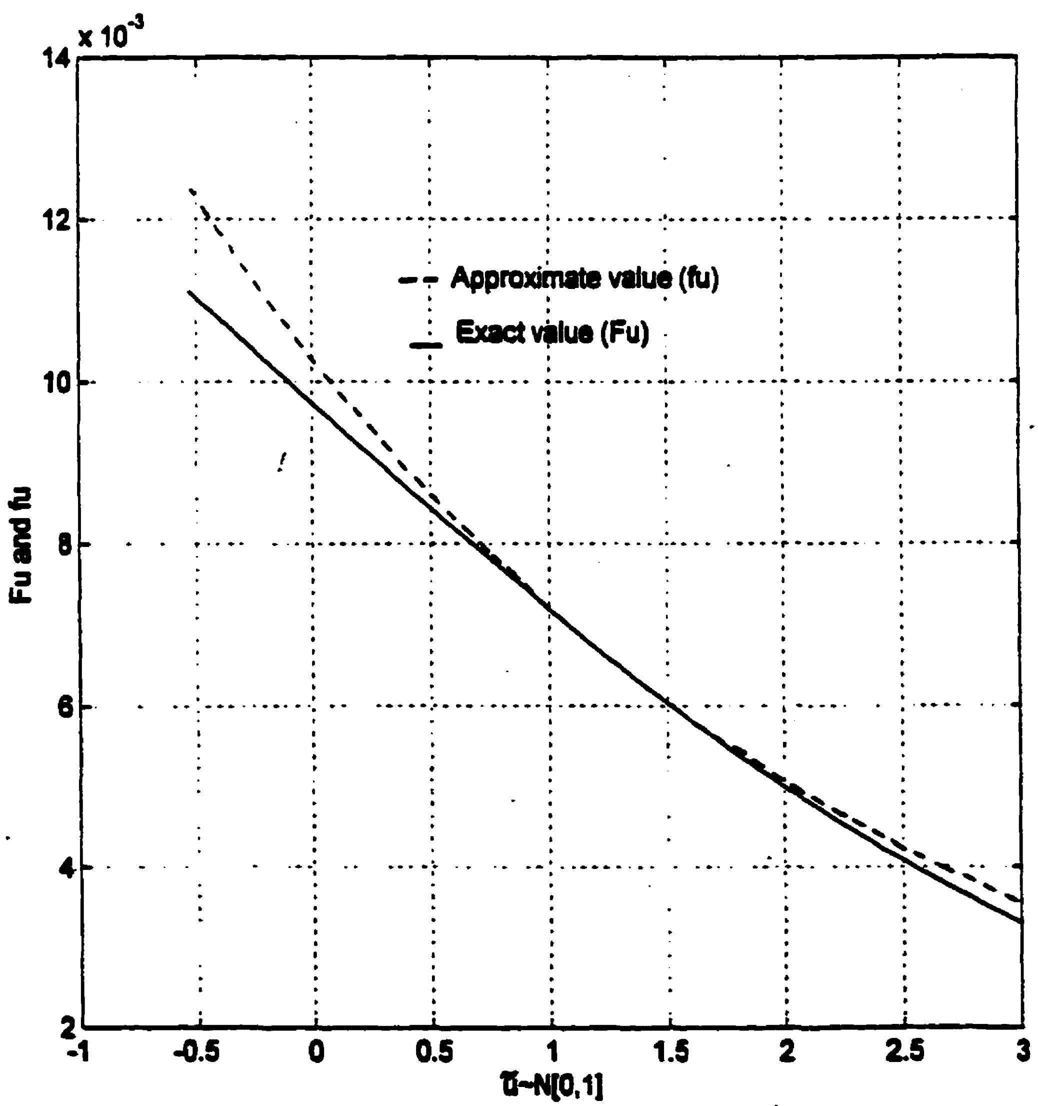


Figure 5

Comparison of the Actual Hyperbolic Integrand and the Approximate Exponential Integrand Functions for the VDB Model: ($D^* = K$)

Calculations of vulnerable call option prices are based on the following parameter values: $S = 30$, $K = 40$, $V = 50$, $D^* = 40$, $T = 1$, $\alpha = 0$, $\sigma_V = 0.1$, $\sigma_s = 0.2$, $\rho_{VS} = 0.5$, $r = 0.05$, $a = 0.5$, $b = 0.08$, $\sigma_r = 0.03$ $\rho_{Vr} = 0$, $\rho_{Sr} = 0.0$. $\rho_{Sr} = 0.0$. The actual hyperbolic integrand function in the variable default boundary model (VDB) is based on 4.4.4.1. The approximate exponential integrand function is based on equation 4.4.4.2. The actual hyperbolic function and approximate exponential function are given by:

$$F(\widetilde{u}) = \frac{1}{D^* - K + \dfrac{S}{B}\exp\left(-\dfrac{s_S^2}{2} + s_S\widetilde{u}\right)} \qquad f(\widetilde{u}) \approx \frac{\exp[g(\widetilde{u} - q)]}{D^* - K + \dfrac{S}{B}\exp\left(-\dfrac{s_S^2}{2} + s_S q\right)}$$

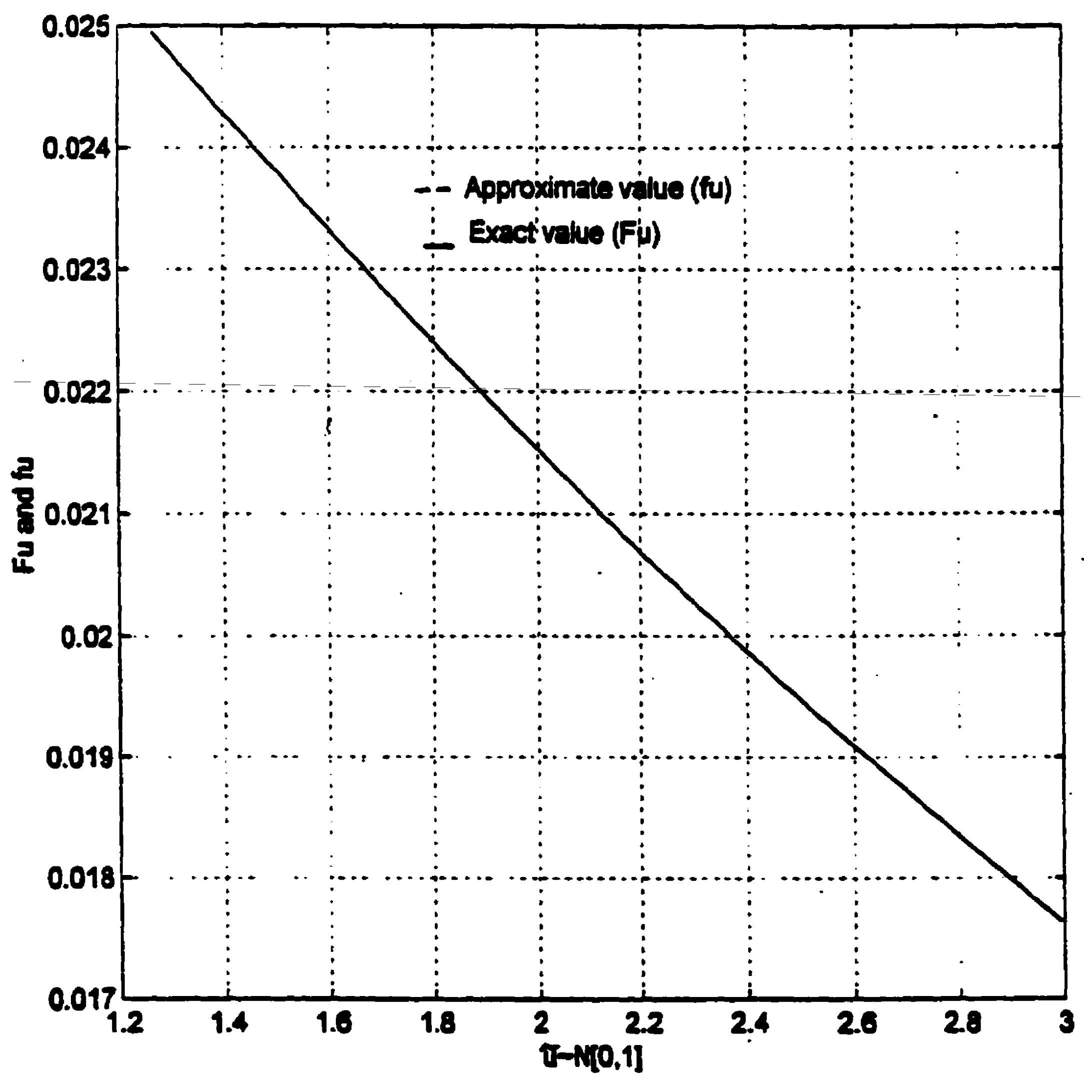
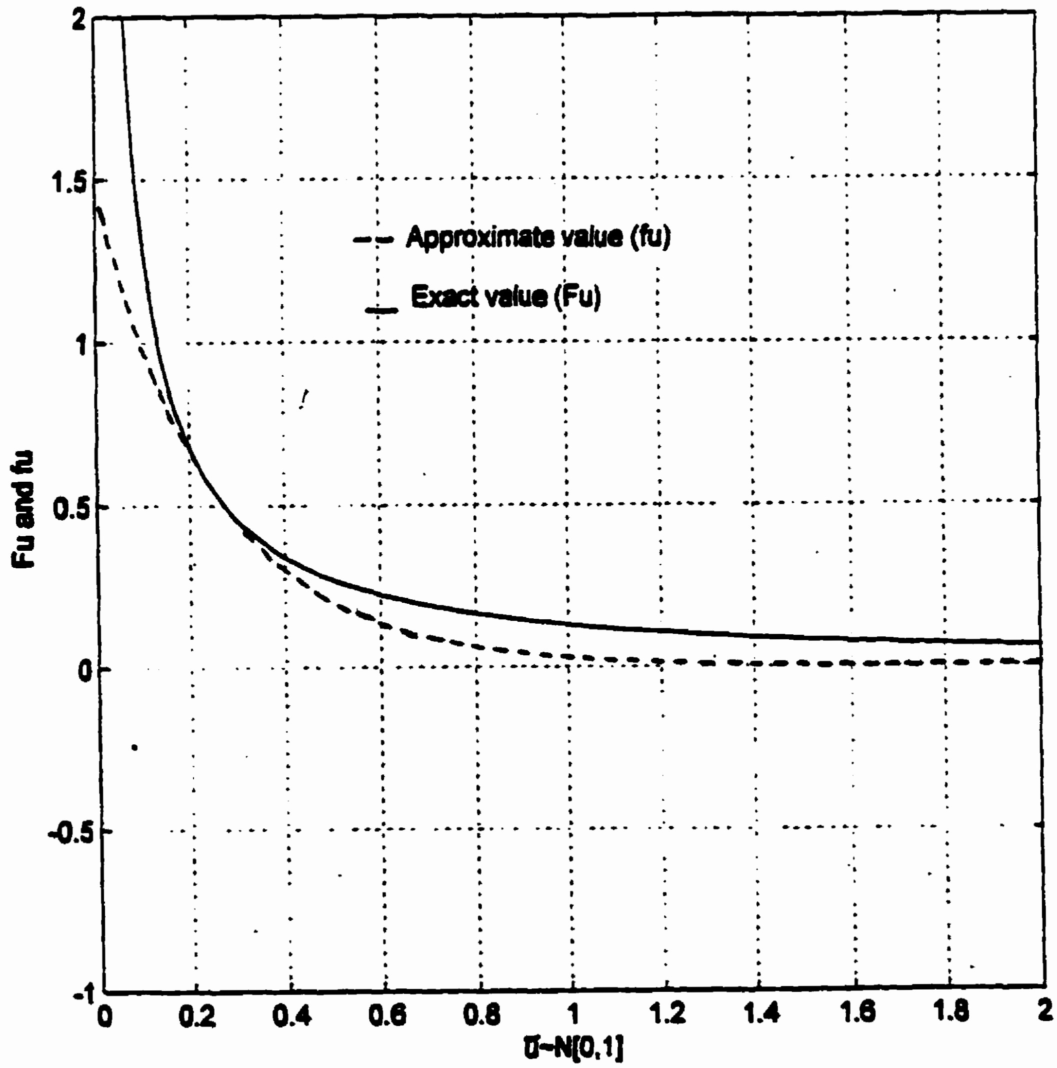


Figure 6

Comparison of the Actual Hyperbolic Integrand and the Approximate Exponential Integrand Functions for the VDB Model: ($D^* < K$)

Calculations of vulnerable call option prices are based on the following parameter values: $S = 40$, $K = 40$, $V = 5$, $D^* = 0$, $T = 0.3333$, $\alpha = 0$, $\sigma_V = 0.3$, $\sigma_S = 0.3$, $\rho_{VS} = 0.0$, $r = 0.05$, $a = 0.1$, $b = 0.05$, $\sigma_r = 0$, $\rho_{Vr} = 0$, $\rho_{Sr} = 0.0$. The actual hyperbolic integrand function in the variable default boundary model (VDB) is based on 4.4.4.1. The approximate exponential integrand function is based on equation 4.4.4.2. The actual hyperbolic function and approximate exponential function are given by:

$$F(\tilde{u}) = \frac{1}{D^* - K + \frac{S}{B}\exp\left(-\frac{s_S^2}{2} + s_S\tilde{u}\right)} \qquad f(\tilde{u}) \approx \frac{\exp[g(\tilde{u} - q)]}{D^* - K + \frac{S}{B}\exp\left(-\frac{s_S^2}{2} + s_S q\right)}$$



bond (i.e. S/B). Integrating $\hat{\sigma}_V^2(T-t)$ and $\hat{\sigma}_S^2(T-t)$ over the life of the option from t to T gives s_V^2 and s_S^2. $\hat{\rho}_{VS}(T-t)$ represents the instantaneous correlation between each of the above normalized variables. Integrating $\hat{\rho}_{VS}(T-t)\hat{\sigma}_V(T-t)\hat{\sigma}_S(T-t)$ over the life of the option gives the integrated covariance s_{VS}, from which the correlation term, $\bar{\rho}_{VS}$ follows.

As in Merton's model the approach in this paper is based on the assumption that the instantaneous variance of the return on a risk-free discount bond depends only on the time to maturity of the bond, and is "otherwise assumed to be non-stochastic and independent of the value of the bond". One of the difficulties with this formulation is that every distinct exercise date for European options requires knowledge not only of the variance of a distinct risk free discount bond with the same maturity, but also knowledge of how this variance changes as the bond moves to maturity. To overcome this problem a model of the term structure of interest rates can be employed.

We use the Vasicek (1977) model for expositional purposes in the remainder of this thesis, but any model satisfying the condition on the instantaneous variance of the bond returns can be employed. The dynamics of the short-term risk-free rate in the Vasicek (1977) term structure model are characterized by:

$$dr = a(b-r)dt + \sigma_r dZ_r \tag{4.5.1}$$

This model assumes that the short term risk-free interest rate is mean reverting to a constant level b at rate a. All the parameters of the model are assumed to be constant. This model suffers from the implicit assumption that, at any given time, the future instantaneous interest rates are normally distributed, resulting in the possibility of negative future interest rates. However, for realistic parameter values the probability of negative interest rates is small. Also, given that the current short rate is positive the expected future short rate will also be positive.

In the Vasicek (1977) framework the price at time zero of a risk-free discount bond B paying one dollar at maturity, T, in terms or the short-term interest rate, r, is given by:

$$B = G(T-t)e^{-H(T-t)r} \tag{4.5.2}$$

where:

$$H(T-t) = \frac{1-e^{-a(T-t)}}{a}$$

$$G(T-t) = \exp\left\{\frac{[H(T-t)-(T-t)][a\phi-\sigma_r^2/2]}{a^2} - \frac{\sigma_r^2 H(T-t)^2}{4a}\right\}$$

and $$\phi = ab - \lambda\sigma_r$$

The parameter λ represents the market price of risk and is assumed to be constant. Once the parameters a, b, λ and σ_r have been chosen all discount bonds can be priced and the entire term structure determined.

At any given time during the life of the option, with remaining life T-t, the volatility of the return on the bond, $\sigma_B(T-t)$, is required in the vulnerable option pricing model, and in the context of the Vasicek (1977) model is given by:

$$\sigma_B(T-t) = \frac{1-e^{-a(T-t)}}{a}\sigma_r$$

(4.5.3)

The integrated variances s_V^2, s_S^2 and covariance, s_{VS}, can now be explicitly determined:

$$s_V^2 = \left[\sigma_V^2 + \frac{\sigma_r^2}{a^2} + \frac{2\sigma_V\sigma_r\rho_{Vr}}{a}\right](T-t) + (e^{-a(T-t)}-1)\left[\frac{2\sigma_r^2}{a^3} + \frac{2\sigma_V\sigma_r\rho_{Vr}}{a^2}\right] - \frac{\sigma_r^2}{2a^3}(e^{-2a(T-t)}-1)$$

(4.5.4)

$$s_S^2 = \left[\sigma_S^2 + \frac{\sigma_r^2}{a^2} + \frac{2\sigma_S\sigma_r\rho_{Sr}}{a}\right](T-t) + (e^{-a(T-t)} - 1)\left[\frac{2\sigma_r^2}{a^3} + \frac{2\sigma_S\sigma_r\rho_{Sr}}{a^2}\right] - \frac{\sigma_r^2}{2a^3}(e^{-2a(T-t)} - 1)$$

(4.5.4)

$$s_{VS} = \left[\rho_{VS}\sigma_V\sigma_S + \frac{\sigma_r^2}{a^2} + \frac{\rho_{Vr}\sigma_V\sigma_r}{a} + \frac{\rho_{Sr}\sigma_S\sigma_r}{a}\right](T-t) + (e^{-a(T-t)} - 1)\left[\frac{\rho_{Vr}\sigma_V\sigma_r}{a^2} + \frac{\rho_{Sr}\sigma_S\sigma_r}{a^2} + \frac{2\sigma_r^2}{a^3}\right]$$
$$- \frac{\sigma_r^2}{2a^3}(e^{-2a(T-t)} - 1)$$

(4.5.5)

4.6 Numerical Solution: Variable Default Boundary Model

Since equation 4.2.3 is not analytically tractable, we employ a Monte Carlo simulation to generate numerical solution to the model, which can be used to assess the accuracy of the approximate analytical solution.

Our final continuous time valuation framework is composed of three stochastic processes, one process for the value of the asset underlying the option, one for the value of the option writer's assets and one for the risk-free rate. In a risk neutral world the processes are given by:

$$dV = rVdt + \sigma_V VdZ_V \qquad (4.6.1)$$

$$dS = rSdt + \sigma_S SdZ_S \qquad (4.6.2)$$

$$dr = a(b-r)dt + \sigma_r dZ_r \qquad (4.6.3)$$

To simulate this system of equations in discrete time we divide the life of the option into N subintervals, each of length Δt. In discrete terms the processes are given by:

$$\Delta V = rVdt + \sigma_V Vx_2\sqrt{\Delta t} \qquad (4.6.4)$$

$$\Delta S = rSdt + \sigma_S S x_1 \sqrt{\Delta t} \tag{4.6.5}$$

$$\Delta r = a(b - r)dt + \sigma_r x_3 \sqrt{\Delta t} \tag{4.6.6}$$

where x_1, x_2 and x_3 are correlated random variables drawn from a standard joint normal distribution with zero means and standard deviations equal to one.

Each value of the call is based on 10,000 simulations, each run over 128 time steps. The 128 time steps are used to generate a good estimate of the average interest rate over the life of the option. An entire matrix of uncorrelated normal random variables is generated once for each of the three random variables: V, S *and* r. Therefore 3*10,000*128 uncorrelated random numbers are drawn from a joint standard normal distribution and stored. The correlation structure of the model is built into the random sample using a Cholesky decomposition of the inverse of the variance/ covariance matrix. This technique results in the following transformation:

$$x_1 = \frac{y_1}{a}\sqrt{\frac{1}{1-\rho_{Sr}}}\left(\rho_{VS} - \rho_{Vr}\rho_{Sr}\right)y_2 + \rho_{Vr}y_3 \tag{4.6.7}$$

$$x_2 = y_2\sqrt{1-\rho_{Sr}^2} + \rho_{Sr}y_3 \tag{4.6.8}$$

$$x_3 = y_3 \tag{4.6.9}$$

where: $$a = \sqrt{\frac{1-\rho_{Sr}^2}{1-\rho_{VS}^2-\rho_{Sr}^2-\rho_{Vr}^2+2\rho_{VS}\rho_{Vr}\rho_{Sr}}}$$

and the y's represent the original uncorrelated sample and the x's represent the correlated random sample. The antithetic variable technique is also used to reduce variance thereby doubling the number of simulations that are performed to 20,000.

The variable default boundary at maturity for the vulnerable call option in our model is given by:

$$D = D^* + \max(S_T - K, 0) \tag{4.6.10}$$

where D is the total obligations of the option writer and D^* corresponds to all other outstanding claims of the option writer. Once the values of V_T, S_T and the average interest rate, $\bar{r}$, over the simulated path are known, the value of the call option for each simulation is given by:

$$c_i = e^{-\bar{r}(T-t)} \begin{cases} \max(S_T - K, 0) & \text{if} \quad V_T \geq D \\ (1-\alpha)\dfrac{V_T}{D}\max(S_T - K, 0) & \text{if} \quad V_T < D \end{cases} \qquad i = 1 \ldots 20{,}000 \qquad (4.6.11)$$

The average value of all the c_i $i = 1 \ldots 20{,}000$, represents an estimate of the call's actual value.

In the next chapter we use numerical examples to demonstrate the effectiveness of our approximation and the importance of including a variable default boundary.

Chapter 5

Numerical Simulations

In this chapter we present a number of numerical examples that demonstrate the properties of both the Klein and Inglis (1999) FDB model and the VDB model presented in this thesis. The results for Merton's (1973) model are also presented for comparison. Section 5.1 presents a sensitivity analysis for each of the models, for three different examples, corresponding to the three cases, $D^* > K$, $D^* = K$ and $D^* < K$. The examples in this section also demonstrate the ability of the approximate analytical solution to accurately match the numerical solutions. Section 5.2 specifically looks at the impact of the three correlation coefficients ρ_{VS}, ρ_{Vr} and ρ_{Sr} on the percentage reduction in the value of the vulnerable call relative to the standard Merton value. The next section shows how to calculate the risk neutral probability of default for each model. It also, estimates the margin required to remove default risk from the vulnerable calls. Finally we look at the issue of hedging a vulnerable call. The deltas and gammas of the calls are calculated numerically with respect to both the value of the assets underlying the option and the value of the option writer's assets. We also demonstrate how well each of the models can hedge a long position in a vulnerable call.

5.1 Model Comparison and Sensitivity Analysis

Tables 2, 3 and 4 present a sensitivity analysis for both the FDB model of Klein and Inglis (1999) and the VDB model presented in this thesis. Merton's (1973) version of Black and Scholes (1973) option pricing model is included for comparison purposes. The example used to generate the results in Table 2 roughly corresponds to the example used in the paper by Klein and Inglis (1999). It involves a highly leveraged firm that writes a long-term (3 year) in-the-money call option against an asset that is uncorrelated with its own assets. The specific parameter values used to generate the results are shown below Table 2. The key to this example is that the face value of the debt of the option writer is greater than the strike price of the option (i.e. $D^* > K$). This is important when estimating the value of the VDB vulnerable call since the solution depends on two design parameters, "p" and "q". As noted in section 4.4, we find the values of the two design parameters, "p" and "q", by minimizing the value of the liabilities of the option writer with respect to "p" and then minimizing the value of the vulnerable call with respect to "q". This procedure is performed for each separate example in table 2.

Four estimates of the value of the call are shown in table 2, the numerical estimate for the VDB, the approximate analytical estimate for the VDB, the analytical estimate for the FDB and the Merton estimate. Also included is the error or percentage difference between each analytical estimate and the corresponding VDB numerical estimate. The standard errors of the numerical estimates are given in parentheses below actual estimate.

First note that the approximate analytical solution to the VDB model tracks the numerical results quite well. The largest error is only 1.03%, while the majority of the errors are less then 0.5%. Note that the approximate solution always overestimates the actual solution, which is consistent with the analysis in section 4.4.

The VDB call values are all considerably less than the corresponding FDB and Merton values. This makes sense since the VDB takes into account the impact of the potential option payoff in triggering default. Also, the actual payoff in the event of default is less then under the VDB model. Finally, the difference between the FDB and Merton values for the call are not significant, although the FDB model values are all lower, as expected. This indicates that even with a quasi-debt ratio (i.e. D^*/V) of 90% the FDB model predicts a relatively small probability of default compared to the VDB model. Quasi-debt ratios in excess of 95% are needed before the FDB model predicts significant decreases in the value of vulnerable calls.

Table 2

Comparison of Fixed vs. Variable Default Boundaries

	VDB Numerical Solution	VDB Approx. Analytical Solution	Error Approx. Analytical Solution	FDB Klein and Inglis, 1999	Error FDB Klein and Inglis, 1999	Merton (1973)	Error Merton (1973)
BaseCas	\$13.07 (4.54)	\$13.12	0.38%	\$19.17	46.67%	\$19.51	49.27%
$S=40$	8.37 (5.49)	8.38	0.12	11.31	35.13	11.51	37.51
$S=6$	17.50 (3.84)	17.68	1.03	27.92	59.54	28.42	62.4
$V=90$	11.64 (3.98)	11.74	0.86	18.53	59.19	19.51	67.61
$V=110$	14.32 (5.16)	14.37	0.35	19.41	35.54	19.51	36.24
$\rho_{VS}=0.5$	14.38 (5.69)	14.47	0.63	19.44	35.19	19.51	35.67
$\rho_{VS}=-0.5$	11.94 (3.26)	11.98	0.34	18.66	56.28	19.51	63.40
$\sigma_S=0.2$	13.58 (3.01)	13.64	0.44	17.50	28.87	17.80	31.08
$\sigma_S=0.4$	12.63 (6.31)	12.69	0.48	21.24	68.17	21.62	71.18
$\sigma_V=0.0$	13.19 (3.74)	13.25	0.45	19.50	47.84	19.51	47.92
$\sigma_V=0.2$	12.55 (5.99)	12.62	0.56	17.68	40.88	19.51	55.46
$\sigma_r=0.0$	13.10 (4.62)	13.16	0.46	19.10	45.80	19.51	48.93
$T-t=2$	11.51 (3.54)	11.55	0.35	16.34	41.96	16.66	44.74
$T-t=4$	14.44 (5.38)	14.51	0.48	21.74	50.55	22.05	52.70

$\alpha = 0.0$	15.67 (6.18)	15.71	0.26	19.43	23.99	19.51	24.51
$\alpha = 0.5$	10.47 (3.94)	10.53	0.57	18.92	80.71	19.51	86.34
$r = 0.03$	12.34 (4.67)	12.39	0.41	18.33	48.54	18.79	52.27
$r = 0.07$	13.81 (4.44)	13.86	0.36	19.99	44.75	20.23	46.49
$\rho_{Vr} = 0.5$	13.13 (4.71)	13.19	0.46	19.05	45.09	19.51	48.59
$\rho_{Vr} = -0.5$	13.01 (4.36)	13.05	0.30	19.31	48.42	19.51	49.96
$\rho_{Sr} = 0.5$	13.35 (4.75)	13.41	0.45	19.55	46.44	19.79	48.24
$\rho_{Sr} = -0.5$	12.79 (4.32)	12.83	0.31	18.77	46.76	19.22	50.27

Calculation of values of vulnerable call options are based on the following parameter values: $S = 50$, $K = 40$, $V = 100$, $D^{*} = 90$, $T = 3$, $\alpha = 0.25$, $\sigma_{V} = 0.1$, $\sigma_{S} = 0.3$, $\rho_{VS} = 0.0$, $r = 0.05$, $a = 0.5$, $b = 0.08$ $\sigma_{r} = 0.03$, $\rho_{Vr} = 0$, $\rho_{Sr} = 0.0$, unless otherwise noted. Standard errors for the numerical estimates are given in parentheses below the actual estimate.

The sensitivity analysis for the most part agrees, at least in sign, with other vulnerable option pricing models proposed in the literature. For example the value of a vulnerable call increases with S and V, which implies that the deltas of a vulnerable call with respect to both S and V (i.e. dc/dS and dc/dV) are positive. These relationships are also illustrated in figures 7 and 8. Figure 7 shows the impact of varying the moneyness (i.e. S/K) of the option on the value of a vulnerable call. First note that the FDB model predicts virtually no reduction in value of the call, even as the call moves significantly into the money. As expected the VDB model does predict larger percentage reductions in value as S increases, since it takes into account the impact of the option of the probability of default. Figure 8 shows the impact of varying the option writer's assets on the value of a vulnerable call. Note that the parameter values used to generate the figures match those used to generate table 2, except that α was set equal to zero. This means that the

Figure 7

Vulnerable Call Values as a Function of Option's Moneyness: A Comparison of the Fixed and Variable Default Boundary Models

Calculations of vulnerable call option prices are based on the following parameter values: $S = 50$, $K = 40$, $V = 100$, $D^{*} = 90$, $T = 3$, $\alpha = 0$, $\sigma_{r} = 0.1$, $\sigma_{s} = 0.3$, $\rho_{VS} = 0$, $r = 0.05$, $a = 0.5$, $b = 0.08$, $\sigma_{r} = 0.03$ $\rho_{Vr} = 0$, $\rho_{Sr} = 0.0$, unless otherwise noted. Numerical solution (circles) of the variable default boundary (VDB) model is based on a Monte-Carlo simulation. Analytical solutions (solid lines) of VDB model are based on equation 4.2.5. The design parameters "p" and "q" are optimized for each value of S, based on the technique outlined in section 4.4. Analytical solutions of the FDB model are based on equation 4.1.1.

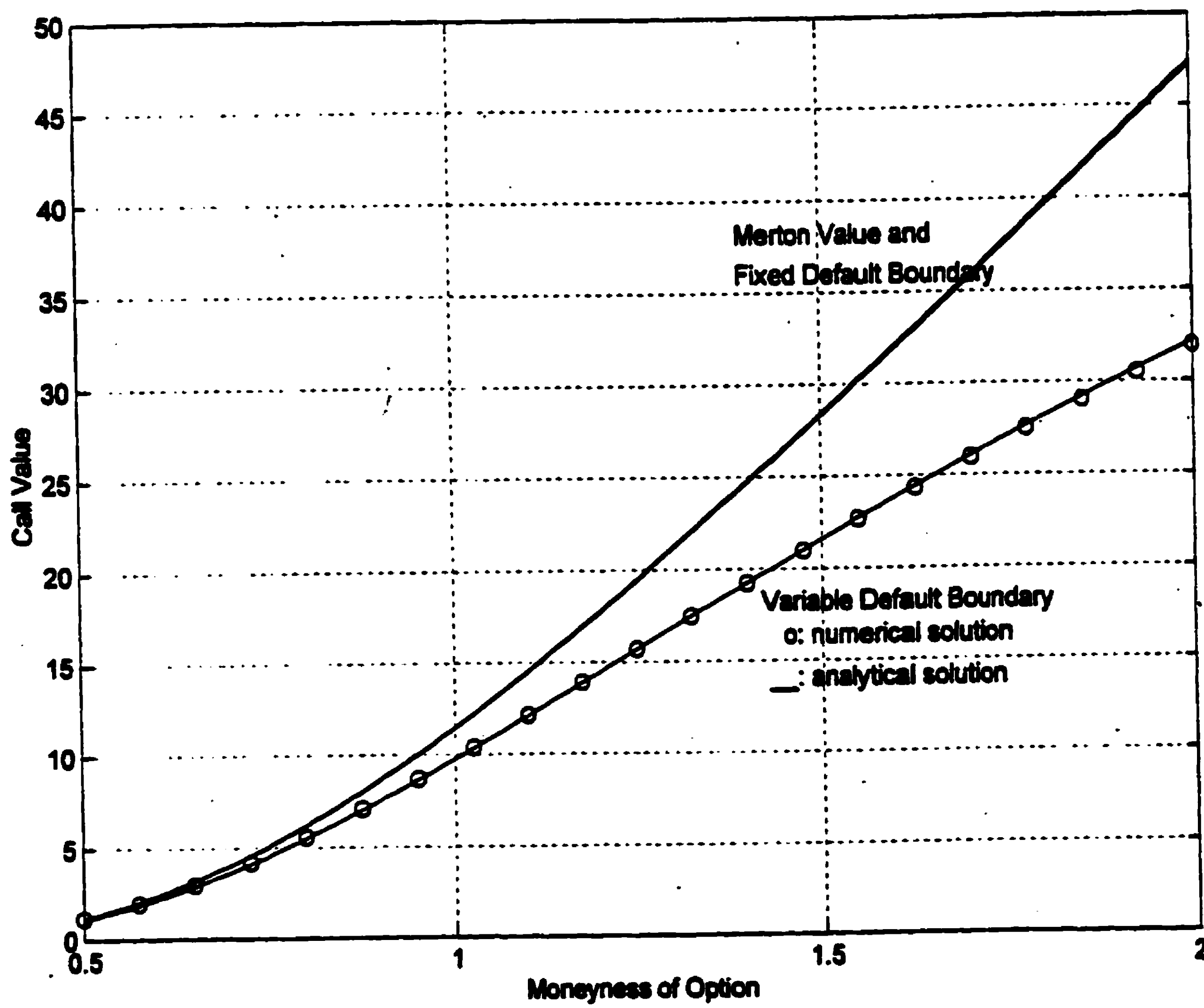
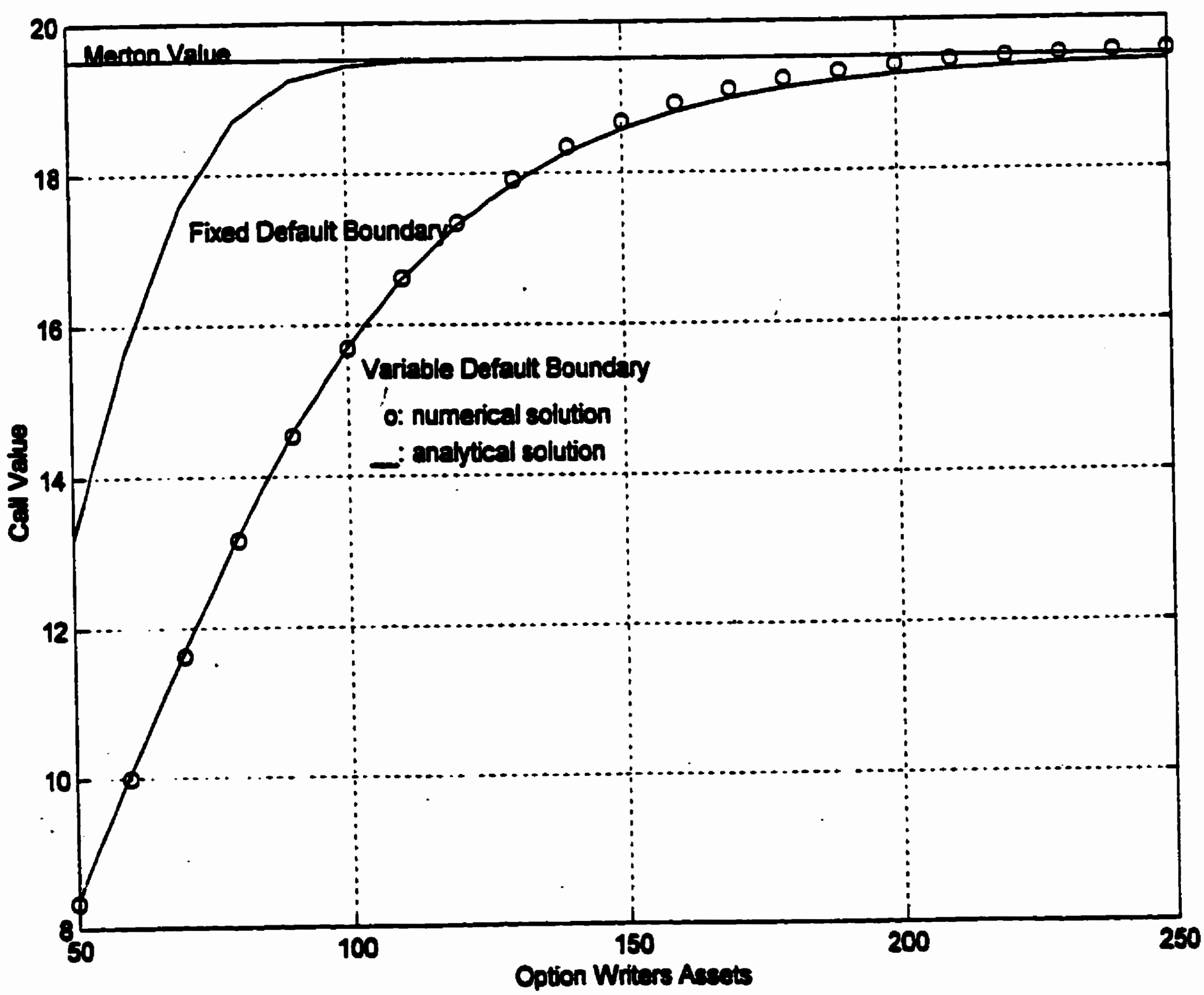


Figure 8

Vulnerable Call Values as a Function of Option Writer's Assets: A Comparison of the Fixed and Variable Default Boundary Models

Calculations of vulnerable call option prices are based on the following parameter values: $S = 50$, $K = 40$, $V = 100$, $D^* = 90$, $T = 3$, $\alpha = 0$, $\sigma_r = 0.1$, $\sigma_s = 0.3$, $\rho_{VS} = 0$, $r = 0.05$, $a = 0.5$, $b = 0.08$, $\sigma_r = 0.03$ $\rho_{Vr} = 0$, $\rho_{Sr} = 0.0$, unless otherwise noted. Numerical solution (circles) of the variable default boundary (VDB) model is based on a Monte-Carlo simulation. Analytical solutions (solid lines) of VDB model are based on equation 4.2.5. The design parameters "p" and "q" are optimized for each value of S, based on the technique outlined in section 4.4. Analytical solutions of the FDB model are based on equation 4.1.1.



creditors of the firm will recover all available assets of the firm, with no reduction due to the direct costs of bankruptcy. In this case the FDB model is virtually indistinguishable from Merton's model unless the quasi-debt ratio is greater then 100% (i.e. the firm has to be on the verge of default before the FDB model will predict even a small decline in the value of a vulnerable call). The VDB model, however predicts small reductions in value for quasi-debt ratios as low as 35%. Also note that both figures indicate that the analytical solution tracks the numerical solution quite well.

An increase in the volatility of the asset underlying the call, σ_S, typically increases the value of an option, as can be seen by the results of Merton's model in Table 2. This result also occurs in the FDB model. However, the VDB model is more complex. Increases in σ_S can either increase or decrease the value of a call. Usually increases in the volatility of the underlying asset will increase the value of a call because of the larger potential payoff. However, for the example in table 2, the value of the vulnerable call actually decreases as σ_S increases. In this case, the value decreases since higher potential payoffs also increase the potential for default in the VDB model. This can be seen more clearly in Figure 9. In this case if σ_S is sufficiently large, further increases in σ_S will actually decrease the value of the call. Therefore, in the VDB model there is a trade-off between higher potential payoffs and greater probability of default as σ_S increases. We look at this phenomenon in greater detail in section 5.3.

Increases in the volatility of the option writer's assets, σ_V, unambiguously decreases the value of the FDB vulnerable call. In this case the probability of the option writer suffering financial distress at the maturity of the option increases. Also, in this particular example the VDB vulnerable call also decreases because of an increase in the probability of default. This result is illustrated in both Table 2 and Figure 10. However, like an increase in σ_S, an increase in σ_V can cause the value of the VDB vulnerable call to either increase or decrease. This result is also studied in greater detail in section 5.3.

Figure 9

Vulnerable Call Values as a Function of the Volatility of the Asset Underlying the Call: A Comparison of the Fixed and Variable Default Boundary Models

Calculations of vulnerable call option prices are based on the following parameter values: $S = 50$, $K = 40$, $V = 100$, $D^* = 90$, $T = 3$, $\alpha = 0$, $\sigma_r = 0.1$, $\sigma_s = 0.3$, $\rho_{VS} = 0$, $r = 0.05$, $a = 0.5$, $b = 0.08$, $\sigma_r = 0.03$ $\rho_{Vr} = 0$, $\rho_{Sr} = 0.0$, unless otherwise noted. Analytical solutions (solid lines) of VDB model are based on equation 4.2.5. The design parameters "p" and "q" are optimized for each value of S, based on the technique outlined in section4.4. Analytical solutions of the FDB model are based on equation 4.1.1.

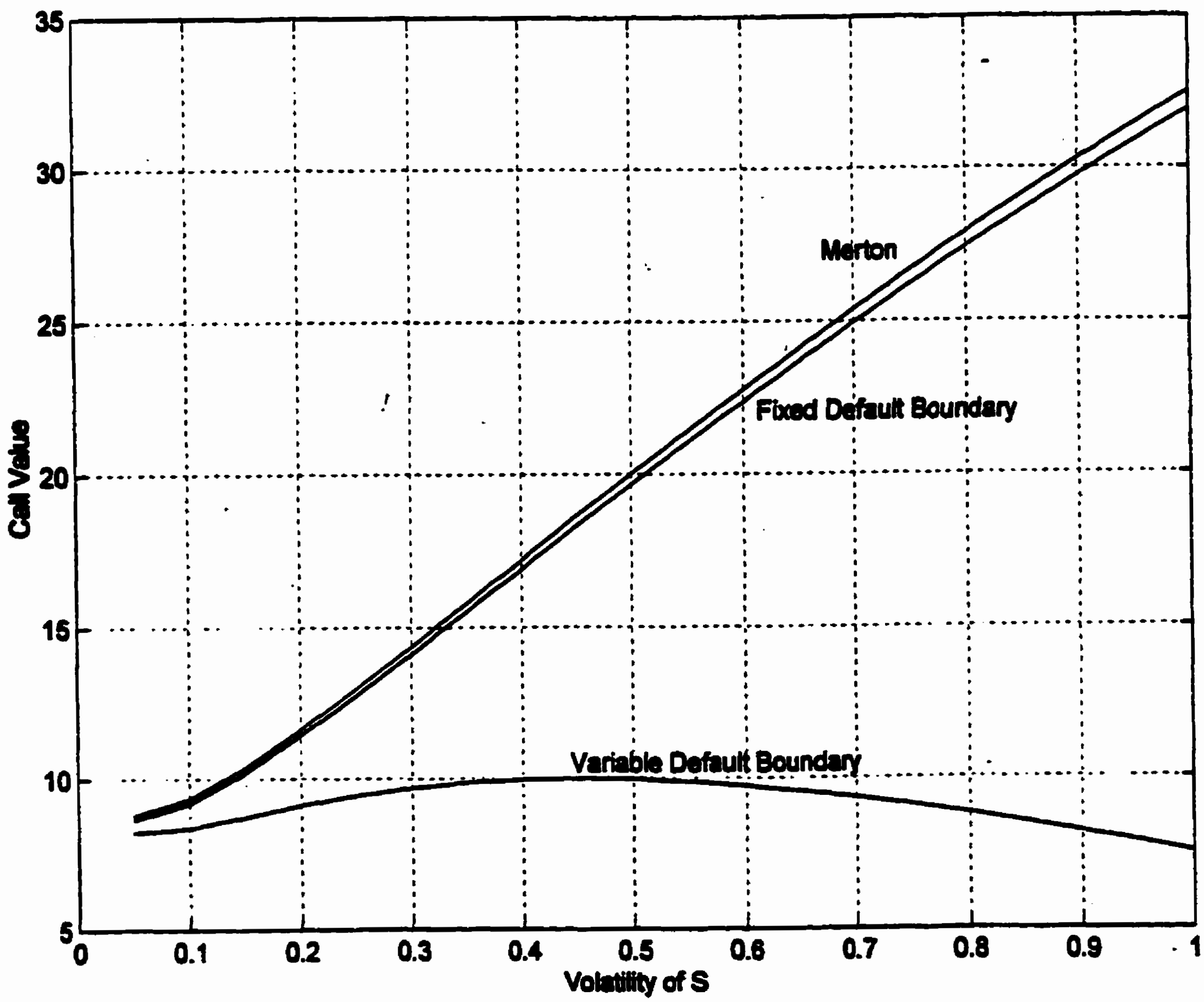
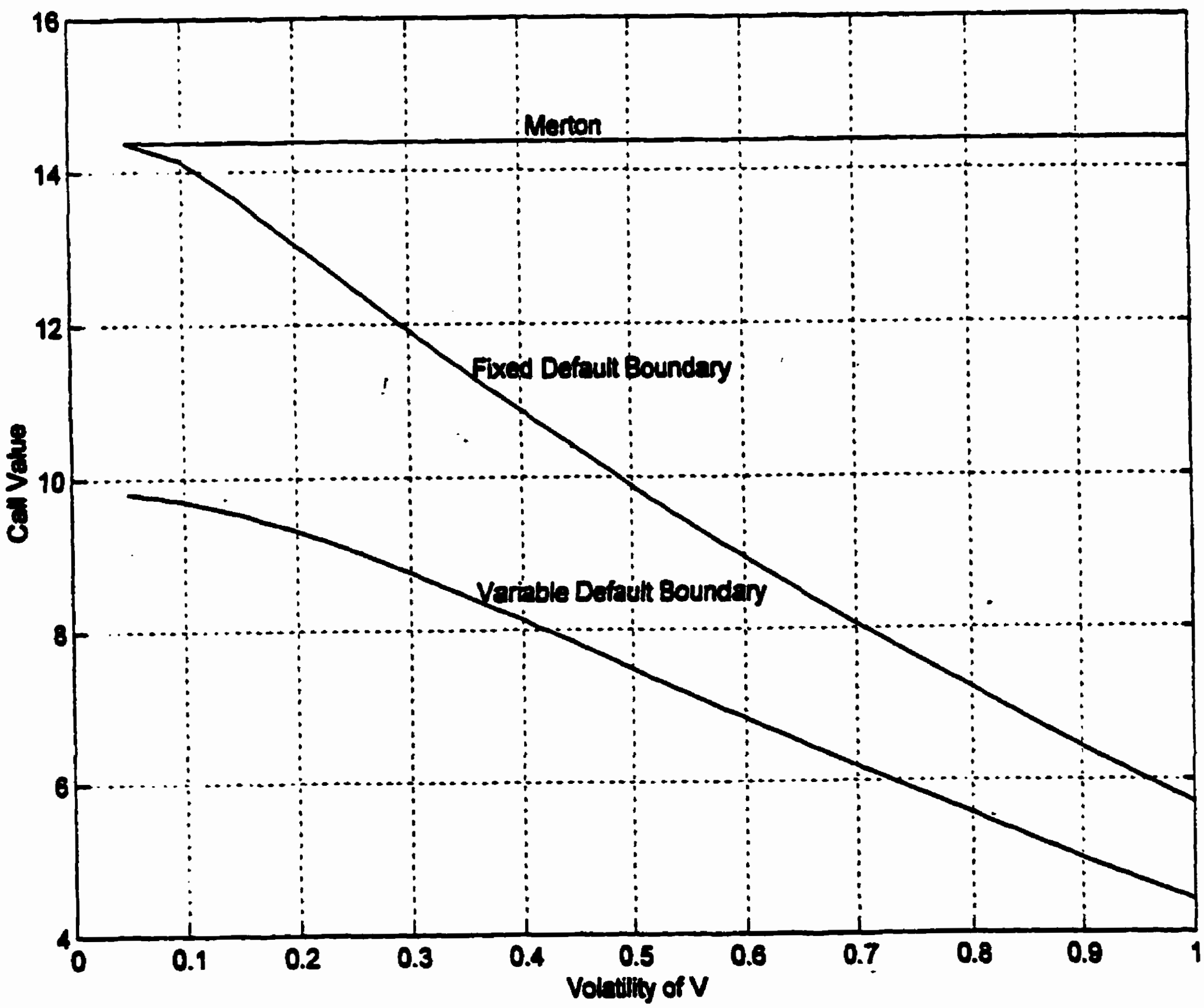


Figure 10

Vulnerable Call Values as a Function of the Volatility of the Option Writer's Assets: A Comparison of the Fixed and Variable Default Boundary Models

Calculations of vulnerable call option prices are based on the following parameter values: $S = 50$, $K = 40$, $V = 100$, $D^* = 90$, $T = 3$, $\alpha = 0$, $\sigma_r = 0.1$, $\sigma_s = 0.3$, $\rho_{VS} = 0$, $r = 0.05$, $a = 0.5$, $b = 0.08$, $\sigma_r = 0.03$ $\rho_{Vr} = 0$, $\rho_{Sr} = 0.0$, unless otherwise noted. Analytical solutions (solid lines) of VDB model are based on equation 4.2.5. The design parameters "p" and "q" are optimized for each value of S, based on the technique outlined in section 4.4. Analytical solutions of the FDB model are based on equation 4.1.1.



As expected the longer the time to maturity the greater the value of the call. However, the non-vulnerable call increases in value the quickest, followed by the FDB model and finally the VDB model which increases in value much more slowly. This is illustrated in Figure 11. Note, that it is possible for the VDB call to decrease in value as the time to maturity increases. Refer to section 5.3 for more details. Also, as expected increasing α, our measure of the direct costs of financial distress, decreases the actual dollar amount of assets that can be recovered, to payoff the liabilities of the firm. As a consequence the value of the vulnerable call decreases.

Increasing the risk-free rate increases the value of vulnerable calls for the same reason that non-vulnerable calls increase. Higher interest rates will result in higher expected growth rates for both S and V, which will lead to a higher expected payoff on the call and a reduced probability of financial distress. Of course, the present value of any future payoff on the call will be less under higher discount rates, but it is well known that this impact is of secondary importance to the higher expected growth rates.

The standard deviation of the interest rate σ_r also has slight impact on call values. For non-vulnerable calls, the higher σ_r, the more expensive the call will be. This makes sense since increasing σ_r will increase s_S^2 (i.e. the integrated variance of the value of the asset underlying the option in units of the discount bond) and as a result increase the value of the call. For vulnerable calls however, σ_r enters the calculations of both s_S^2 and s_V^2 which can have offsetting impacts on the value of the call. Increasing s_S^2 will usually increase the value of a call, whereas increasing s_V^2 will decrease the value of the call. In Table 2, we observe that increasing σ_r decreases the value of FDB call, but slightly increases the value of the VDB call.

Tables 3 and 4 perform the same numerical analysis for two additional examples. The primary difference is between the examples presented in this section is the relationship between D^* and K.

Figure 11

Vulnerable Call Values as a Function of Time to Maturity:
A Comparison of the Fixed and Variable Default Boundary Models

Calculations of vulnerable call option prices are based on the following parameter values: $S = 50$, $K = 40$, $V = 100$, $D^* = 90$, $T = 3$, $\alpha = 0$, $\sigma_v = 0.1$, $\sigma_s = 0.3$, $\rho_{VS} = 0$, $r = 0.05$, $a = 0.5$, $b = 0.08$, $\sigma_r = 0.03$ $\rho_{Vr} = 0$, $\rho_{Sr} = 0.0$, unless otherwise noted. Analytical solutions (solid lines) of VDB model are based on equation 4.2.5. The design parameters "p" and "q" are optimized for each value of S, based on the technique outlined in section 4.4. Analytical solutions of the FDB model are based on equation 4.1.1.

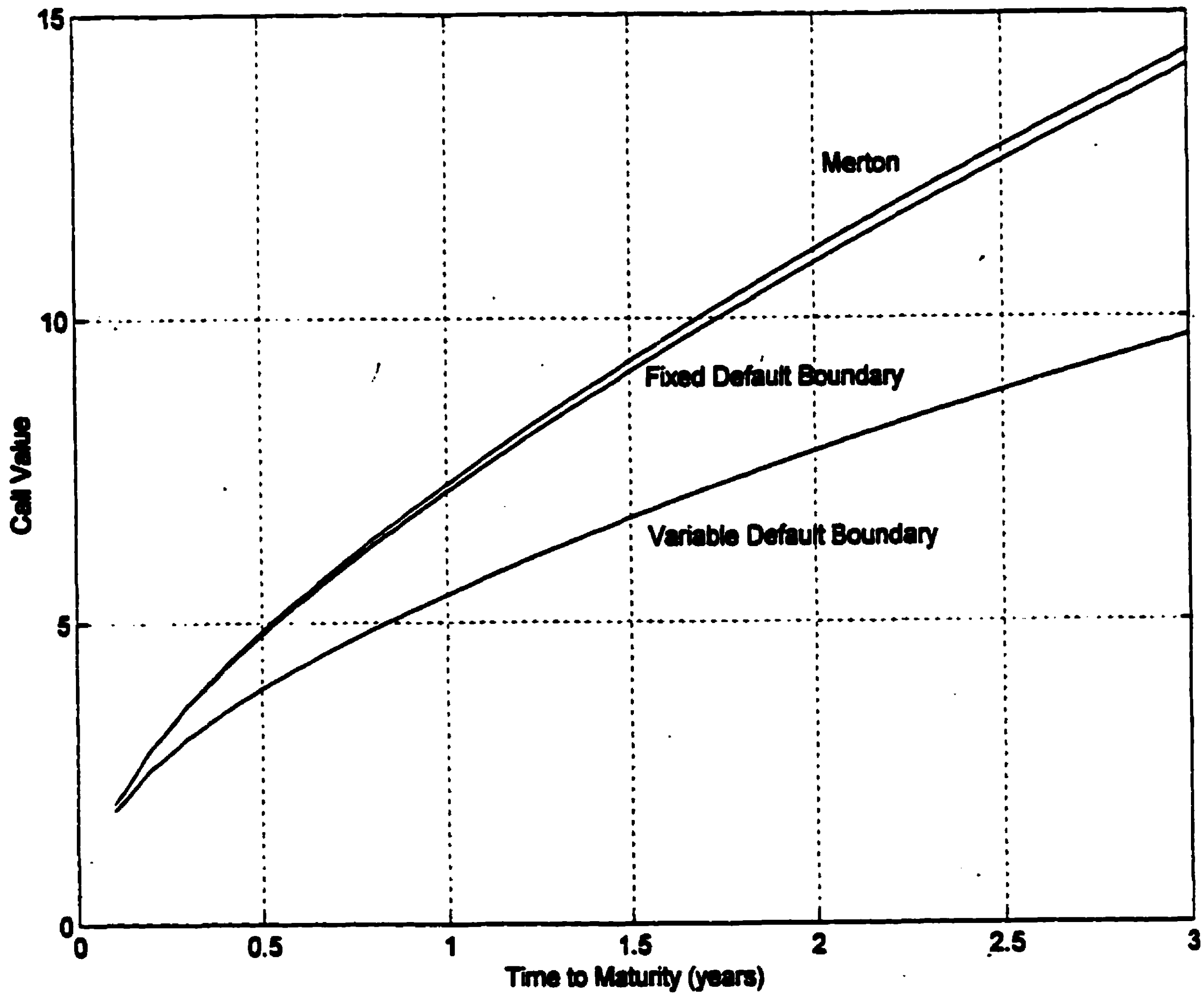


Table 3

Comparison of Fixed vs. Variable Default Boundaries

	VDB Numerical Solution	VDB Approx. Analytical Solution	Error Approx. Analytical Solution	FDB Klein and Inglis, 1999	Error FDB Klein and Inglis, 1999	Merton (1973)	Error Merton (1973)
BaseCas	\$0.406 (1.13)	\$0.409	0.74%	\$0.415	2.22%	\$0.415	2.22%
$S = 40$	3.910 (2.21)	3.880	-0.77	4.322	10.54	4.322	10.54
$S = 50$	9.218 (1.11)	9.228	0.11	12.511	35.72	12.512	35.73
$V = 40$	0.343 (0.90)	0.344	0.29	0.410	19.53	0.415	20.99
$V = 60$	0.412 (1.15)	0.415	0.07	0.415	0.07	0.415	0.07
$\rho_{VS} = 0.0$	0.383 (1.02)	0.385	0.52	0.414	8.09	0.415	8.36
$\rho_{VS} = -0.5$	0.348 (0.89)	0.348	0.00	0.411	18.10	0.415	19.25
$\sigma_S = 0.1$	0.013 (0.12)	0.013	0.00	0.013	0.00	0.013	0.00
$\sigma_S = 0.3$	1.154 (2.18)	1.147	-0.61	1.272	10.23	1.272	10.23
$\sigma_V = 0.0$	0.403 (1.10)	0.406	0.74	0.415	2.98	0.415	2.98
$\sigma_V = 0.2$	0.403 (1.13)	0.409	1.24	0.414	2.48	0.415	2.78
$\sigma_r = 0.0$	0.403 (1.12)	0.407	0.99	0.412	2.23	0.412	2.23
$T-t = 0.5$	0.062 (0.35)	0.062	0.00	0.06	0.00	0.062	0.00
$T-t = 2$	1.531 (2.47)	1.520	-0.72	1.601	4.57	1.601	4.57

$\alpha = 0.0$	0.411 (1.15)	0.414	0.73	0.415	0.97	0.415	0.97
$\alpha = 0.5$	0.403 (1.12)	0.405	0.50	0.415	2.98	0.415	2.98
$r = 0.03$	0.349 (1.05)	0.351	0.57	0.356	2.01	0.356	2.01
$r = 0.07$	0.474 (1.21)	0.476	0.42	0.481	1.48	0.481	1.48
$\rho_{Vr} = 0.5$	0.407 (1.13)	0.410	0.74	0.415	1.97	0.415	1.97
$\rho_{Vr} = -0.5$	0.407 (1.13)	0.409	0.49	0.415	1.97	0.415	1.97
$\rho_{Sr} = 0.5$	0.450 (1.20)	0.453	0.67	0.459	2.00	0.459	2.00
$\rho_{Sr} = -0.5$	0.364 (1.05)	0.366	0.55	0.371	1.92	0.371	1.92

Calculation of values of vulnerable call options are based on the following parameter values: $S = 30$, $K = 40$, $V = 50$, $D^* = 40$, $T - t = 1$, $\alpha = 0.25$, $\sigma_V = 0.1$, $\sigma_S = 0.2$, $\rho_{VS} = 0.5$, $r = 0.05$, $a = 0.5$, $b = 0.08$, $\sigma_r = 0.03$, $\rho_{Vr} = 0$, $\rho_{Sr} = 0.0$, unless otherwise noted. Standard errors for the numerical estimates are given in parentheses below the actual estimate.

In Table 3, D^* is chosen to be equal to K. As pointed out in section 4.4.2, if $D^* = K$ then the approximate analytical solution to the VDB model is exact. Thus, the numerical solution and the analytical solution should be the same. As illustrated in Table 3, the analytical solution is always within plus or minus 1% of the numerical estimates. In this example, the option is out-of-the money, so it is unlikely that the option will add significantly to the probability of default. Therefore, the VDB model results closely follow those of the FDB model. Also, the quasi-debt ratio (i.e. D^* / V) is only 80%, so the probability of default is relatively small in this example. Thus, the Merton model results are close to both the FDB and VDB model results. The only exceptions occur when S increases and when ρ_{VS} decreases. In the first case, increasing S in the VDB model causes the value of the call to increase but at a slower rate then in either the FDB

Table 4

Comparison of Fixed vs. Variable Default Boundaries

	VDB Numerical Solution	VDB Approx. Analytical Solution	Error Approx. Analytical Solution (%)	FDB Klein and Inglis, (1999)	Error FDB Klein and Inglis, (1999) (%)	Merton (1973)	Error Merton (1973) (%)
BaseCas	1.59 (0.63)	1.58	-0.63	3.08	-93.71	3.08	-93.71
$S = 30$	0.12 (0.42)	0.12	-0.16	0.15	22.95	0.15	22.95
$S = 50$	3.23 (0.77)	3.06	-5.26	10.95	239.01	10.95	239.01
$V = 1$	0.37 (0.06)	0.31	-16.22	3.08	732.43	3.08	732.43
$V = 10$	2.55 (1.45)	2.54	-0.39	3.08	20.78	3.08	20.78
$\rho_{VS} = 0.0$	1.54 (0.63)	1.46	-5.19	3.08	100.00	3.08	100.00
$\rho_{VS} = -0.5$	1.50 (0.60)	1.35	-10.00	3.08	105.33	3.08	105.33
$\sigma_S = 0.2$	1.56 (0.62)	1.54	-1.28	2.18	39.74	2.18	39.74
$\sigma_S = 0.4$	1.59 (0.65)	1.57	-1.26	3.99	150.94	3.99	150.94
$\sigma_V = 0.2$	1.62 (0.67)	1.55	-4.32	3.08	90.12	3.08	90.12
$\sigma_V = 0.4$	1.69 (0.77)	1.61	-4.73	3.08	82.25	3.08	82.25
$\sigma_r = 0.03$	1.66 (0.72)	1.58	-4.82	3.08	85.54	3.08	85.54
$T - t = 0.25$	1.58 (0.73)	1.52	-3.80	2.63	66.46	2.63	66.46
$T - t = 0.5$	1.75 (0.72)	1.65	-5.71	3.85	120.00	3.85	120.00
$\alpha = 0.0$	1.98 (0.97)	1.89	-4.55	3.08	55.56	3.08	55.56

$\alpha = 0.5$	1.32 (0.59)	1.27	-3.79	3.08	133.33	3.08	133.33
$r = 0.03$	1.60 (0.77)	1.53	-4.38	2.95	84.38	2.95	84.38
$r = 0.07$	1.70 (0.67)	1.63	-4.12	3.21	88.82	3.21	88.82
$\rho = -0.5$	1.68 (0.72)	1.58	-5.95	3.08	83.33	3.08	83.33
$\sigma_r = 0.03$ $\rho_{Vr} = -0.5$	1.68 (0.72)	1.58	-5.95	3.08	83.33	3.08	83.33
$\sigma_r = 0.03$ $\rho_{Sr} = 0.5$	1.68 (0.70)	1.58	-5.95	3.10	84.52	3.10	84.52
$\sigma_r = 0.03$ $\rho_{Sr} = -0.5$	1.64 (0.74)	1.58	-3.66	3.06	86.59	3.06	86.59

Calculation of vulnerable call values are based on the following parameter values: $S = 40$, $K = 40$, $V = 5$, $D^* = 0$, $T - t = 0.3333$, $\alpha = 0.25$, $\sigma_V = 0.3$, $\sigma_S = 0.3$, $\rho_{VS} = 0.5$, $r = 0.05$, $a = 0.5$, $b = 0.0$ $\sigma_r = 0$, $\rho_{Vr} = 0$, $\rho_{Sr} = 0$, unless otherwise noted. Standard errors for the numerical estimates are given in parentheses below the actual estimate.

model or Merton's model. This is because increasing S increases the probability of default in the VDB model which mitigates the increase in the value of the call. We will discuss the impact of the correlation coefficient in the next section.

In Table 4, D^* is chosen to be less than K. As noted in section 4.4, if $D^* < K$ we can find the values of the two design parameters, "p" and "q", by first maximizing the value of the liabilities of the option writer with respect to "p" and second maximizing the value of the vulnerable call with respect to "q". This example is analogous to the example of Johnson and Stulz (1987), which involves a firm, with no other liabilities, writing an at-the-money option. If the option writer defaults, the option holder receives all the assets of the writer.

First, note that the approximate analytical solution is always less then the corresponding numerical solution. This is expected since the approximate analytical solution will always underestimate the actual value of the call as discussed in chapter 4.

The errors are generally in the -0.5% to -5% range. The approximate analytical solution tends to have more difficulty with problems where D^* is approaching zero. Figures 12 and 13 also illustrate this point. The reason for this can be seen in figures 3 and 6. As D^* approaches zero both the default boundary and the hyperbolic integrand become much more non-linear and the linear approximations used in the approximate analytical solution become much less accurate. In particular when the value of the firms assets drops toward zero, the error becomes much larger at -16.22%.

There is no difference between the vulnerable call value estimated by the FDB and Merton Models. Since there are no other liabilities in this example (i.e. $D^* = 0$), the probability of default under the FDB model is zero and there is no reduction in the value vulnerable call.

5.2 Impact of ρ_{VS}, ρ_{Vr} and ρ_{Sr} on the value of Vulnerable Calls

The impact of the three correlation coefficients, ρ_{VS}, ρ_{Vr} and ρ_{Sr} on the value of vulnerable calls is presented in this separate subsection so that an explicit comparison to the results of Klein and Inglis (1999) can be performed. The impact of these correlation coefficients under the assumptions of the FDB model was the focus of the numerical results in Klein and Inglis (1999).

The call increases in value as the correlation between S and V, ρ_{VS}, increases assuming everything else remains constant. This is expected since a high value of S will usually be associated with a high value of V - so in the case where the option is in-the-money and thus likely to be exercised at maturity there is also likely to be a high degree of creditworthiness on the part of the option writer and thus a low probability of default. Likewise, a low value of S will usually be associated with a low value of V. In this case the option writer is likely to default but the call is likely to be out of the money anyway. Therefore, the higher the correlation between S and V the smaller the reduction

Figure 12

Vulnerable Call Values as a Function of Option's Moneyness: A Comparison of the Fixed and Variable Default Boundary Models

Calculations of vulnerable call option prices are based on the following parameter values: $S = 40$, $K = 40$, $V = 5$, $D^{*} = 0$, $T = 0.3333$, $\alpha = 0$, $\sigma_{V} = 0.3$, $\sigma_{S} = 0.3$, $\rho_{VS} = 0.5$, $r = 0.05$, $a = 0.1$, $b = 0.05$, $\sigma_{r} = 0$, $\rho_{Vr} = 0$, $\rho_{Sr} = 0.0$, unless otherwise noted. Numerical solution (circles) of the variable default boundary (VDB) model is based on a Monte-Carlo simulation. Analytical solutions (solid lines) of VDB model are based on equation 4.2.5. The design parameters "p" and "q" are optimized for each value of S, based on the technique outlined in section 4.4. Analytical solutions of the FDB model are based on equation 4.1.1.

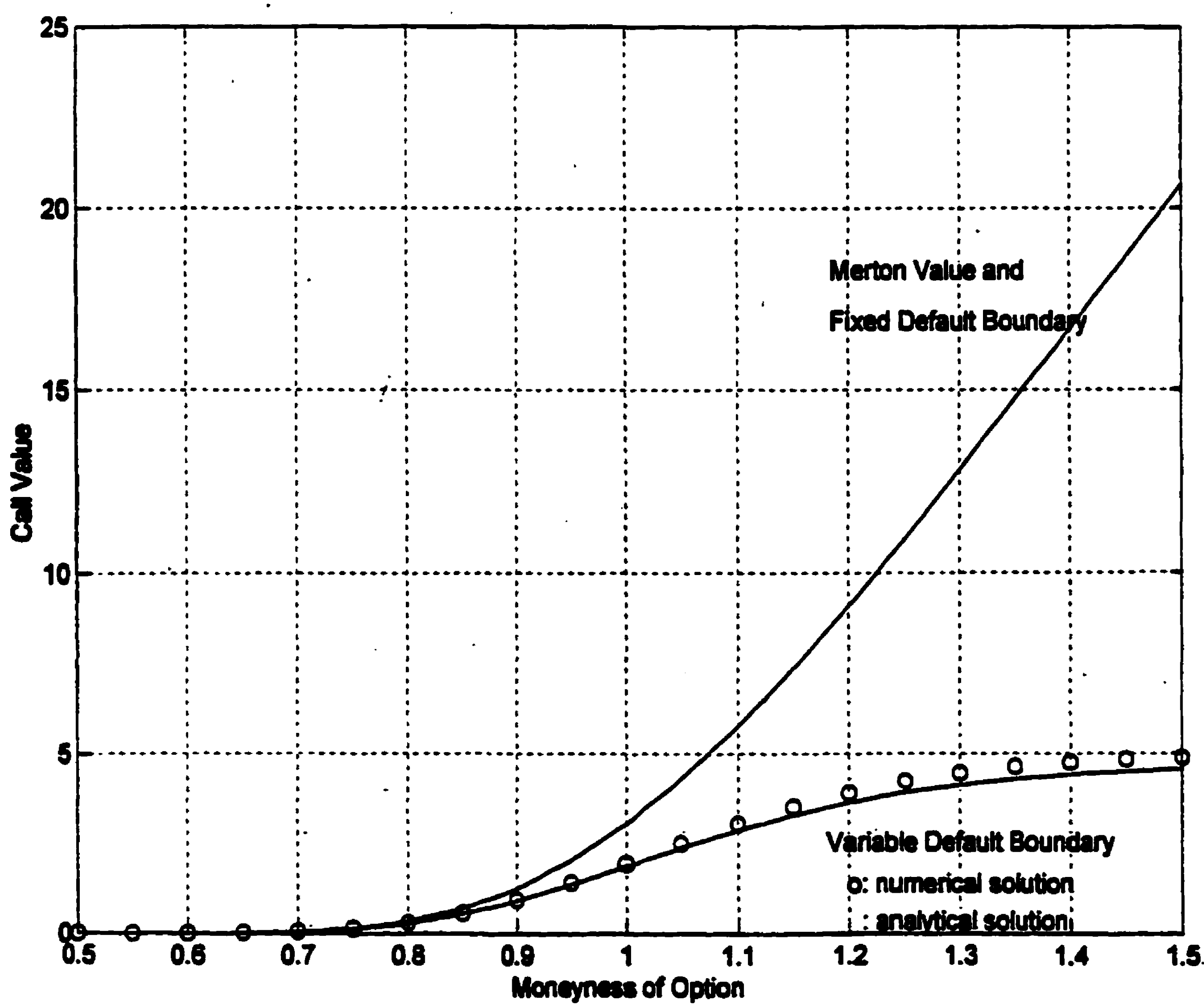
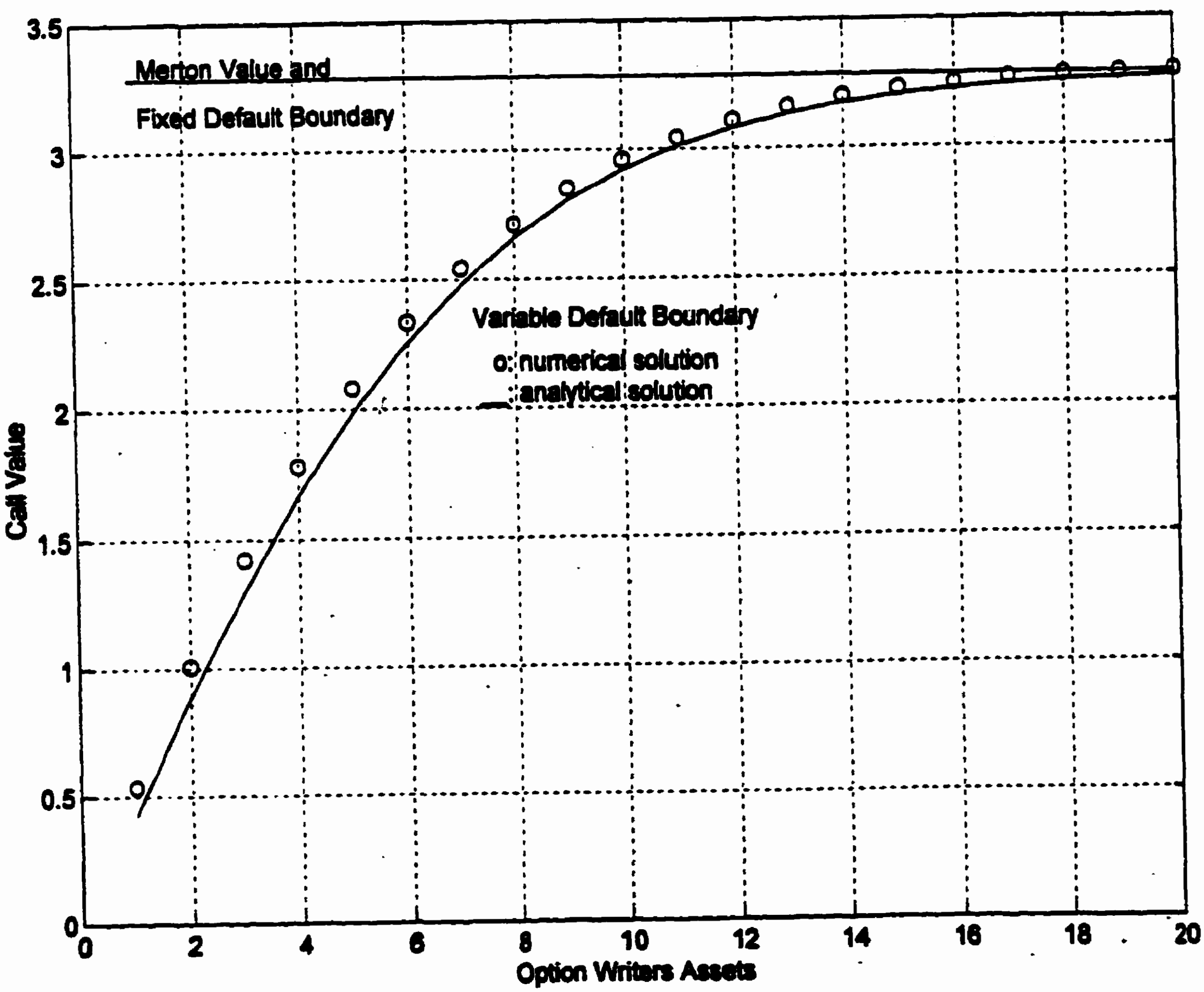


Figure 13

Vulnerable Call Values as a Function of Option Writer's Assets:
A Comparison of the Fixed and Variable Default Boundary Models

Calculations of vulnerable call option prices are based on the following parameter values: $S = 40$, $K = 40$, $V = 5$, $D^* = 0$, $T = 0.3333$, $\alpha = 0$, $\sigma_V = 0.3$, $\sigma_s = 0.3$, $\rho_{VS} = 0.5$, $r = 0.05$, $a = 0.1$, $b = 0.05$, $\sigma_r = 0$ $\rho_{Vr} = 0$, $\rho_{Sr} = 0.0$, unless otherwise noted. Numerical solution (circles) of the variable default boundary (VDB) model is based on a Monte-Carlo simulation. Analytical solutions (solid lines) of VDB model are based on equation 4.2.5. The design parameters "p" and "q" are optimized for each value of S, based on the technique outlined in section 4.4. Analytical solutions of the FDB model are based on equation 4.1.1.



in value of a vulnerable call relative to a non-vulnerable call[1]. Note, that FDB model predicts almost no discount off the Merton value when S and V are highly correlated, which is consistent with anecdotal evidence from the capital markets[2]. However, the VDB predicts a much larger reduction in value since the higher stock price will increase the potential required payoff and therefore increase the probability of default. The FDB model ignores this fact. This could be a potentially important result for traders dealing with highly leveraged option writers where the option contract is a significant portion of the writer's liabilities. Figure 14 illustrates the impact of ρ_{VS} on the value of vulnerable calls under the assumptions of the FDB and VDB models, for the example presented in table 2.

Figures 15 and 16 also illustrate the impact of ρ_{VS} on the value of vulnerable options. The parameters used to generate these figures correspond to those of the 3 year in-the-money call, where $D^{*} > K$. Again note the general conclusion that the more negative the correlation the greater the reduction in the value of a vulnerable call relative to its equivalent non-vulnerable counterpart.

Figure 15 shows the percentage reduction in the value of the call (from the corresponding Merton value) as a function of the moneyness of the option. The dashed lines represent the results of the FDB model and duplicate the results found in Klein and Inglis (1999). The solid lines are the corresponding results from the VDB model. First note that the correlation coefficient has a greater impact on out-of-the-money calls (in percentage terms) then in-the-money calls. The change in the percentage reduction as the money-ness of the option changes is a function of the values of the vulnerable and non-vulnerable call and their respective deltas (with respect to S).

[1] Consider a gold producer who has written a call option on gold on part of is output or inventory. If the price of gold remains below the strike price the gold producer generates income; if the price goes above the strike price than the producer has sold his gold for the strike price. Obviously, the firm's assets and the asset underlying the option are positively correlated. Also, there is little chance of default since the gold producer presumably has an inventory of gold with which to fulfill his obligation to deliver gold.

[2] Correspondence with Peter Klein.

Figure 14

Vulnerable Call Values as a Function of the Correlation between the Asset Underlying the Option and the Option Writer's Assets: A Comparison of the Fixed and Variable Default Boundary Models

Calculations of vulnerable call option prices are based on the following parameter values: $S = 50$, $K = 40$, $V = 100$, $D^{*} = 90$, $T = 3$, $\alpha = 0$, $\sigma_r = 0.1$, $\sigma_s = 0.3$, $\rho_{VS} = 0$, $r = 0.05$, $a = 0.5$, $b = 0.08$, $\sigma_r = 0.03$ $\rho_{Vr} = 0$, $\rho_{Sr} = 0.0$, unless otherwise noted. Analytical solutions (solid lines) of VDB model are based on equation 4.2.5. The design parameters "p" and "q" are optimized for each value of S, based on the technique outlined in section 4.4. Analytical solutions of the FDB model are based on equation 4.1.1.

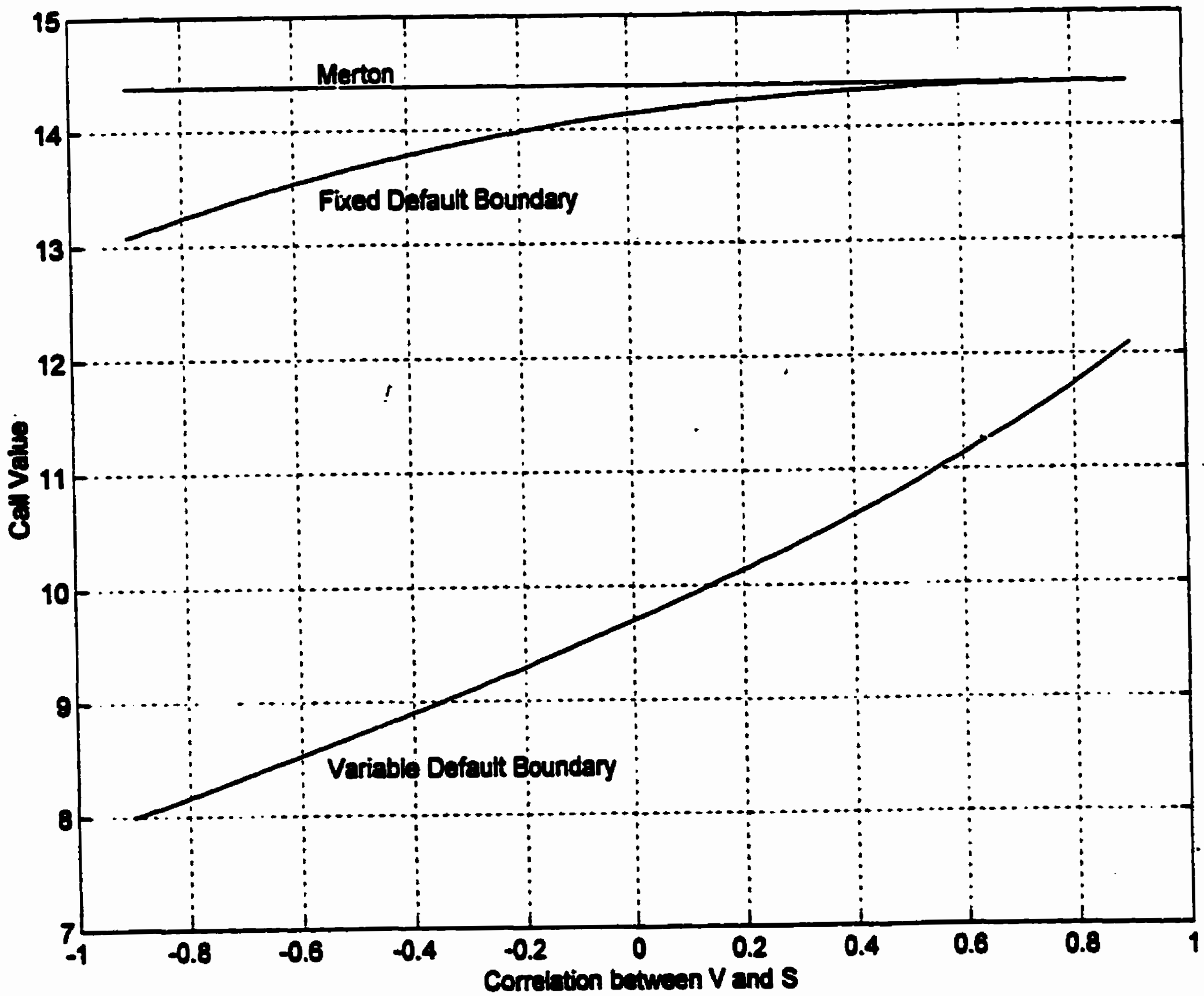


Figure 15

Percentage Reduction in the Value of Vulnerable European Calls as a function of Option's Moneyness: A Comparison of the Fixed and Variable Default Boundary Models Under Different Assumptions about ρ_{VS}

Calculations of vulnerable call option prices are based on the following parameter values: $S = 50$, $K = 40$, $V = 100$, $D^* = 90$, $T = 3$, $\alpha = 0$, $\sigma_V = 0.1$, $\sigma_S = 0.3$, $r = 0.05$, $a = 0.5$, $b = 0.08$, $\sigma_r = 0.03$ $\rho_{Vr} = 0$, $\rho_{Sr} = 0.0$, unless otherwise noted. Analytical solutions of VDB model are based on equation 4.2.5. The design parameters "p" and "q" are optimized for each value of S, based on the technique outlined in section 4.4. Analytical solutions of the FDB model are based on equation 4.1.1. Percentage reduction in both cases is based on the first relationship in equation 5.2.1.

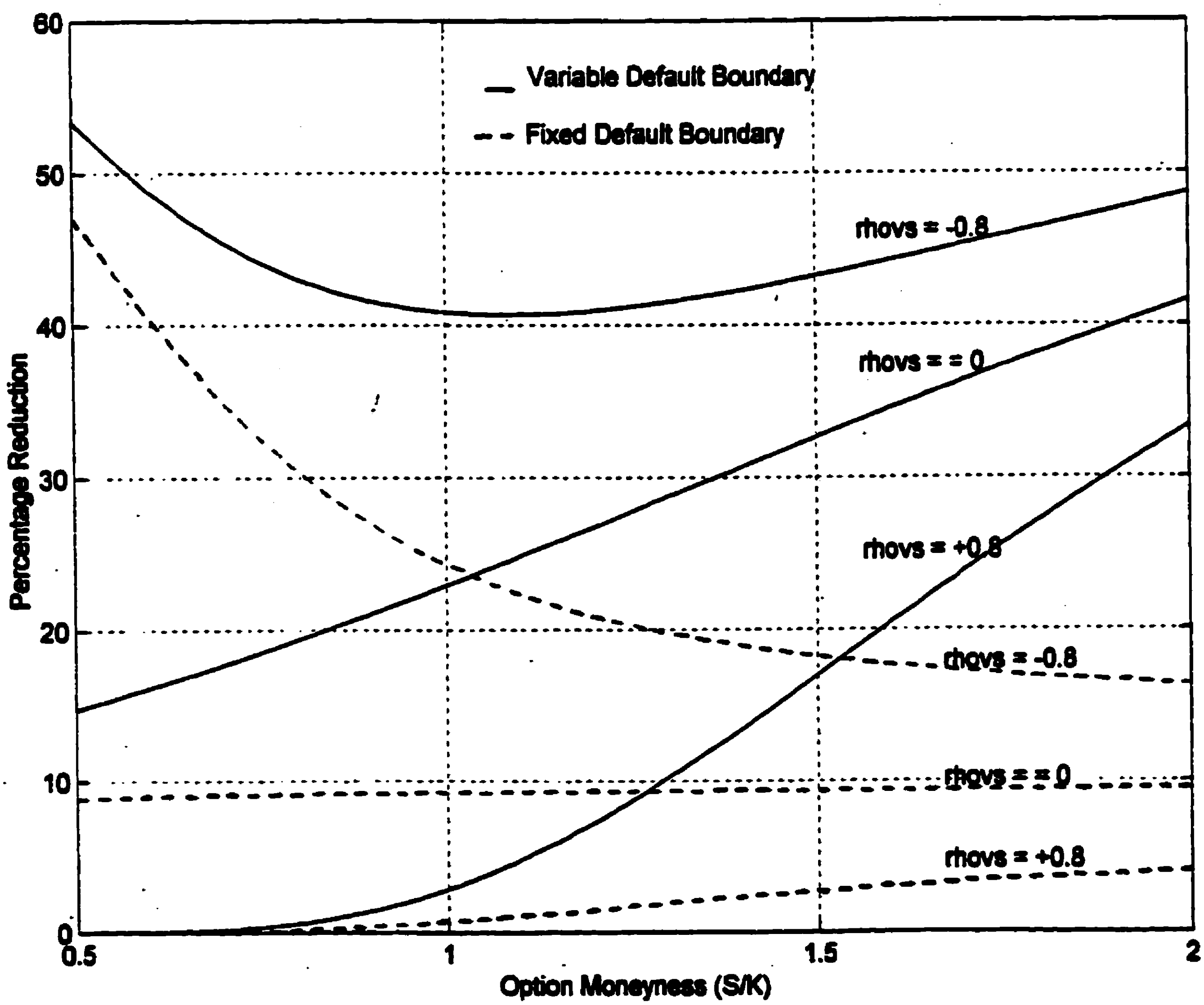
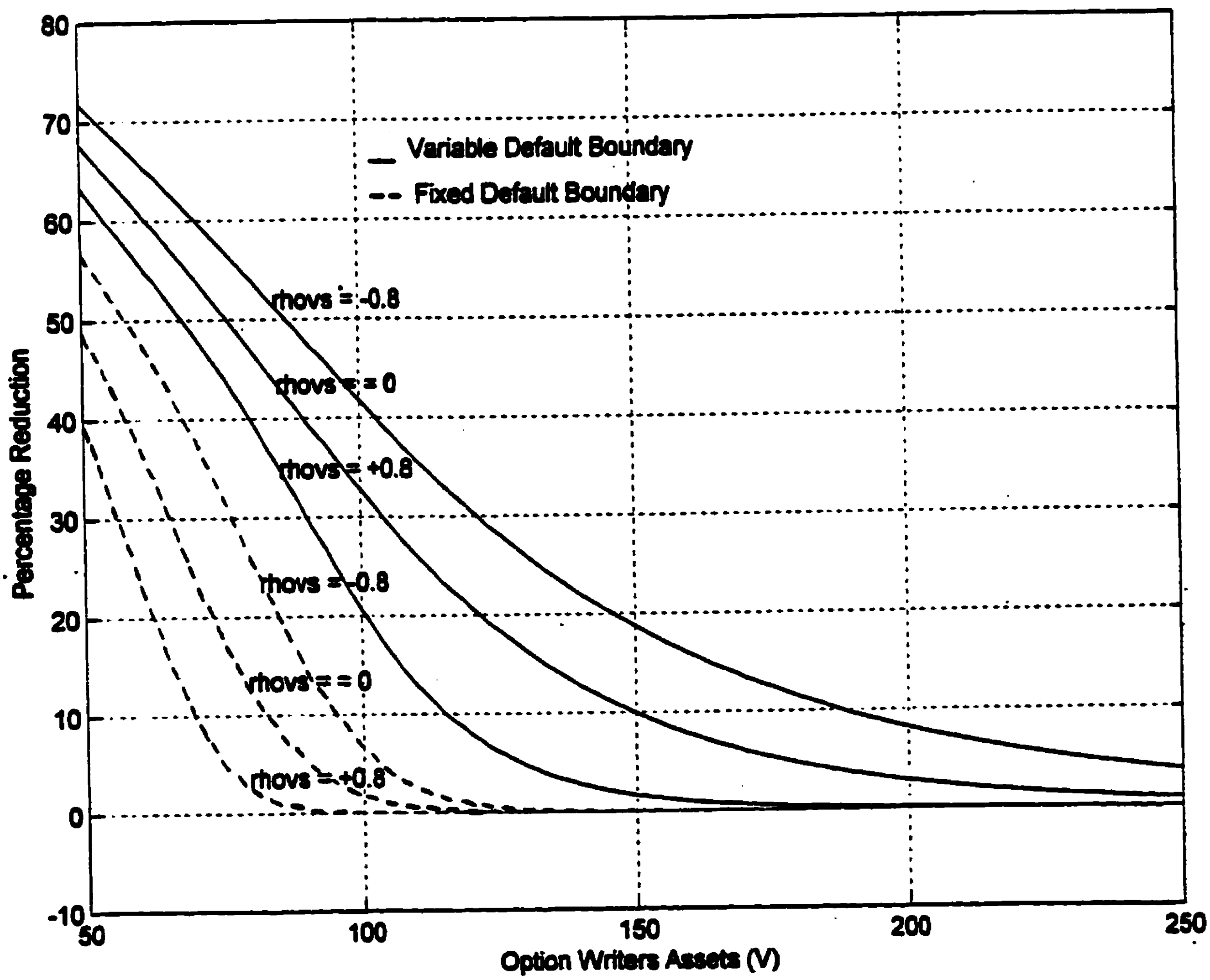


Figure 16

Percentage Reduction in the Value of Vulnerable European Calls as a function of Option Writer's Assets: A Comparison of the Fixed and Variable Default Boundary Models Under Different Assumptions about ρ_{VS}

Calculations of vulnerable call option prices are based on the following parameter values: $S = 50$, $K = 40$, $V = 100$, $D^{*} = 90$, $T = 3$, $\alpha = 0$, $\sigma_{V} = 0.1$, $\sigma_{s} = 0.3$, $r = 0.05$, $a = 0.5$, $b = 0.08$, $\sigma_{r} = 0.03$ $\rho_{Vr} = 0$, $\rho_{Sr} = 0.0$, unless otherwise noted. Analytical solutions of VDB model are based on equation 4.2.5. The design parameters "p" and "q" are optimized for each value of S, based on the technique outlined in section 4.4. Analytical solutions of the FDB model are based on equation 4.1.1. Percentage reduction in both cases is based on the first relationship in equation 5.2.1.



This can be shown as follows:

$$\% \text{ Re}\,duction = R = \frac{C_{Merton} - C_{Vu\ln erable}}{C_{Merton}}$$

$$\frac{\partial \mathrm{R}}{\partial \mathrm{S}} = \frac{1}{C_{Merton}}\left[C_{Vu\ln erable}\frac{\partial C_{Merton}}{\partial S} - \frac{\partial C_{Vu\ln erable}}{\partial S}\right]$$

(5.2.1)

Therefore the rate of change of the percentage reduction (i.e. the slope) can be either positive or negative depending on the relative sizes of $C_{Vulnerable}$ and the deltas of the vulnerable and non-vulnerable calls. Looking at figure 15, the only time the slope (i.e. dR/dS) is negative is when the correlation between S and V is negative. For the FDB model the slope is negative throughout the entire range of S. However, for the VDB model the slope is only negative for out-of-the-money calls. Otherwise, the percentage reduction generally becomes larger with the money-ness of the call. However, it increases much more quickly in the case of the VDB model. This is expected since the probability of default increases as S increases in the VDB model, but does not change with increases in S in the FDB model.

Figure 16 uses the same parameter values as above but plots the percentage reduction as a function of the option writer's assets V. As expected, the percentage reduction decreases and eventually approaches zero as the option writer's assets increase, although the decrease is much slower for the VDB model. In the FDB model, the percentage reduction is zero even for quasi-debt ratios (i.e. D^* / V) equal to 100% if the correlation between S and V is 0.8. The reduction only rises to ~ 16% if the correlation is -0.8. The corresponding numbers for the VDB model are 27% and 47% respectively. This seems to be a much more realistic prediction, given the high quasi-debt ratios.

Referring to table 2 we can see that a negative correlation between V and r (i.e. $\rho_{Vr} < 0$) decreases the value of a vulnerable call relative to the base case in the VDB model, but increases the value in the FDB case. Figures 17 and 18 also illustrate the impact of the correlation coefficient ρ_{Vr} on vulnerable calls. Generally the more negative the correlation the greater the reduction in value. However, there is a trade-off between changes in the interest rate and changes in the assets of the option writer. Intuitively, if interest rates were to increase and this resulted in a decrease in the option writer's assets (i.e. a negative correlation) then the probability of the option writer experiencing financial distress would increase. This would cause the vulnerable option to decrease in value. However, this impact would be offset by the increase in the value of the call when interest rates increase. This trade-off is clearly shown in the FDB model results, where the impact of ρ_{Vr} is reversed for deep-in-the-money options and lower quasi-debt ratios.

Figure 17 shows the percentage reduction as a function of the call's money-ness. Moving the call into the money has little impact in the FDB model, but a large impact in the VDB model. Since the VDB model takes into account the liability due to the written option, the deeper the call is in-the-money the greater the probability of default and the greater the reduction in price. Again, deep out-on-the-money options seem to be the most sensitive to ρ_{Vr}.

Figure 18 shows the percentage reduction as a function of the option writer's assets. The general conclusion is that increasing the writer's assets reduces the probability of default and the therefore the reduction in price. This figure indicates that the choice of ρ_{Vr} is of secondary importance in valuing the call, relative to ρ_{VS}.

Finally, we can see from table 2 that decreasing the correlation between the asset underlying the call and the risk-free rate, ρ_{Sr}, decreases the value of the vulnerable call. This is true for both the VDB and FDB model. Figures 19 and 20 also illustrate the impact of this correlation coefficient. The same general conclusions can be drawn in this case. First, deep-out-of-money options are most sensitive to changes in ρ_{Vr} and ρ_{Sr}.

Figure 17

Percentage Reduction in the Value of Vulnerable European Calls as a function of Option's Moneyness: A Comparison of the Fixed and Variable Default Boundary Models Under Different Assumptions about ρ_{Vr}

Calculations of vulnerable call option prices are based on the following parameter values: $S = 50$, $K = 40$, $V = 100$, $D^{*} = 90$, $T = 3$, $\alpha = 0$, $\sigma_r = 0.1$, $\sigma_s = 0.3$, $r = 0.05$, $a = 0.5$, $b = 0.08$, $\sigma_r = 0.03$ $\rho_{Vr} = 0$, $\rho_{Sr} = 0.0$, unless otherwise noted. Analytical solutions of VDB model are based on equation 4.2.5. The design parameters "p" and "q" are optimized for each value of *S*, based on the technique outlined in section 4.4. Analytical solutions of the FDB model are based on equation 4.1.1. Percentage reduction in both cases is based on the first relationship in equation 5.2.1.

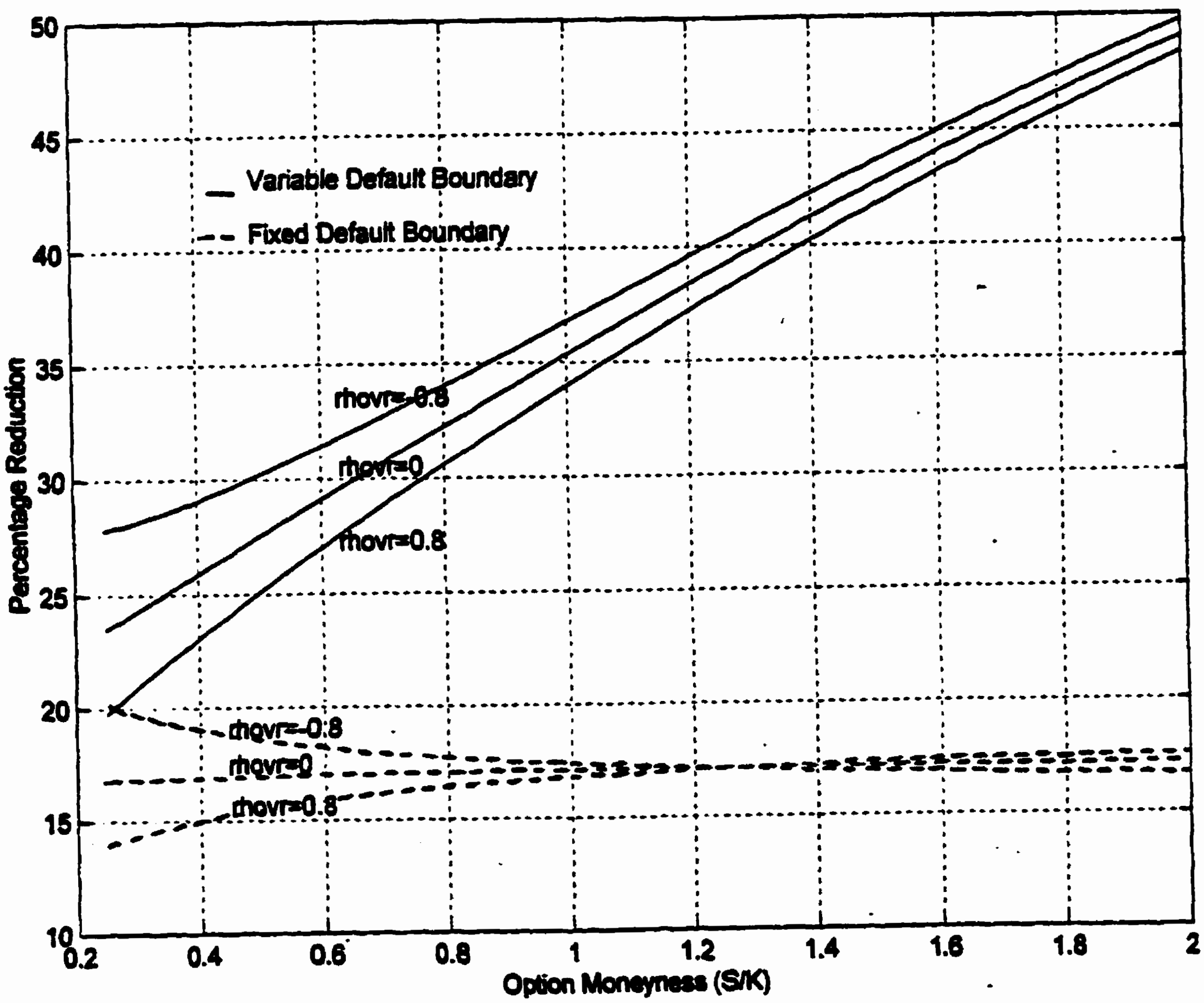


Figure 18

Percentage Reduction in the Value of Vulnerable European Calls as a function of Option Writer's Assets: A Comparison of the Fixed and Variable Default Boundary Models Under Different Assumptions about ρ_{Vr}

Calculations of vulnerable call option prices are based on the following parameter values: $S = 50$, $K = 40$, $V = 100$, $D^* = 90$, $T = 3$, $\alpha = 0$, $\sigma_V = 0.1$, $\sigma_s = 0.3$, $r = 0.05$, $a = 0.5$, $b = 0.08$, $\sigma_r = 0.03$ $\rho_{Vr} = 0$, $\rho_{Sr} = 0.0$, unless otherwise noted. Analytical solutions of VDB model are based on equation 4.2.5. The design parameters "p" and "q" are optimized for each value of *S*, based on the technique outlined in section 4.4. Analytical solutions of the FDB model are based on equation 4.1.1. Percentage reduction in both cases is based on the first relationship in equation 5.2.1.

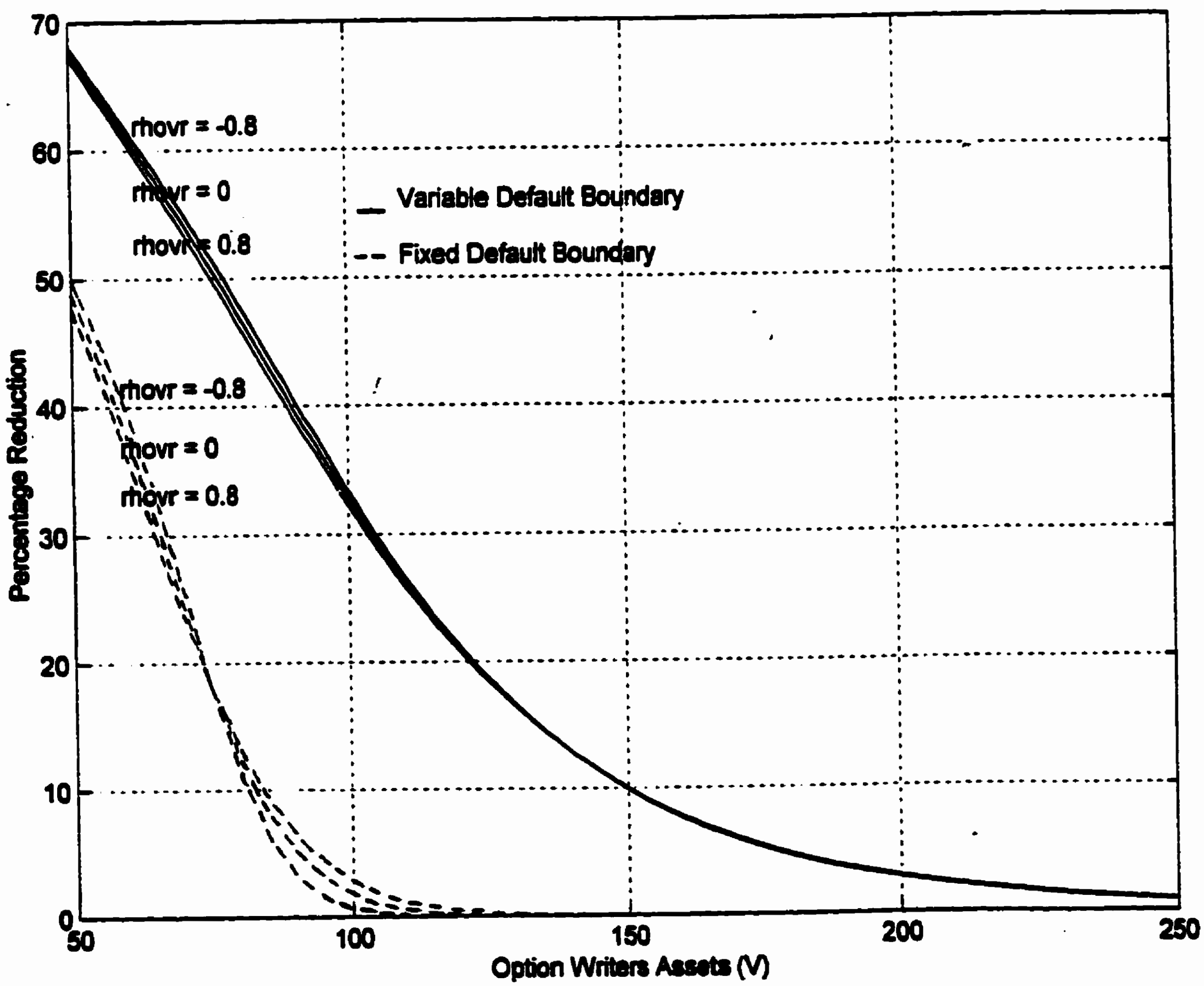


Figure 19

Percentage Reduction in the Value of Vulnerable European Calls as a function of Option's Moneyness: A Comparison of the Fixed and Variable Default Boundary Models Under Different Assumptions about ρ_{Sr}

Calculations of vulnerable call option prices are based on the following parameter values: $S = 50$, $K = 40$, $V = 100$, $D^{*} = 90$, $T = 3$, $\alpha = 0$, $\sigma_{r} = 0.1$, $\sigma_{s} = 0.3$, $r = 0.05$, $a = 0.5$, $b = 0.08$, $\sigma_{r} = 0.03$ $\rho_{Vr} = 0$, $\rho_{Sr} = 0.0$, unless otherwise noted. Analytical solutions of VDB model are based on equation 4.2.5. The design parameters "p" and "q" are optimized for each value of S, based on the technique outlined in section 4.4. Analytical solutions of the FDB model are based on equation 4.1.1. Percentage reduction in both cases is based on the first relationship in equation 5.2.1.

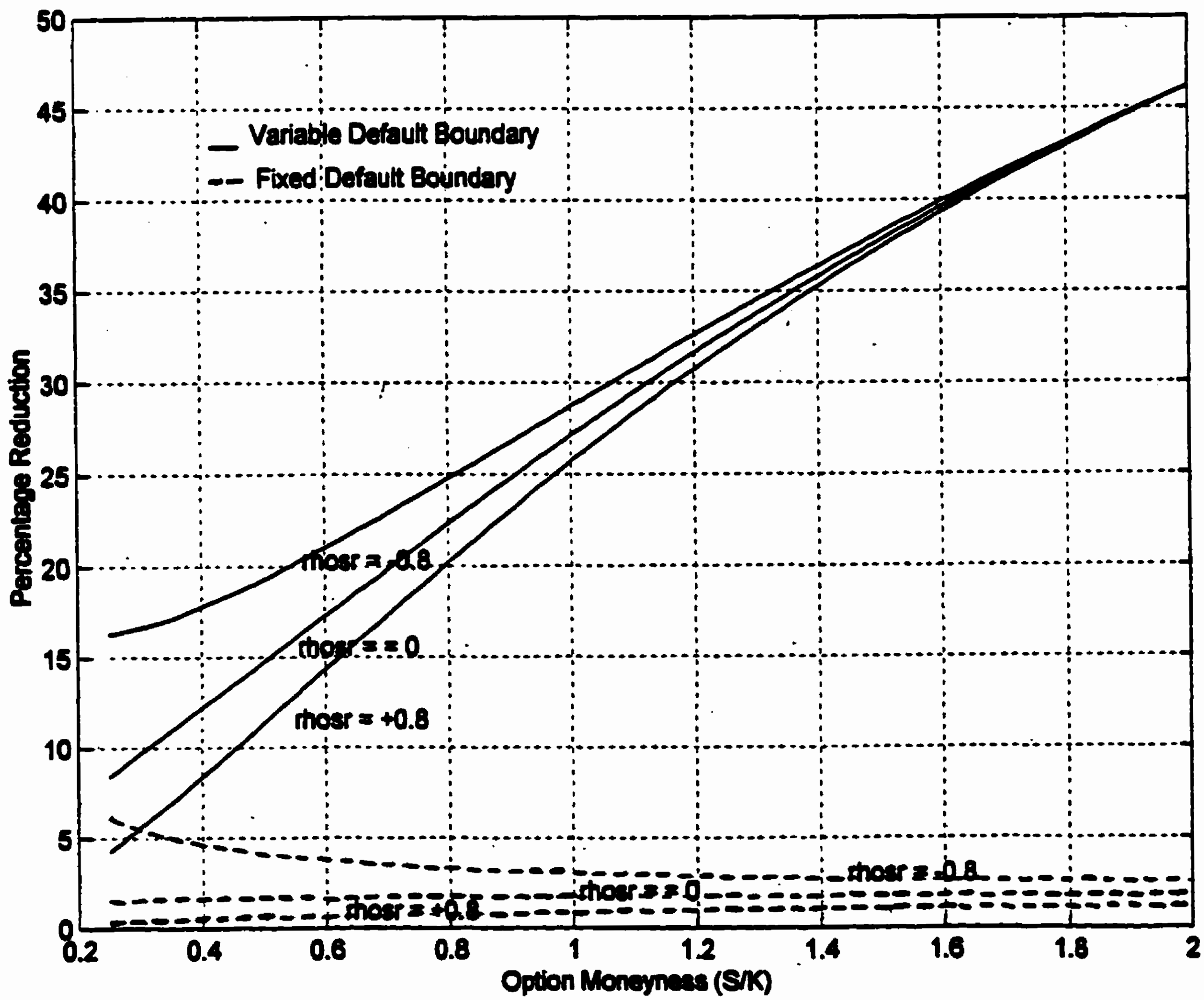
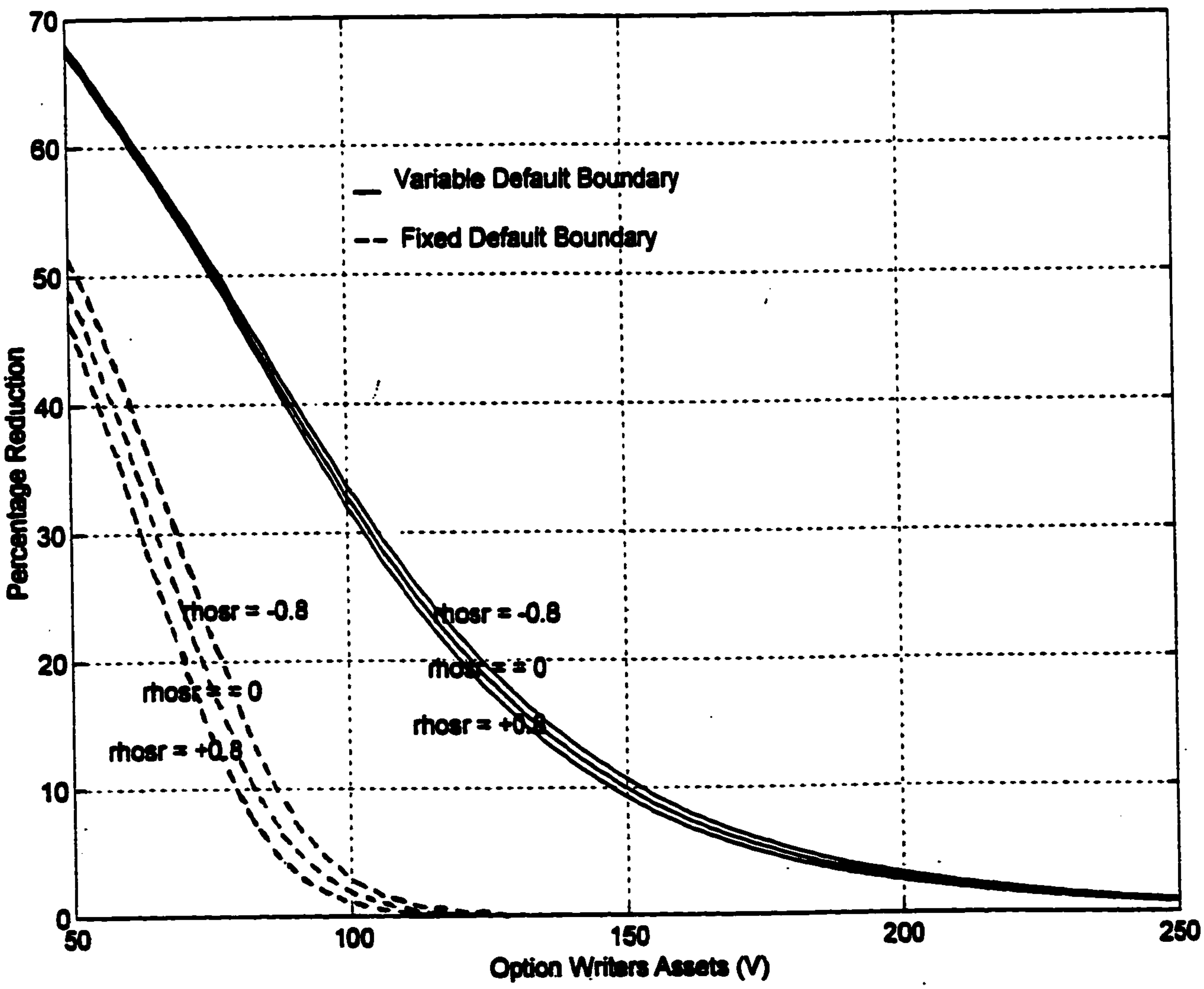


Figure 20

Percentage Reduction in the Value of Vulnerable European Calls as a function of Option Writer's Assets: A Comparison of the Fixed and Variable Default Boundary Models Under Different Assumptions about ρ_{Sr}

Calculations of vulnerable call option prices are based on the following parameter values: $S = 50$, $K = 40$, $V = 100$, $D^{*} = 90$, $T = 3$, $\alpha = 0$, $\sigma_{r} = 0.1$, $\sigma_{s} = 0.3$, $r = 0.05$, $a = 0.5$, $b = 0.08$, $\sigma_{r} = 0.03$ $\rho_{Vr} = 0$, $\rho_{Sr} = 0.0$, unless otherwise noted. Analytical solutions of VDB model are based on equation 4.2.5. The design parameters "p" and "q" are optimized for each value of S, based on the technique outlined in section 4.4. Analytical solutions of the FDB model are based on equation 4.1.1. Percentage reduction in both cases is based on the first relationship in equation 5.2.1.



Second, generally this coefficient is not of primary importance in estimating the value of the vulnerable call.

5.3: Probability of Default and Margin Requirements

In this section we look at the probability of the option writer defaulting and the cost of eliminating the risk of default through the posting of margin. The risk-neutral probability that the option writer will default can be calculated in a relatively straightforward way. The real world probabilities would be calculated in the same way, except that we would substitute the expected growth rate of the underlying asset for the risk-free rate. For the FDB model the probability of default is the probability that $V_T < D^*$, while for the VDB the probability of default is the probability that $V_T < D^* + \max(S_T - K)$. These probabilities can be shown to be: (See appendix H and appendix I for the derivations)
For the FDB Model:

$$\Pr[Default] = N_1(-\eta) \tag{5.3.1}$$

Where η is defined as:

$$\eta = \frac{\ln\left(\frac{V}{BD^*}\right) - \frac{s_V^2}{2}}{s_V} = b_2 \qquad \text{(from section 4.1)}$$

And for the VDB Model:

$$\Pr[Default] = N_2(-b_1, -\eta, \bar{\rho}_{VS}) + N_2(b_1, \lambda, -\delta) \tag{5.3.2}$$

Where:

$$b_1 = \frac{\ln\left(\frac{S}{BK}\right) - \frac{s_S^2}{2}}{s_S}$$ (from section 4.1 and 4.2)

$$\eta = \frac{\ln\left(\frac{V}{BD^*}\right) - \frac{s_V^2}{2}}{s_V} = b_2$$ (from section 4.1)

$$\lambda = -\frac{b - mp}{\sqrt{1 - 2\bar{\rho}_{VS} m + m^2}} = b_2$$ (from section 4.2)

$$\delta = \frac{\bar{\rho}_{VS} - m}{\sqrt{1 - 2\bar{\rho}_{VS} m + m^2}}$$ (from section 4.2)

and the other parameters are as defined in section 4.2.

The second part of this section deals with the "discount" embedded in vulnerable call options due to the possibility of default. This discount can be used to estimate the amount of collateral (i.e. margin) that would be required to remove the default risk. To simplify the analysis of this section, assume that the option holder, in the event of default by the writer, will make no recoveries. This corresponds to setting $\alpha = 1$. In this case the last two terms in both the FDB and VDB valuation formulas (i.e. equations 4.1.4 and 4.2.5), which deal with expected recoveries in the event of default, disappear. The valuation equations can then be rewritten in a slightly different and more intuitive format by using the following property of the cumulative bi-variate normal distribution function:

$$N_2(q_1, q_2, \rho) = N(q_1) - N_2(q_1, -q_2, -\rho) \tag{5.3.3}$$

Note that for this section we have redefined the names of the parameters to make comparisons between the FDB and VDB models easier.

The FDB model valuation formula can be rewritten as:

$$c = [SN_1(\gamma_1) - KBN_1(\gamma_2)] - [SN_2(\gamma_1, -\xi_1, -\bar{\rho}_{VS}) - KBN_2(\gamma_2, -\xi_2, -\bar{\rho}_{VS})$$

(5.3.4)

where:

$$\gamma_1 = \frac{\ln\left(\frac{S}{BK}\right) + \frac{s_S^2}{2}}{s_S} \qquad \gamma_2 = \frac{\ln\left(\frac{S}{BK}\right) - \frac{s_S^2}{2}}{s_S}$$

are the parameters of the standard non-vulnerable Merton model. The additional parameters are given by:

$$\xi_1 = \frac{\ln\left(\frac{V}{BD^*}\right) - \frac{s_V^2}{2} + s_V s_S \bar{\rho}_{VS}}{s_V} \qquad \xi_2 = \frac{\ln\left(\frac{V}{BD^*}\right) - \frac{s_V^2}{2}}{s_V}$$

and

$$\bar{\rho}_{VS} = \frac{s_{VS}}{s_V s_S}$$

The VDB model valuation formula can be rewritten as:

$$c = [SN_1(\gamma_1) - KBN_1(\gamma_2)] - [SN_2(\gamma_1, -\zeta_1, -\delta) - KBN_2(\gamma_2, -\zeta_2, -\delta)$$

(5.3.5)

where:

$$\gamma_1 = \frac{\ln\left(\frac{S}{BK}\right) + \frac{s_S^2}{2}}{s_S} \qquad \gamma_2 = \frac{\ln\left(\frac{S}{BK}\right) - \frac{s_S^2}{2}}{s_S}$$

are again the parameters of the standard non-vulnerable Merton model. The additional parameters are given by:

$$\zeta_1 = -\frac{b - mp}{\sqrt{1 - 2\bar{\rho}_{VS} m + m^2}} + \delta s_S \qquad \zeta_2 = -\frac{b - mp}{\sqrt{1 - 2\bar{\rho}_{VS} m + m^2}}$$

and

$$\delta = \frac{\bar{\rho}_{VS} - m}{\sqrt{1 - 2\bar{\rho}_{VS} m + m^2}}$$

Equations 5.3.4 and 5.3.5 have a great deal of intuitive appeal. The two terms in the first set of square brackets in each equation is simply the Black-Scholes-Merton valuation formula for a call option. The terms in the second set of square brackets represent the value of the option writer's option to default. This is the discount embedded in vulnerable call options due to the possibility of default (i.e. the default premium).

If an over-the-counter (OTC) option holder wants to eliminate the default risk of the writer then we can use the default premium as an objective measure of the collateral (i.e. margin) that should be required. The option holder should be indifferent between a non-vulnerable call and a vulnerable call that is secured with collateral equal to the default premium. Note that the collateral required would vary over time with changing conditions.

This result provides a simple way of determining whether the margin required in option markets is representative of fair market value. If margin requirements are market determined, then the value of the writer's option to default should roughly correspond to the margin required to write an exchange traded option. In Canada, the current practice mandates that a customer must deposit the option premium plus 10% of the underlying stock value for a uncovered at-the-money index option. Table 5 reports the risk-neutral probability of default and the default premium (as a percentage of the underlying stock index value) predicted by both the FDB and VDB models, for an at-the-money call option written by a highly leveraged writer.

Table 5

Probability of Default and Estimated Margin Requirements for at-the-Money

V	D^*	$\frac{V}{D^*}$	σ_V	σ_S	ρ_{VS}	T	Prob. of Default (FDB) (%)	Prob. Of Default (VDB) (%)	Margin (FDB) (% of S)	Margin (VDB) (% of S)	Margin Rich (1996) (% of S)
105.0	100	1.05	0.10	0.30	0.25	0.75	8.29	28.71	0.59	9.01	6.34
210.0	200	1.05	0.10	0.30	0.25	0.75	8.29	16.66	0.59	4.45	6.34
102.5	100	1.03	0.10	0.30	0.25	0.75	13.40	36.31	1.05	10.30	8.70
105.0	100	1.05	0.10	0.25	0.25	0.75	8.29	26.50	0.54	6.93	5.48
105.0	100	1.05	0.15	0.30	0.25	0.75	18.73	35.42	1.56	8.40	7.67
105.0	100	1.05	0.10	0.30	0.75	0.75	8.29	22.35	0.03	6.54	4.25
105.0	100	1.05	0.10	0.30	-0.50	0.75	8.29	31.20	2.85	11.10	9.23
105.0	100	1.05	0.10	0.30	0.25	1.00	7.52	28.26	0.64	10.80	8.06

Additional parameters not specified in the table are: $S = K = 50$, $r = 0.10$, $\sigma_r = 0.0$, $\rho_{Vr} = \rho_{Sr} = 0.0$, $\alpha = 1.0$.

This example is adapted from Rich (1996) and his results are reproduced here for comparison. The models presented here only allow for default at the maturity of the option, while Rich's model allows for default any time V drops below a stochastic default boundary. However, the payout ratio in Rich's model is a predefined proportion of the writer's assets at the time of default, while our models relate the payout ratio to the firm's assets at the maturity of the option. Therefore, the models are not directly comparable, however we choose all relevant parameters so as to make the models as comparable as possible.

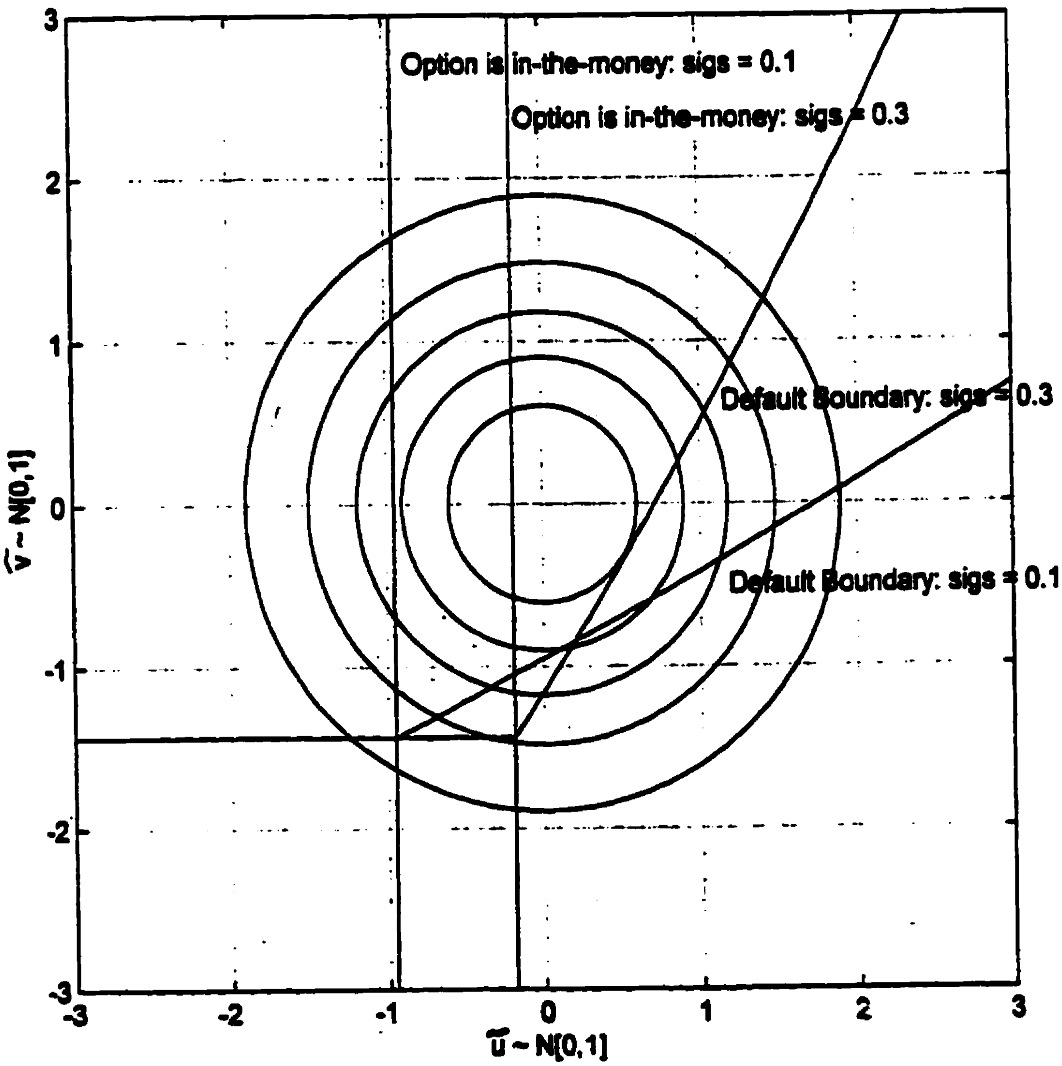
First note that the FDB model always predicts a lower probability of default and a lower margin requirement (i.e. default premium) than the corresponding VDB model. This is expected since the example was chosen so that the vulnerable call option is relatively large compared to the writer's total assets. Also, so long as the ratio of V / D^* remains constant the probability of default and the default premium remain constant in the FDB model. This means that the FDB model is homogeneous of degree zero with respect to V and D^*. Therefore, V and D^* can be scaled by any constant without altering the results. However, this is definitely not true of the VDB model. If the size of V and D^* are changed relative to the size of S, then both the probability of default and the default premium are affected. Note that in this regard, Rich's model has the same characteristics as the FDB model. So, unlike the VDB model, the FDB model and Rich's model do not incorporate the fact that the call option may actually cause the writer to default.

Decreasing the volatility of the asset underlying the option, σ_S, does not impact the probability of default in the FDB model. Again in this model the characteristics of the underlying asset have no impact on whether default occurs or not. However decreasing σ_S does decrease the margin requirement by a small amount, since the chance of large payoffs on the call decrease. This is also true of Rich's model. As noted in section 5.1, a change in σ_S can cause the probability of default to either increase or decrease. Likewise, the default premium can either increase or decrease depending on the specific parameters used in the model. The reason for this result is illustrated in figure 21. This figure shows the VDB boundaries expressed in terms of the variables v and u, which correspond to the variables V_T and S_T after the standard log-normal transformation are been performed. Two sets of boundaries are shown. The first set of boundaries

Figure 21
Integration Region for Vulnerable European Calls for Different Values of σ_s

Calculations of vulnerable call option prices are based on the following parameter values: $S = 50$, $K = 50$, $V = 10$, $D^* = 100$, $T = 0.75$, $\alpha = 1$, $\sigma_V = 0.1$, $\rho_{VS} = 0.25$, $r = 0.10$, $a = 0.5$, $b = 0.10$, $\sigma_r = 0$, $\rho_{Vr} = 0$, $\rho_{Sr} = 0.0$. Figure shows the boundaries of the integration region superimposed on the contour lines of a bi-variate normal probability distribution. The default boundary for the VDB model is based on the linear approximation given in Appendix D. The default boundary for the fixed default boundary model (FDB) is based on the boundary function shown in Appendix C. Note that:

$$\tilde{u} = \frac{\ln(S_T) - \ln\left(\frac{S}{B}\right) + \frac{s_S^2}{2}}{s_S} \qquad \tilde{v} = \frac{\ln(V_T) - \ln\left(\frac{V}{B}\right) + \frac{s_V^2}{2}}{s_V}$$



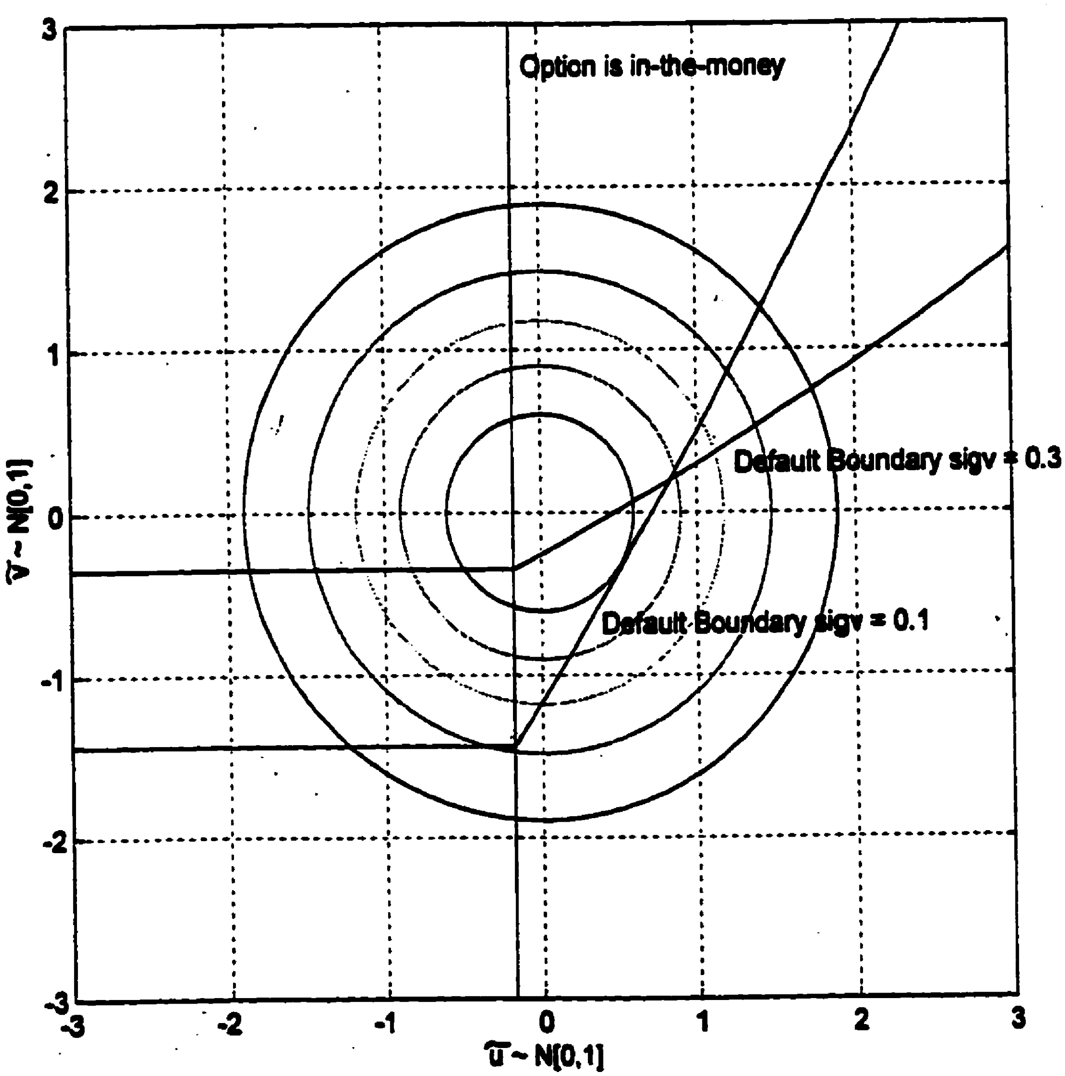
corresponds to $\sigma_S = 0.1$ and the second set corresponds to $\sigma_S = 0.3$. Note that the change in σ_S is larger in the figure then in table 5, to make the impact on the boundaries more obvious. The vertical lines represent the boundary between the call being in-the-money (i.e. to the right of the line) and out-of-the-money (i.e. to the left of the line). This boundary shifts to the right as σ_S increases which indicates a reduction in the probability of the option being exercised. The other boundary lines represent the default boundary. The horizontal line to the left of the vertical lines corresponds to D^* while the upward sloping lines to the right of the vertical lines corresponds to $D^* + S_T - K$. As can be seen in the figure, increasing σ_S shifts the intercept to the right and increases the slope of the default boundary to the right of the vertical line. Therefore the probability of default can either increase or decrease as σ_S increases depending on whether the integral of the bi-variate normal distribution over the area below the default boundaries increases or decreases. Similarly, the default premium can either increase or decrease as σ_S increases. In this case it depends on the value of the integral of option's payoff times the normal distribution over the area to the right of the at-the-money line (i.e. the vertical line) below the default boundary. In the example in Table 5, decreasing σ_S decreases both the probability of default and the default premium.

Increasing the volatility of the option writer's assets, σ_V, increases both the probability of default and the default risk premium in the FDB model. However, this result is not as clear in the VDB model. Although, the probability of default increases as expected with an increase in σ_V, the default premium (i.e. the amount you would have to theoretically post as margin) actually decreases since the expected loss decreases. This counter intuitive result can be explained by looking at figure 22. This figure is similar in design to figure 21, discussed above. However, in this example the two sets of boundary conditions correspond to two different values of σ_V. The dividing line between in-the-

Figure 22
Integration Region for Vulnerable European Calls for Different Values of σ_r

Calculations of vulnerable call option prices are based on the following parameter values: $S = 50$, $K = 50$, $V=10$, $D^{*}=100$, $T=0.75$, $\alpha = 1$, $\sigma_s = 0.3$, $\rho_{VS} = 0.25$, $r = 0.10$, $a = 0.5$, $b = 0.10$, $\sigma_r = 0$, $\rho_{Vr} = 0$, $\rho_{Sr} = 0.0$. Figure shows the boundaries of the integration region superimposed on the contour lines of a bi-variate normal probability distribution. The default boundary for the VDB model is based on the linear approximation given in Appendix D. The default boundary for the fixed default boundary model (FDB) is based on the boundary function shown in Appendix C. Note that:

$$\tilde{u} = \frac{\ln(S_T) - \ln\left(\dfrac{S}{B}\right) + \dfrac{s_S^2}{2}}{s_S} \qquad \tilde{v} = \frac{\ln(V_T) - \ln\left(\dfrac{V}{B}\right) + \dfrac{s_V^2}{2}}{s_V}$$



money and out-of-the money calls is not affected by changes in σ_V. The default boundary for the VDB model shifts upwards in the left hand side of the diagram and has a higher intercept, but lower slope. Remember that in this case the major factor controlling the slope, is the ratio σ_S/σ_V. As σ_V increases the ratio and therefore the slope of the default boundary on the right hand side of the figure become smaller.
The probability of default is given by integrating the bi-variate normal distribution function (the contour lines of the distribution are shown in the figure for reference) over the area below the default boundary. Looking at the figure it is relatively easy to see that the probability of default will increase in this example. However, it is possible to find a set of parameters for the model where the opposite result would hold. The Black-Scholes-Merton value of the call is given by integrating the payoff on the call times the normal distribution function over the in-the-money region (i.e. the area to the right of the vertical line). This value can be split into two components, the value of the call in the area above the default boundary is the value of the VDB call. The value below the default boundary is the default premium. Since the default boundaries for the two cases presented in the figure (i.e. $\sigma_V = 0.1$ and $\sigma_V = 0.3$) cross, the default premium can either increase or decrease as σ_V increases depending on the particular parameters chosen.

Changing the correlation coefficient, ρ_{VS}, does not shift either of the boundaries in either figure 21 or 22. However, it does affect the bi-variate normal distribution function and therefore has an effect on both the probability of default and the default premium. From table 5 we can see that increasing ρ_{VS} decreases the default premium (and therefore increases the value of the vulnerable call) in all cases for the reasons discussed in section 5.1. However, it is interesting to note that the probability of default in not affected in the FDB model, but decreases in the VDB model, as ρ_{VS} increases.

Finally, in this example, increasing the time to maturity of the option decreases the probability of default but increases the default premium. However, in the case of the VDB model, this depends on the actual parameters used in the model. It is possible to generate completely the opposite results. The reasoning is the same as in figures 21 and 22. By changing T, the boundaries are shifted in such as way that it is possible that the probability of default and the default premium may either increase or decrease.

As a final point, observe that the default premium (i.e. the amount of margin that would have to be posted to remove the default risk) is smallest for the FDB model and largest for the VDB model. Rich's model falls in the middle. Of the three models, the VDB model comes closest to predicting the actual margin requirements demanded by exchanges. Note that if recoveries were possible then all of the models would predict even smaller margin requirements. These models all suggest that exchanges require excessive margin deposits.

5.4 Hedging Vulnerable Options

In this section we look at the hedging parameters of Merton's model, the FDB model and the VDB model. Specifically we numerically calculate the the delta and gamma (ie. dc / dS and dc^2 / dS^2) of all three models. In addition, for the FDB model and the VDB model we calculate (i.e. dc / dV and dc^2 / dV^2). We then perform a simple hedging experiment to illustrate how each of the models could be used to hedge a long position in a vulnerable call.

5.4.1 Hedge Parameters

The example chosen for this section of chapter 5 corresponds to the example in table 3. The base case involves a moderately leveraged (i.e. $D^* / V = 0.8$) firm writing a 1 year out-of-the-money option. The only difference from the example in table 3 it that we assume here that there are no recoveries (i.e. $\alpha = 1$). Note that in this case $D^* = K = 40$, which means that the VDB valuation formula is independent of the design parameters "p" and "q". Therefore the valuation formula gives exact results. Figures 23 ,24 and 25 illustrate respectively, the value of the call, the delta of the call and the gamma of the call as a function of the options moneyness. On each figure we present Merton's model, the FDB model and the VDB model. The first point to note is that there is virtually no difference between Merton's model and the FDB model. There are two reasons for this: First the firm is only moderately leveraged as measured by the quasi-debt ratio (i.e. $D^{*} / V = 0.8$). The FDB model does not significantly decrease the value of a vulnerable call until this ratio is greater than 90%. Secondly, we chose $\alpha = 1$. It turns out that this parameter can have a significant influence in determining the actual value of the vulnerable call. In fact, the parameter α could be used as a tuning parameter to generate the correct call value. The VDB model call values, deltas and gammas are significantly different from those of the first two models. As the option moves more and more into the money the value of the call continues to increase in this example but at a decreasing rate. Note that the reduction in value at $S / K = 1$ is relatively small (i.e. less then 5%) but at $S / K = 2$ the reduction in value is quite significant (i.e. greater then 50%), reflecting the impact of the written call on the probability of default.

The delta (i.e. dc / dS) of the calls were calculated numerically as follows:

$$\frac{dc}{dS} = \frac{c_{plus} - c_{minus}}{2\varepsilon} \qquad (5.4.1.1)$$

where:

c_{plus} = value of call evaluated at $S + \varepsilon$
c_{minus} = value of call evaluated at $S + \varepsilon$
$\varepsilon = 0.00001$

Figure 23

Vulnerable Call Values as a Function of Option's Moneyness: A Comparison of the Fixed and Variable Default Boundary Models

Calculations of vulnerable call option prices are based on the following parameter values: $S = 30$, $K = 40$, $V = 50$, $D^* = 40$, $T = 1$, $\alpha = 0$, $\sigma_r = 0.1$, $\sigma_s = 0.2$, $\rho_{VS} = 0.5$, $r = 0.05$, $a = 0.5$, $b = 0.08$, $\sigma_r = 0.03$ $\rho_{Vr} = 0$, $\rho_{Sr} = 0.0$ unless otherwise noted. Analytical solutions (solid lines) of VDB model are based on equation 4.2.5. The design parameters "p" and "q" are optimized for each value of S, based on the technique outlined in section 4.4. Analytical solutions of the FDB model are based on equation 4.1.1.

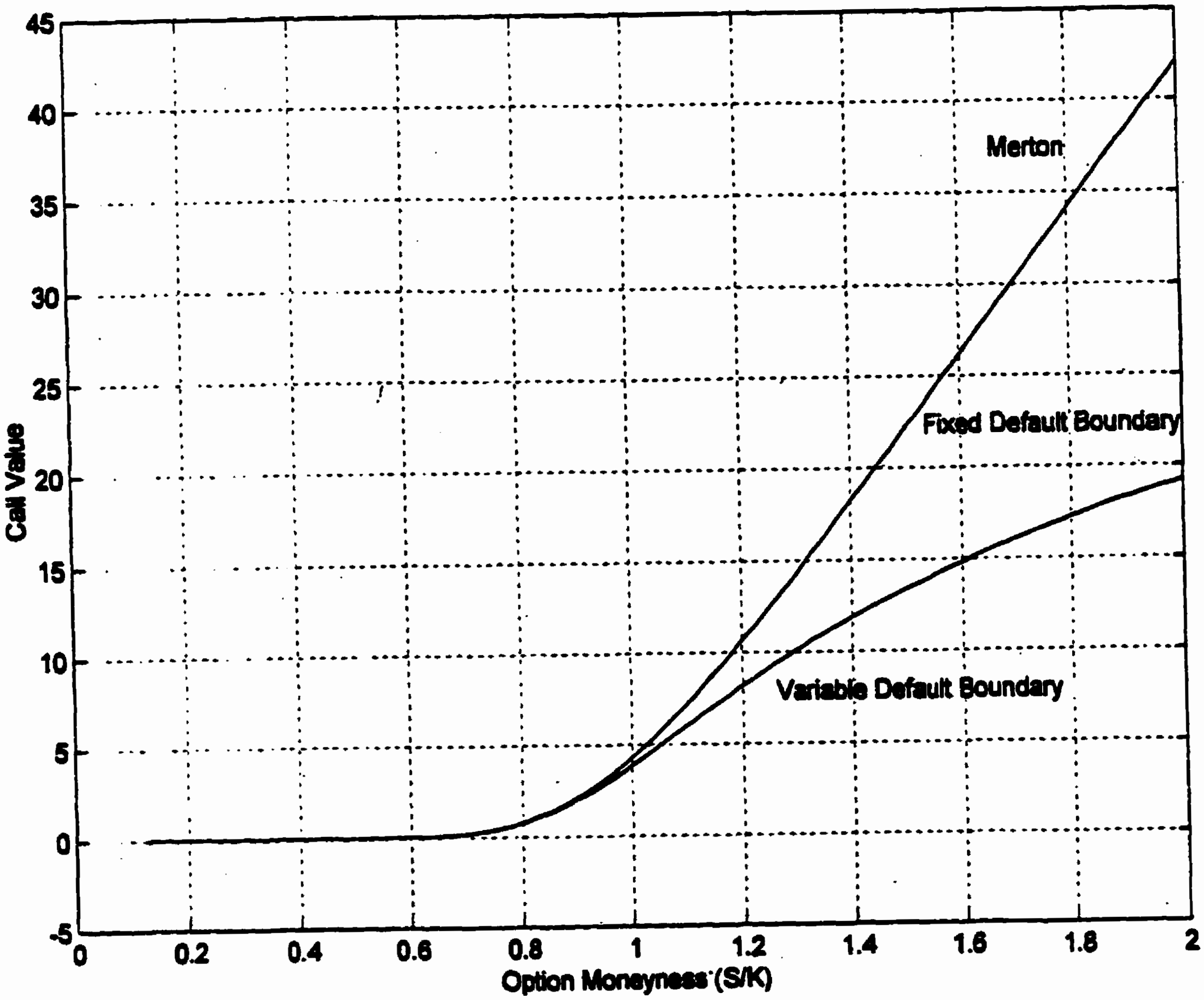


Figure 24

Vulnerable Call Deltas (w.r.t S) as a Function of Option's Moneyness: A Comparison of the Fixed and Variable Default Boundary Models

Calculations of vulnerable call option prices are based on the following parameter values:

$S = 30$, $K = 40$, $V = 50$, $D^* = 40$, $T = 1$, $\alpha = 0$, $\sigma_V = 0.1$, $\sigma_s = 0.2$, $\rho_{VS} = 0.5$, $r = 0.05$ $a = 0.5$, $b = 0.08$, $\sigma_r = 0.03$ $\rho_{Vr} = 0$, $\rho_{Sr} = 0.0$ unless otherwise noted. Analytical solutions of VDB model are based on equation 4.2.5. The design parameters "p" and "q" are optimized for each value of S, based on the technique outlined in section 4.4. Analytical solutions of the FDB model are based on equation 4.1.1. The deltas were calculated numerically using equation 5.4.1.1.

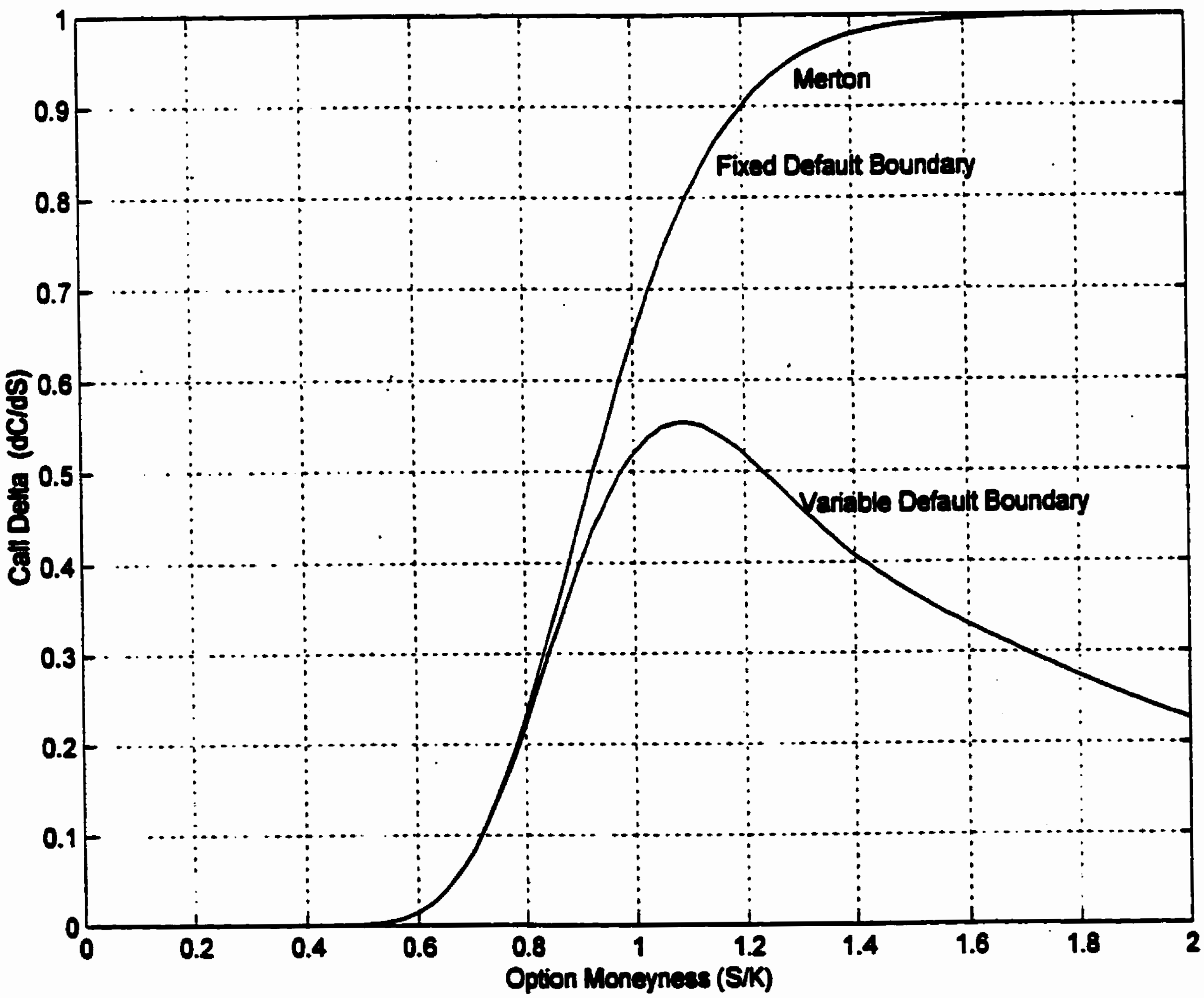
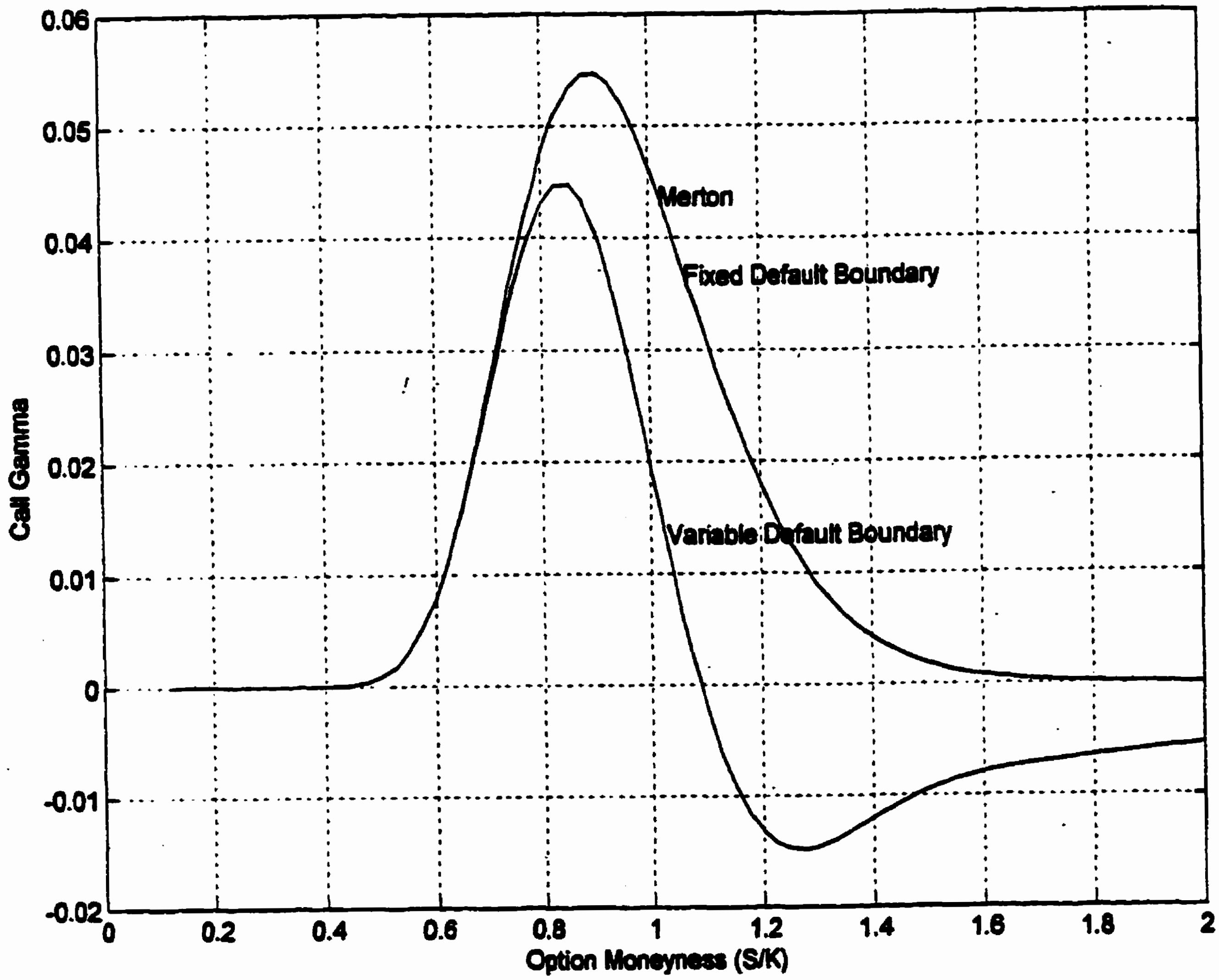


Figure 25

Vulnerable Call Gammas (w.r.t S) as a Function of Option's Moneyness: A Comparison of the Fixed and Variable Default Boundary Models

Calculations of vulnerable call option prices are based on the following parameter values: $S = 30$, $K = 40$, $V = 50$, $D^* = 40$, $T = 1$, $\alpha = 0$, $\sigma_r = 0.1$, $\sigma_s = 0.2$, $\rho_{VS} = 0.5$, $r = 0.05$, $a = 0.5$, $b = 0.08$, $\sigma_r = 0.03$, $\rho_{Vr} = 0$, $\rho_{Sr} = 0.0$, unless otherwise noted. Analytical solutions of VDB model are based on equation 4.2.5. The design parameters "p" and "q" are optimized for each value of S, based on the technique outlined in section 4.4. Analytical solutions of the FDB model are based on equation 4.1.1. The gammas were calculated numerically using equation 5.4.1.2.



Again, the Merton and FDB models generate the same results. However, the VDB model again is significantly different. First note that the VDB delta is always less than the corresponding delta from the first two models. Also, in this case the delta of the call increases as S increases, but only up to point, beyond which the delta starts to decrease. This indicates that hedging a long position in a vulnerable call will require a smaller short position in the underlying asset when S increases beyond a certain point (approximately $S / K = 1.1$ in this case. This makes sense since the probability of default increases as the money moves deeper into-the-money and the ability of the option writer to payoff its debts decreases. Therefore you need a smaller position in the underlying asset to hedge the risk due to changes in the value of that asset. As another point, it is actually possible to generate negative deltas in the VDB model. For example if we chose $\alpha = 0$ we would see negative deltas for in-the-money calls. Thus, it is not possible to sign the delta of a vulnerable call under the VDB model.

The gamma (i.e. d^2c / dS^2) of the calls were calculated numerically as follows:

$$\frac{d^2c}{dS^2} = \frac{c_{plus} + c_{minus} - 2c}{\varepsilon^2} \qquad (5.4.1.2)$$

where:

$$\begin{aligned} c &= \text{value of the call evaluated at S} \\ c_{plus} &= \text{value of call evaluated at S} + \varepsilon \\ c_{minus} &= \text{value of call evaluated at S} + \varepsilon \\ \varepsilon &= 0.00001 \end{aligned}$$

Figure 25 illustrates the gammas for all three models. The Merton and FDB model gammas are again the same, while the absolute value of gamma in the VDB model is less than or equal to the Merton and FDB gamma so long as $S / K < 1.3$. The VDB gamma is greater whenever $S / K > 1.3$. This indicates that the VDB model is less sensitive to a sudden change in S whenever $S / K < 1.3$ and more sensitive whenever $S / K > 1.3$. Also note that since gamma represents the rate of change of delta, the gamma of the VDB model becomes negative as the delta function starts to decrease as S increases.

Figures 26, 27 and 28 illustrate the call value, dc / dV and d^2c / dV^2 as functions of the value of the option writer's assets. Again, the values were calculated numerically using formulas similar to those presented in equations 5.4.1 and 5.4.2. As expected Merton's call value is not affected by V, and both the FDB and VDB vulnerable call values increase as V increases. This occurs since the probability of default will decrease as V increases. Also, the value of the VDB call is always below the FDB value. In fact the FDB model does not predict any significant reduction in the value of a call until the quasi-debt ratio (D^* / V) exceeds 100%. The VDB model starts to predict significant reductions in value when the quasi-debt ratio is greater then 80%.

The hedge ratio dc / dV is used to mitigate the risk of default. In this case dc / dV represents the percentage of the option writer's assets that would have to be shorted to remove the default risk from holding a position in a long vulnerable call. First note that at no time does this hedge parameter exceed 2%, which indicates that there is not a significant amount of default risk in this example. This makes sense since the quasi-debt ratio is only 80% and the option is well out of the money. Also, dc / dV is never negative since the value of a vulnerable call always increases with increasing V and dc / dV represents the slope of the call value for different values of V. Obviously as V gets very large dc / dV approaches zero for both models since default risk diminishes to zero. However, it takes considerably more assets to completely remove the default risk from the VDB model, even though the option is out-of-the-money and its current value does not represent a large component of the writer's liabilities.

The hedge parameter d^2c / dV^2 which represents the sensitivity of dc / dV to changes in V can be either positive or negative. Also, in general the FDB model seems have potentially greater sensitivity to changes in V.

The figures corresponding to the examples in tables 2 and 4 in section 5.1 show

Figure 26

Vulnerable Call Values as a Function of the Option Writer's Assets: A Comparison of the Fixed and Variable Default Boundary Models

Calculations of vulnerable call option prices are based on the following parameter values: $S=30$, $K=40$, $V=50$, $D^*=40$, $T=1$, $\alpha=0$, $\sigma_V=0.1$, $\sigma_S=0.2$, $\rho_{VS}=0.5$, $r=0.05$, $a=0.5$, $b=0.08$, $\sigma_r=0.03$ $\rho_{Vr}=0$, $\rho_{Sr}=0.0$ unless otherwise noted. Analytical solutions (solid lines) of VDB model are based on equation 4.2.5. The design parameters "p" and "q" are optimized for each value of V, based on the technique outlined in section 4.4. Analytical solutions of the FDB model are based on equation 4.1.1.

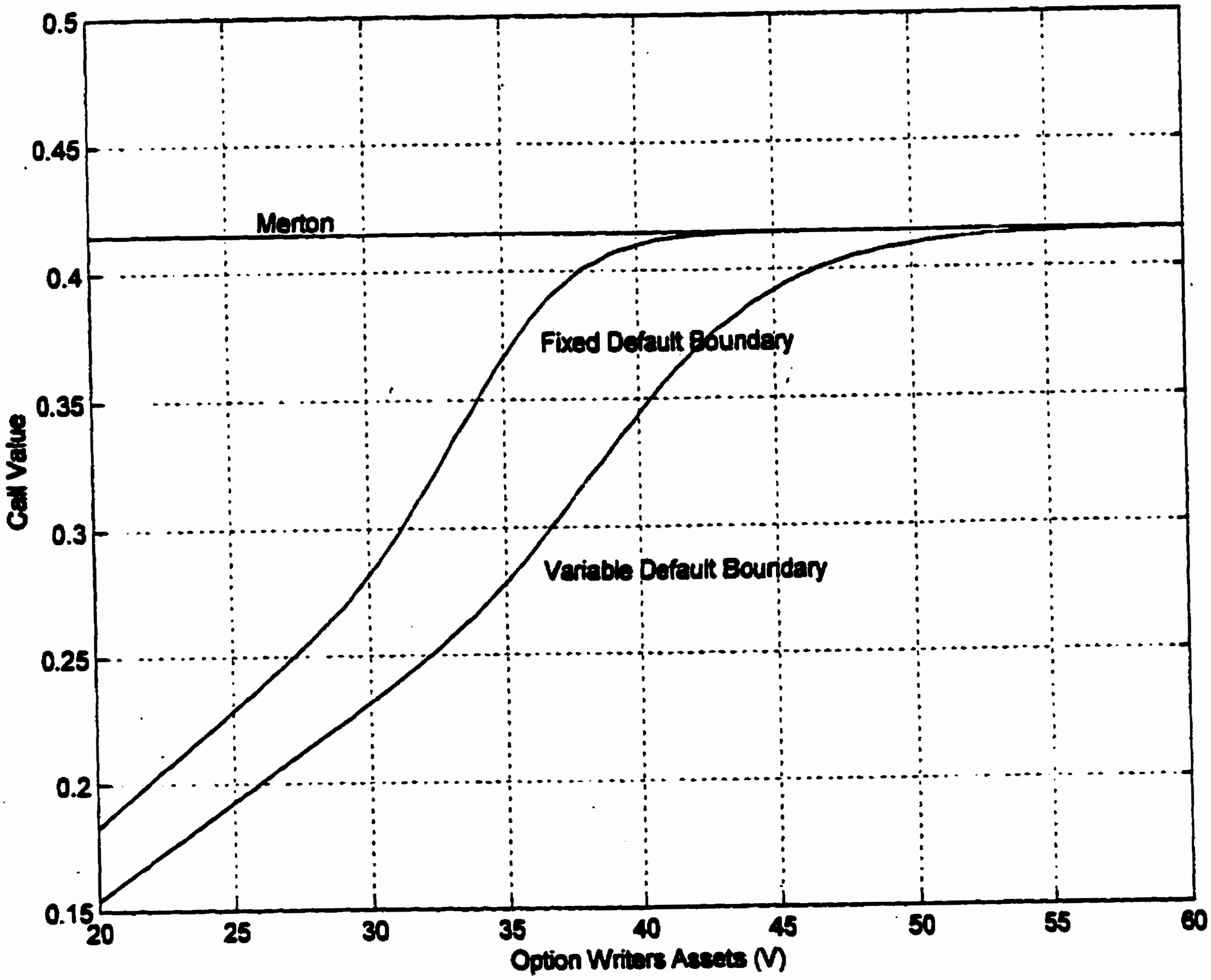


Figure 27

Vulnerable Call Deltas (w.r.t V) as a Function of the Option Writer's Assets: A Comparison of the Fixed and Variable Default Boundary Models

Calculations of vulnerable call option prices are based on the following parameter values:

$S = 30$, $K = 40$, $V = 50$, $D^* = 40$, $T = 1$, $\alpha = 0$, $\sigma_V = 0.1$, $\sigma_s = 0.2$, $\rho_{VS} = 0.5$, $r = 0.05$, $a = 0.5$, $b = 0.08$, $\sigma_r = 0.03$, $\rho_{Vr} = 0$, $\rho_{Sr} = 0.0$ unless otherwise noted. Analytical solutions of VDB model are based on equation 4.2.5. The design parameters "p" and "q" are optimized for each value of V, based on the technique outlined in section 4.4. Analytical solutions of the FDB model are based on equation 4.1.1. The deltas were calculated numerically using equation 5.4.1.1.

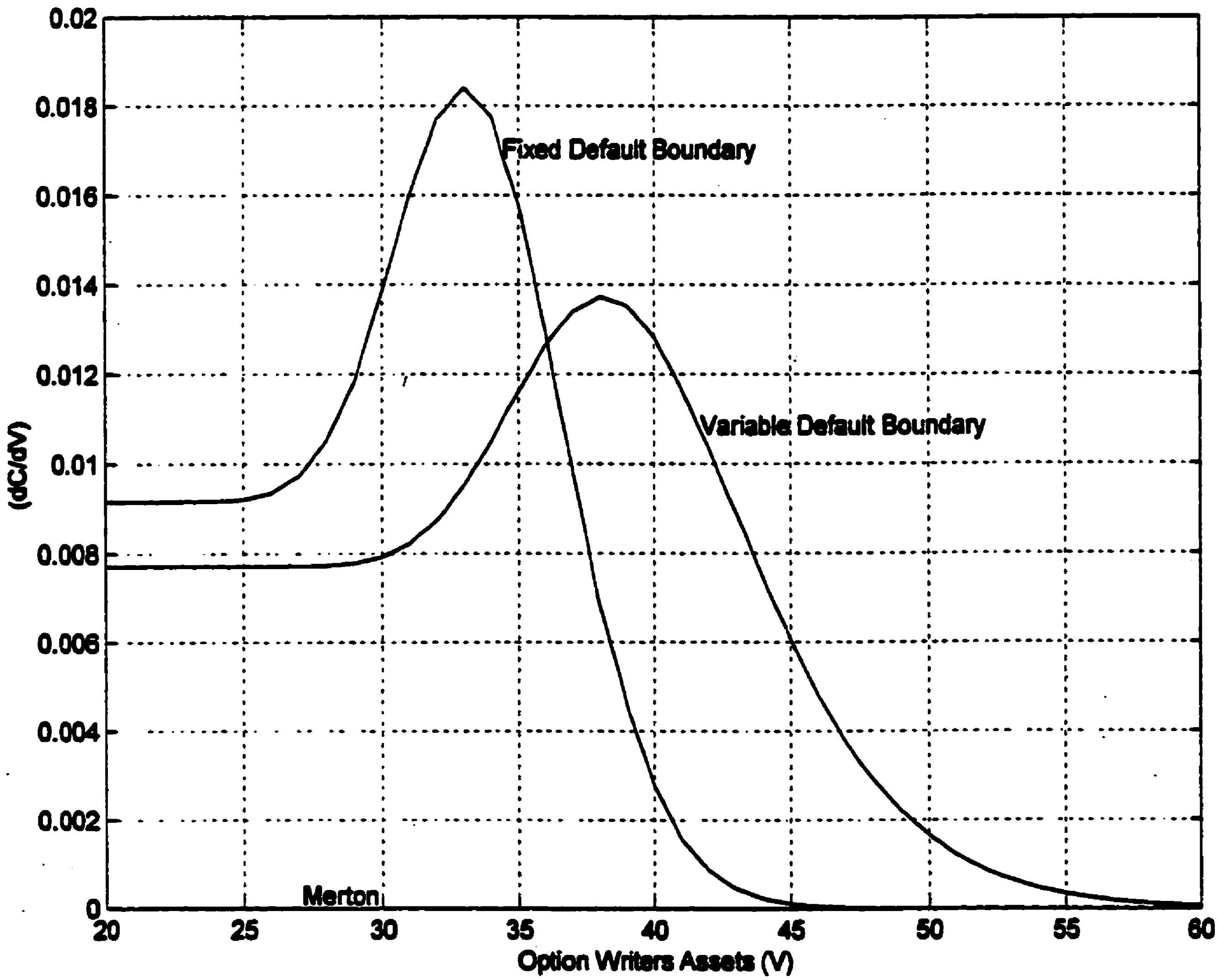
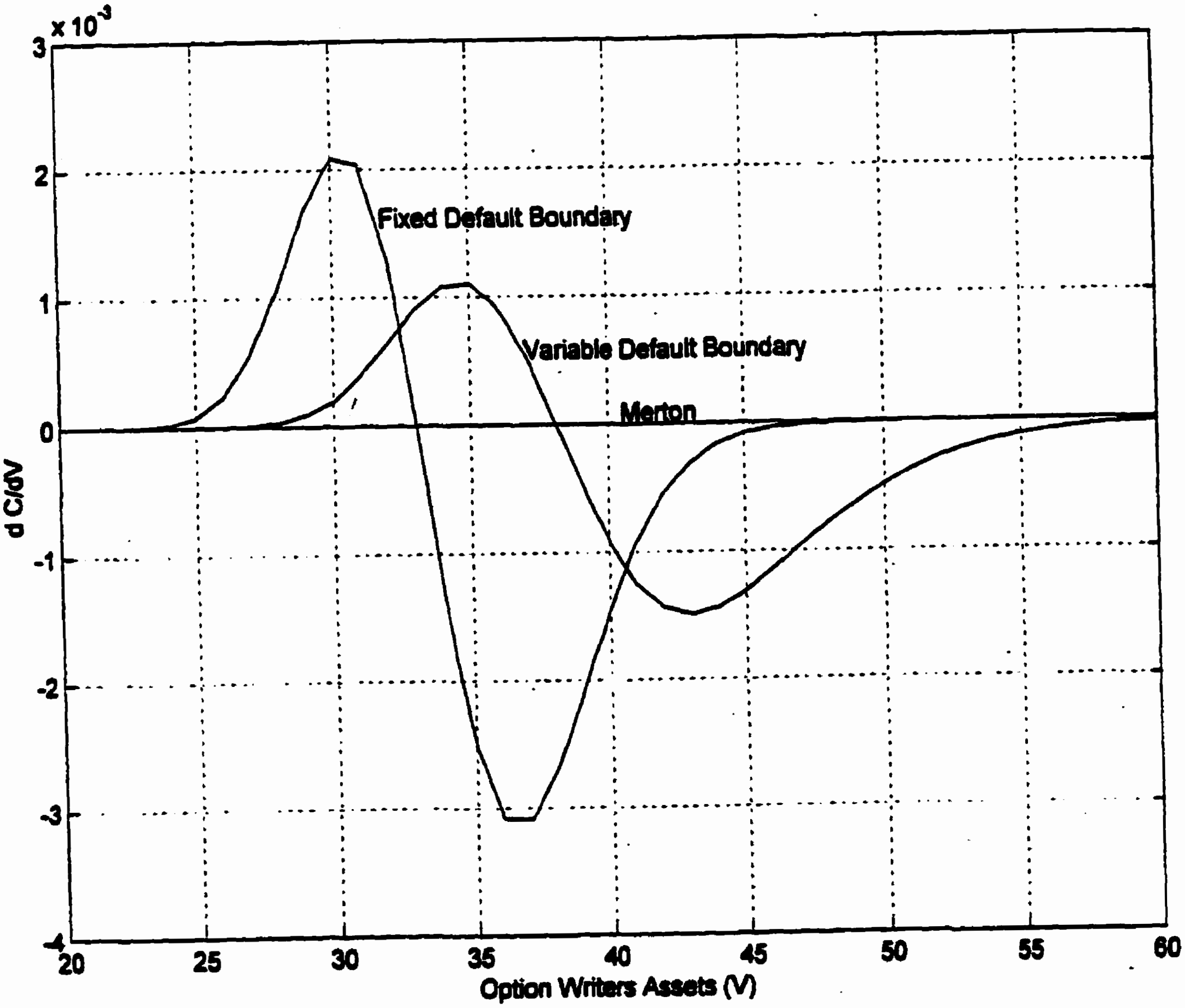


Figure 28

Vulnerable Call Gammas (w.r.t V) as a Function of the Option Writer's Assets: A Comparison of the Fixed and Variable Default Boundary Models

Calculations of vulnerable call option prices are based on the following parameter values: $S = 30$, $K = 40$, $V = 50$, $D^* = 40$, $T = 1$, $\alpha = 0$, $\sigma_r = 0.1$, $\sigma_s = 0.2$, $\rho_{VS} = 0.5$, $r = 0.05$, $a = 0.5$, $b = 0.08$, $\sigma_r = 0.03$, $\rho_{Vr} = 0$, $\rho_{Sr} = 0.0$, unless otherwise noted. Analytical solutions of VDB model are based on equation 4.2.5. The design parameters "p" and "q" are optimized for each value of V, based on the technique outlined in section 4.4. Analytical solutions of the FDB model are based on equation 4.1.1. The gammas were calculated numerically using equation 5.4.1.2.



similar patterns to those above and are therefore not presented here. However, table 6 summarizes the general conclusions about the specific hedge parameters mentioned above.

5.4.2 Hedging a Long Position in a Vulnerable Call

In this section we investigate the ability of each of the three models to hedge their own payoff as well as the payoff associated with the VDB model. We take the point of view of the counter-party buying an in-the-money vulnerable call from the highly leveraged option writer. The specific parameters used in this example are shown below table 7.

We simulate the price of the asset underlying the call, *S*, the assets of the option writer, *V*, and the interest rate *r*, over 52 time periods of one week each. At time zero and at the end of each time period we form an instantaneously risk-free portfolio consisting of a long position in one vulnerable option, a short position in $dc \,/\, dS$ shares of *S* and a short position in $dc \,/\, dV$ units of the option writers assets. The hedge parameter $dc \,/\, dS$ removes the risk due to the uncertainty of S, while the hedge parameter $dc \,/\, dV$ accounts for the default risk. This approach assumes that we can freely trade in the assets of the option writer. At each time step we keep track of the number of units of S and V that must be sold short to hedge the vulnerable call. We also, keep track of the cumulative proceeds as well as any interest earned or paid during each time step. At the maturity of the option we calculate the net profit or loss on the entire position. The algorithm is outlined below:

At time zero, the units of *S* and *V* to short, $U_{S,t=0}$ and $U_{V,t=0}$ are given by:

$$U_{S,t=0} = \frac{dc}{dS}$$
$$U_{V,t=o} = \frac{dc}{dV} \qquad (5.4.2.1)$$

Table 6

Summary of Behaviour of Hedge Parameters

Parameter	*Merton's Model*	*FDB Model*	*VDR Model*
$c(S)$	↑ as S ↑	↑ as S, but at a slower rate then Merton's Model. No upper boundary	↑ as S, but at a slower rate then FDB Model. Eventually plateaus
dc / dS	↑ as S ↑ $0 \leq dc / dS \leq 1$	↑ as S ↑ Equals zero for deep out-of-the-money calls. Upper boundary can be less then one.	↑ or ↓ as S ↑ Equals zero for deep in-the-money and deep out-of-the-money calls.
d^2c / dS^2	Always positive	Always positive	Can be positive or negative
$c(V)$	Not relevant	↑ as V ↑	↑ as V ↑
dc / dV	Not relevant	Always positive Approaches zero as V ↑	Always positive Approaches zero as V ↑, but at slower rate then FDB Model
d^2c / dV^2	Not relevant	Can be positive or negative. Equals zero for deep in-the-money and deep out-of-the-money calls.	Can be positive or negative Equals zero for deep in-the-money and deep out-of-the-money calls.

At time zero, the cumulative proceeds, $CP_{t=0}$, from shorting S and V is given by:

$$CP_{t=0} = U_{S,t=0}S_{t=0} + U_{V,t=0}V_{t=0} - c \qquad (5.4.2.2)$$

At the end of any subsequent time step units of S and V are bought or sold to bring the portfolio back to its instantaneously risk-free state. This depends on the updated hedge parameters. Therefore the cumulative proceeds at any time t, including any interest that is earned during the period is given by:

$$CP_t = CP_{t-1} * \exp(r_{t-1}\Delta t) + \left(U_{S,t} - U_{S,t-1}\right)S_t + \left(U_{V,t} - U_{V,t-1}\right)V_t \tag{5.4.2.3}$$

At the maturity of the call, the net profit on the entire portfolio is calculated as follows:

$$NP_T = Payoff_T + CP_{T-1} * \exp(r\Delta t) - U_{S,T-1}S_T - U_{V,T-1}V_T \tag{5.4.2.4}$$

The actual payoff on the call depends on the particular type of option being examined. For the standard Black-Scholes-Merton call the payoff is $\max(S_T - K, 0)$. The payoffs for the FDB and VDB models are given in tables 1a and 1b. We run the entire experiment 1,000 times and calculate the net profit, NP_T, for each run.

Presented below, in table 7, is the mean and standard deviation of the 1,000 runs for a number of different simulations:

Table 7

Mean and Standard Deviation of Net Profit

	Merton Model Hedging a Black-Scholes Payoff	***Merton Model Hedging a VDB Payoff***	***FDB Model Hedging a FDB Payoff***	***FDB Model Hedging a VDB Payoff***	***VDB Model Hedging a VDB Payoff***
Mean	-0.0111	-3.6448	-0.0066	-3.6058	0.0503
Standard Deviation	0.6835	6.7890	0.6954	6.7438	1.3296

Parameters used in the above simulations: $S = 55$, $K = 50$, $V = 60$, $D^* = 50$, $T = 1$, $\alpha = 0.25$, $\sigma_V = 0.1$, $\sigma_S = 0.3$, $\rho_{VS} = 0.0$, $r = 0.05$, $a = 0.5$, $b = 0.0$ $\sigma_r = 0.0$, $\rho_{Vr} = 0$, $\rho_{Sr} = 0.0$

Figures 29, 30 and 31 show the net profit for each of the 1,000 runs for each of the simulations presented in table 7. Theoretically, if we re-balanced the portfolio at every instant of time the net profit of hedging the vulnerable option should be zero, assuming we are using the correct model to calculate the hedge parameters. However, in these results we are only re-balancing the portfolio once per week, which results in a small amount of hedging error. As expected, hedging a Black-Scholes payoff with the Black-Scholes-Merton model results in an average net profit that is close to zero, however there is some variation in the net profit from experiment to experiment although the variation, as measured by the standard deviation of the net profits is quite low. This result is also true when the FDB model is used to hedge a FDB payoff. Using the VDB model to hedge the VDB payoff also results in a average profit close to zero, however, the standard deviation is slightly larger, which can be explained by the extra source of risk present in this simulation.

Note that both the Merton model and the FDB model do not hedge the VDB payoff very well. In both cases the average profit is negative, indicating that we are paying too much for the option up-front, given the level of default risk. This can also, be seen in the bottom half of figures 29 and 30, where the net profit is biased downward. Also, the standard deviation indicates considerable uncertainty about the average net profit.

The purpose of this section was to demonstrate how one could hedge a vulnerable call using the models presented. The general conclusion is that so long as the vulnerable options payoff is the same as assumed by the model used to calculate the hedge parameters, the standard hedging technique should work reasonably well. However, using an incorrect model will result in large hedging errors.

Figure 29

Net Profit on 1000 Individual Trials of a Hedging Experiment, using Merton's Model

The parameters of the vulnerable call models used to generate the above simulations are: $S = 55$, $K = 50$, $V = 60$, $D^{*} = 50$, $T = 1$, $\alpha = 0.25$, $\sigma_r = 0.1$, $\sigma_s = 0.3$, $\rho_{VS} = 0$, $r = 0.05$, $a = 0.5$, $b = 0.05$, $\sigma_r = 0$, $\rho_{Vr} = 0$, $\rho_{Sr} = 0.0$ The net profit for each trial is given by equation 5.4.2.4.

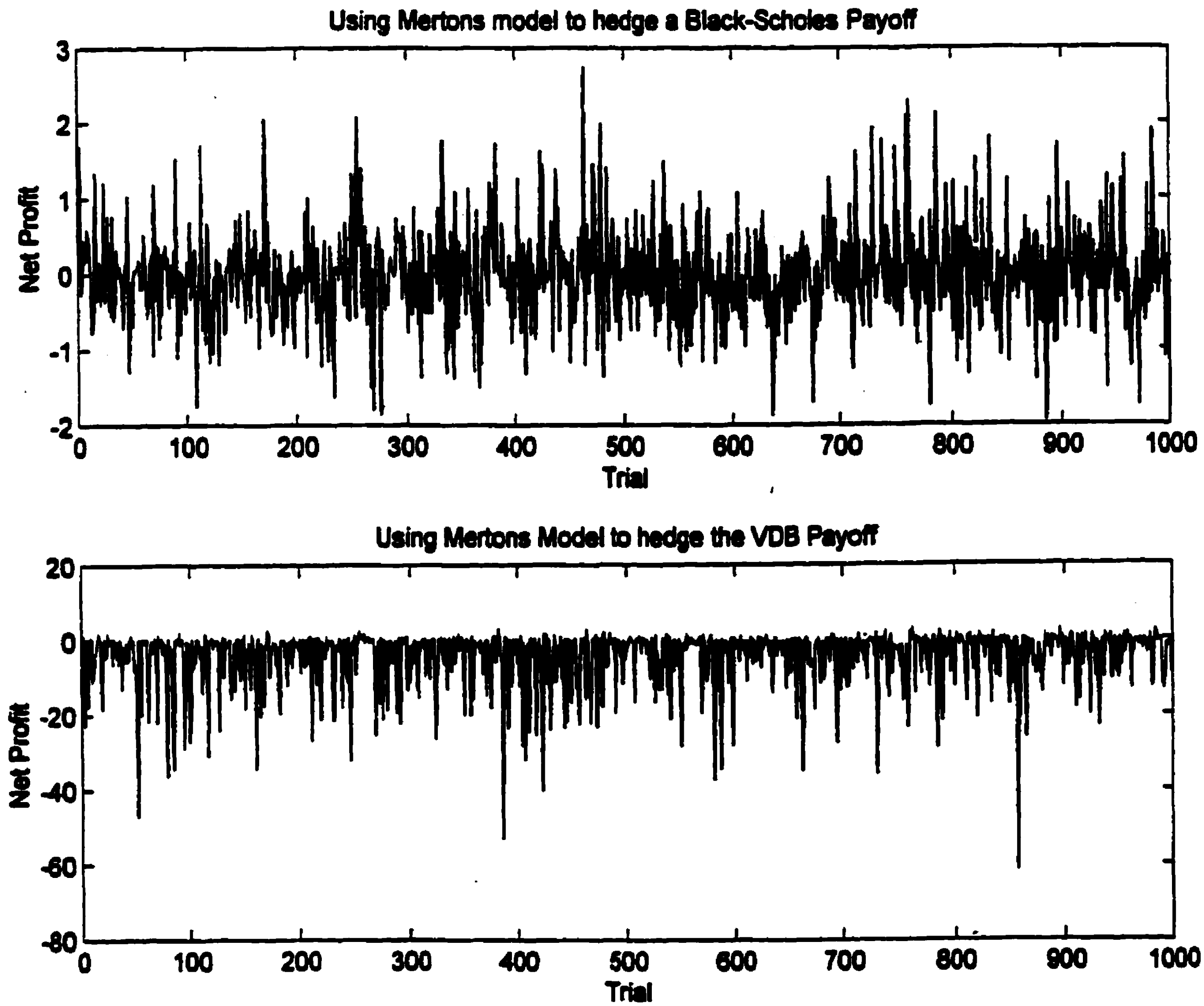
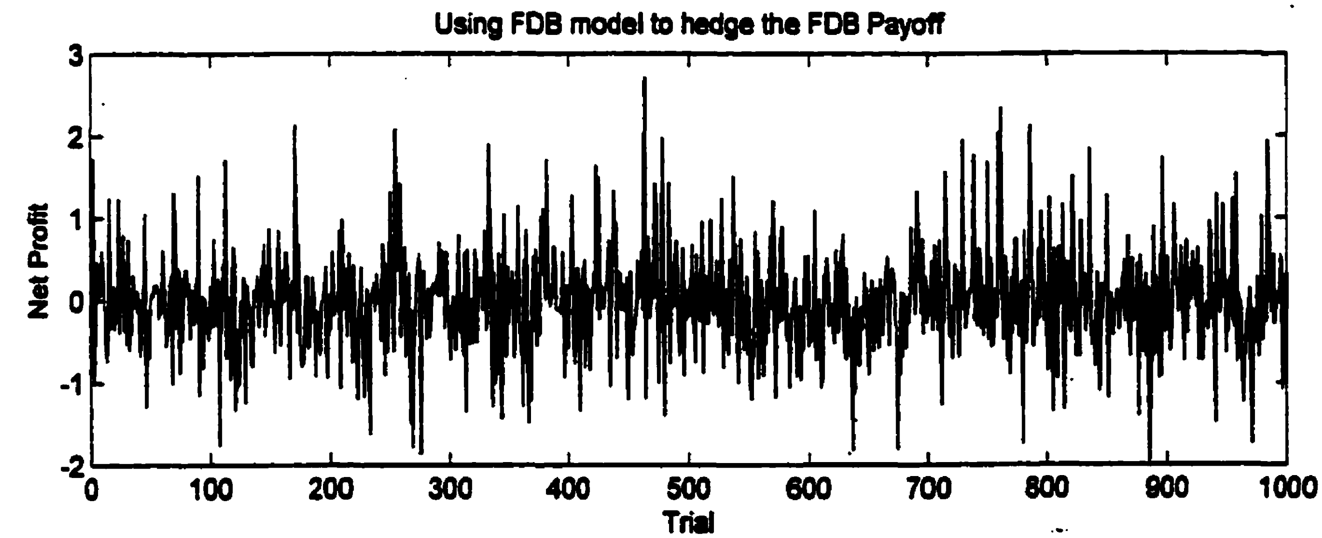


Figure 30

Net Profit on 1000 Individual Trials of a Hedging Experiment, using the FDB Model

The parameters of the vulnerable call models used to generate the above simulations are:
$S = 55$, $K = 50$, $V = 60$, $D^* = 50$, $T = 1$, $\alpha = 0.25$, $\sigma_V = 0.1$, $\sigma_s = 0.3$, $\rho_{VS} = 0$, $r = 0.05$, $a = 0.5$, $b = 0.05$, $\sigma_r = 0$, $\rho_{Vr} = 0$, $\rho_{Sr} = 0.0$

Analytical solutions of the FDB model are based on equation 4.1.1. The net profit for each trial is given by equation 5.4.2.4.



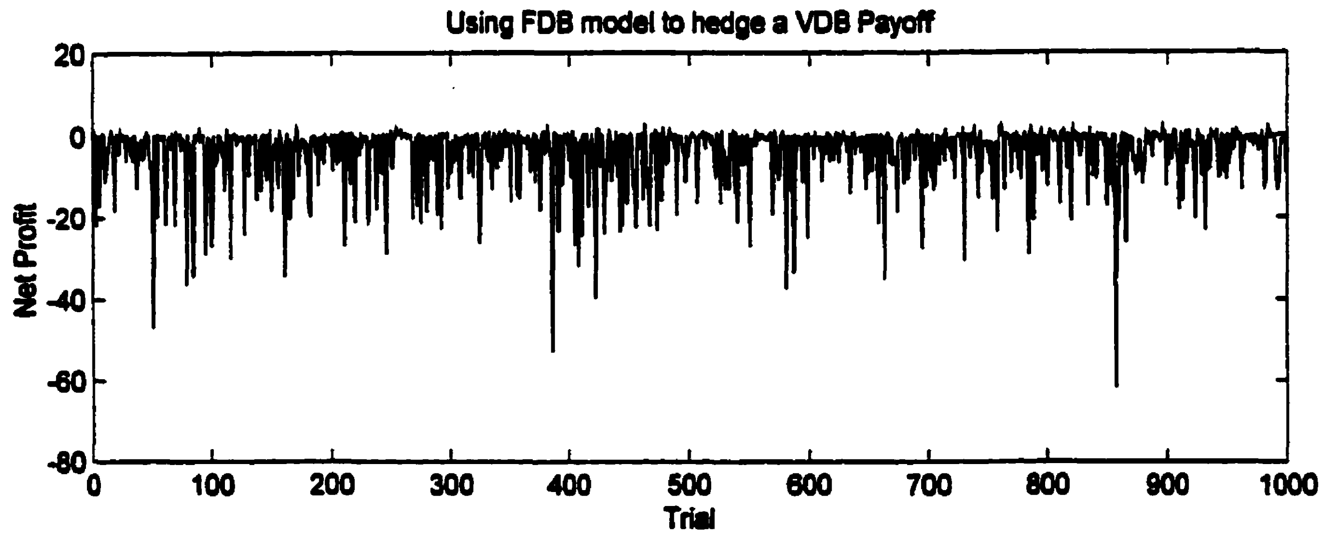
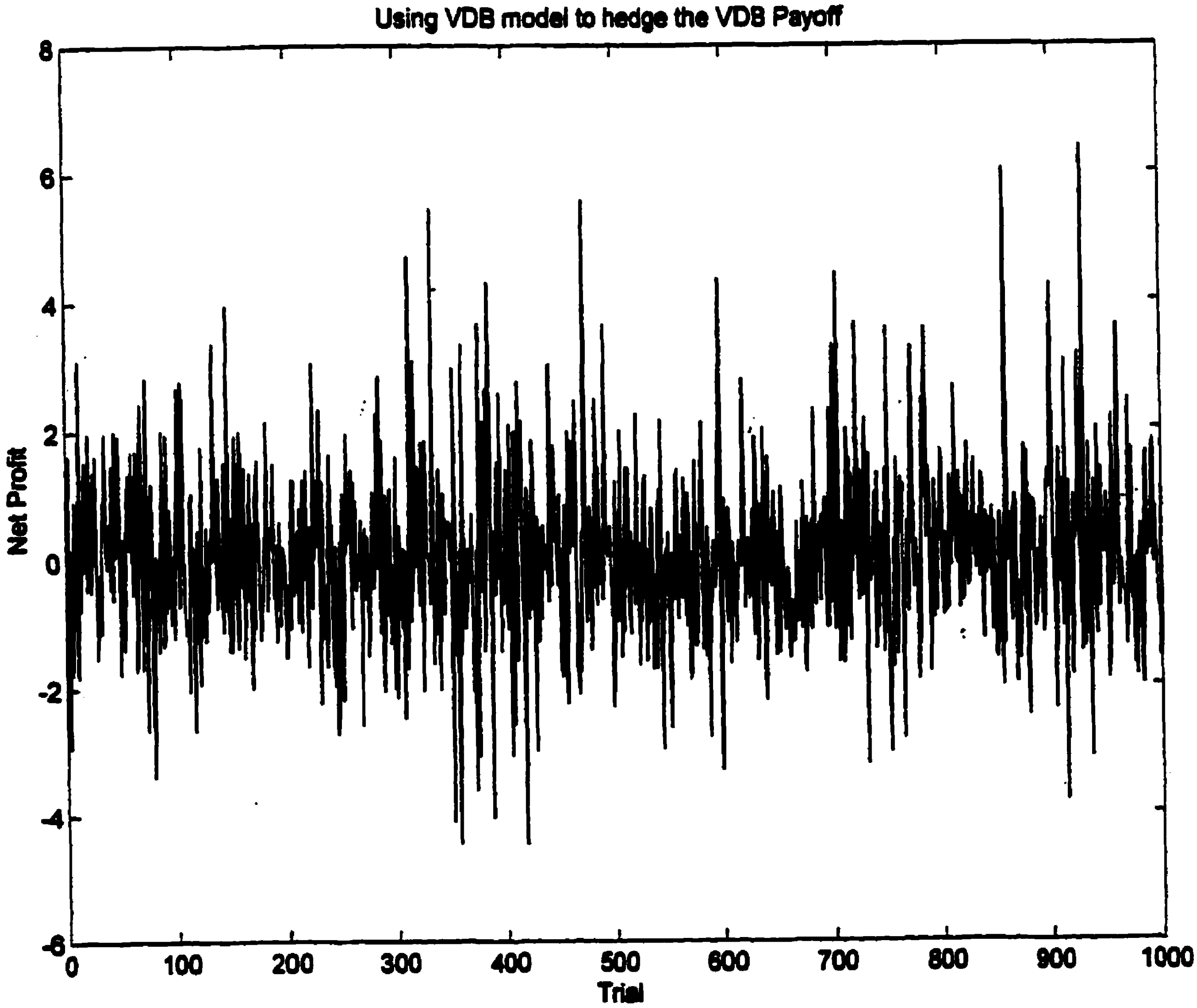


Figure 31

Net Profit on 1000 Individual Trials of a Hedging Experiment, using the VDB Model

The parameters of the vulnerable call models used to generate the above simulations are:
$S = 55$, $K = 50$, $V = 60$, $D^{*} = 50$, $T = 1$, $\alpha = 0.25$, $\sigma_{r} = 0.1$, $\sigma_{s} = 0.3$, $\rho_{VS} = 0$,
$r = 0.05$, $a = 0.5$, $b = 0.05$, $\sigma_{r} = 0$, $\rho_{Vr} = 0$, $\rho_{Sr} = 0.0$

Analytical solutions of VDB model are based on equation 4.2.5. The design parameters "p" and "q" are optimized for each value of V, based on the technique outlined in section 4.4. The net profit for each trial is given by equation 5.4.2.4.



Chapter 6

Summary and Future Research

6.1: Summary

This thesis develops a simple continuous time framework for valuing European options that incorporates market risk, default risk and interest rate risk. In particular we present two models for vulnerable European call options. Both of the models fall into the category labeled "structural models" by Jarrow, Lanso and Turnbull (1997), since default is defined to occur when the value of the option writer's assets fall below some boundary.

The first model, referred to as the fixed default boundary (FDB) model, allows default to occur only at the maturity of the option if the value of the writer's assets falls below a fixed, predetermined value, usually associated with the writer's debt obligations. Also intuitively, the payout in the event of default is directly related to the nominal claims of the creditors and the value of the firms remaining assets. This model is a direct extension of the model proposed by Klein (1996), to include stochastic interest rates.

The second model developed in this thesis is referred to as the variable default boundary (VDB) model. This model allows default to occur at the maturity of the option if the value of the writer's assets is below the sum of a fixed value related to the writers debt and a variable value equal to payoff on the call option. As a result, default risk arises not only from the possible deterioration of the writer's assets, but from the possible increase in the value of the call option. Like the FDB model, the VDB model relates the payout in the event of default directly to the nominal claim of the creditors and the value of the option writer's assets. An advantage of this approach over the FDB model is that it guarentees that there will be sufficient assets available to make the payouts assumed by the model. The FDB model does not guarantee this result in all cases. The VDB model can be seen as an extension of the FDB model to allow default to be driven not only by a reduction in the writer's assets but by a increase in the value of the option. It could also be seen as an extension of Johnson and Stulz (1987) to include debt in the writer's assets and stochastic interest rates.

In chapter 4 we develop analytical valuation formulas for both the FDB and VDB models. The FDB model is analytically tractable, but the VDB model is not. However, by employing two linear Taylor series expansions; one for the non-linear default boundary and one for the hyperbolic integrand in the valuation equation, we develop a simple approximate analytical formula. Using the two Taylor series expansions results in the introduction of two design parameters into the solution. We develop a simple algorithm, requiring the numerical solution of two single variable optimization problems, to estimate these two design parameters. This algorithm required the development of a valuation equation for the total liabilities of the option writer. Since this equation only depends on one of the design parameters we are able to separate one multivariate problem into two univariate problems. Since, no exact valuation formula exits for the VDB model we also employ a Monte Carlo simulation to generate numerical results for comparison purposes.

Chapter 5, presents a number of of numerical examples, which are used to illustrate the different properties of the two valuation models. In addition to comparing the FDB and VDB models, comparisons to the Black-Scholes-Merton, Johnson and Stulz (1987) and Rich (1996) models are also presented. Note, that since our model is a generalization of the Johnson and Stulz (1987) model, our approximate analytical solution provides approximate solutions for their model as well.

We perform a sensitivity analysis of both the FDB and VDB models with respect to all of the parameters in the models, for three significantly different examples. In all cases the approximate analytical solution to the VDB model matched the result generated by the Monte-Carlo simulation quite well. The sensitivity analysis shows that the VDB model can behave quite differently then models based on a fixed default boundary. In particular the values of vulnerable calls predicted by the VDB model are almost always significantly lower than their corresponding FDB model equivalents. This is intuitive since the VDB model will almost always predict a higher probability of default. Also, the volatility of the asset under-lying the option, the volatility of the option writer's assets and the correlation between them can have unpredictable impacts on the value of a vulnerable call. As a first estimate one would predict that increasing σ_S, σ_V and ρ_{VS} would increase, decrease and increase the value of a vulnerable call respectively. However, simulations show that the call could actually either increase or decrease depending on the particular set of parameters.

Klein and Inglis (1999) showed that stochastic interest rates and in particular the correlation coefficients ρ_{Vr} and ρ_{Sr} could have an important impact on the valuation of longer term vulnerable options, especially if they were deep-out-of-the money options. The VDB model supports this conclusion, although the impacts can be reversed in certain situations. Specifically, increasing ρ_{Vr} will decrease the value of the FDB call but increase the value of a VDB call. However, as a general conclusion it would be much more important to accurately estimate ρ_{VS} than either ρ_{Vr} or ρ_{Sr}.

We also looked at the probability of default and the amount of margin that would have to be posted to remove the default risk from the option. The VDB model predicts both higher risk neutral probabilities of default and higher margin requirements. Rich's (1996) model predicted values between those of the FDB and VDB models. In fact the VDB model predicted margin requirement that are only slightly lower then those required in the Canadian index options market. Both the FDB model and Rich's model predict even lower required margin.

Finally, we looked at the hedging parameters for both the FDB and VDB model. Again, the VDB model can predict significantly different hedging parameters. The most obvious example is that the VDB model predicts that the delta of a vulnerable call will first increase and then eventually decrease as the moneyness of the option increases, a prediction the FDB does not make. Hedging experiments demonstrate that using either the Black-Scholes-Merton model or the FDB model to hedge the payoff on a long position in a vulnerable call will result in significant loss of money.

In summary this thesis has developed a simple extension of the Black-Scholes-Merton option pricing model, that will take into account not only market risk, but also default risk and interest rate risk. The Black-Scholes model has been adopted as a benchmark by both academics and pratitioners alike because of its simple and easily understood assumptions, despite the fact that these assumptions are not a perfect description of reality. The proposed model for vulnerable calls also rests on some overly simplistic yet easily understood assumptions that preserve the intuitive appeal of the Black-Scholes-Merton model and yet provide some insights into how default risk affects vulnerable option valuation.

6.2: Future Research

There are a number of extensions of this work that could easily be undertaken. For example the the capital structure of the firm could be extended to include not only debt obligations and a written call, but a written put as well. If we assumed for simplicity that the strike prices of the call and put were the same then it would be relatively easy to extent the payoff table (see Table 8) and write down the valuation equations. However, the choice of the design parameters would be complicated by this addition. Of course requiring different strike prices would also considerably complicate the situation. This may be one of the greatest disadvantages of the structural approach, since the capital structure of any but the simplest firms would be considerably more complicated than those proposed in this thesis or in Table 8. Despite this downfall, the formulas presented in this paper may still be of practical use if they capture the major factors influencing the value of vulnerable options.

Since all of the securities of the option writer can be valued using the techniques presented in this thesis, we could value the debt and equity of the firm in addition to the liabilities and vulnerable call. Some work has been done in this area. Figure 32 shows

Table 8:

Payoff Table for Variable Default Boundary Model with both a Written Call and Written Put in the Writer's Capital Structure

Moneyness of Call Option	$S_T \le K$ (out-of-the-money)		$S_T > K$ (in-the-money)	
Default Status of Option Writer	$V_T \le D^* + K - S_T$ (default)	$V_T > D^* + K - S_T$ (no-default)	$V_T \le D^* + S_T - K$ (default)	$V_T > D^* + S_T - K$ (no-default)
Debt	$(1-\alpha)\frac{V_T}{D^* + K - S_T} D^*$	D^*	$(1-\alpha)\frac{V_T}{D^* + S_T - K} D^*$	D^*
Call	0	0	$(1-\alpha)\frac{V_T}{D^* + S_T - K}(S_T - K$	$S_T - K$
Put	$(1-\alpha)\frac{V_T}{D^* + K - S_T}(K - S_T$	$K - S_T$	0	0
Liabilities	$(1-\alpha)V_T$	$D^* + K - S_T$	$(1-\alpha)V_T$	$D^* + S_T - K$
Equity	0	$V_T - (D^* + K - S_T$	0	$V_T - (D^* + S_T - K$
Total Assets (net of direct default costs)	$(1-\alpha)V_T$	V_T	$(1-\alpha)V_T$	V_T

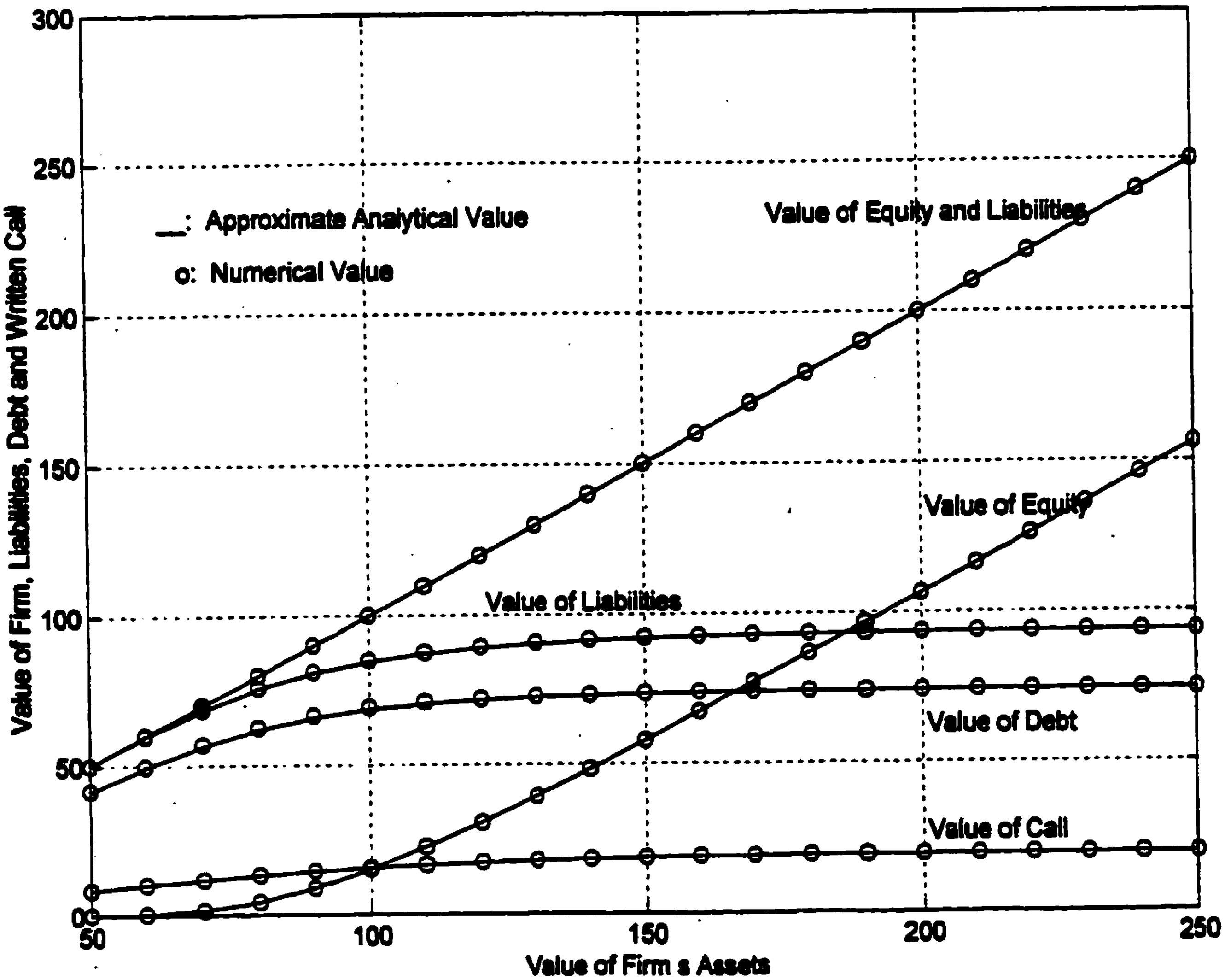
the value of all of the option writer's securities using the parameter values corresponding to table 2. Both the numerical and approximate analytical values are shown in the Figure. Future research could look at the credit spread predictions from this valuation framework and compare them to empirically observed results.

Another extension, suggested by an anonymous reviewer of an earlier working paper, would involve making the debt obligations of the option writer, D^*, stochastic. We could imagine the debt of the option writer being of relatively long term compared to the written call, whose value would be driven by the same Wiener process that drives the

Figure 32

Value of the Option Writer's Securities as a Function of Writer's Assets: Variable Default Boundary Model

Calculations of security prices are based on the following parameter values: $S = 50$, $K = 40$, $V = 100$, $D^* = 90$, $T = 3$, $\alpha = 0$, $\sigma_r = 0.1$, $\sigma_s = 0.3$, $\rho_{VS} = 0$, $r = 0.05$, $a = 0.5$, $b = 0.08$, $\sigma_r = 0.03$, $\rho_{Vr} = 0$, $\rho_{Sr} = 0.0$, unless otherwise noted.



risk-free bond. Default might then occur for three reasons; the assets of the firm diminish; the asset underlying the option increases or the the value on the debt increases. Adding this second source of risk to the default boundary may have a beneficial value to the overall structure of the model. This is left to future research.

As pointed out by Collin-Dufresne and Goldstein (2001), the probability of default in fixed default boundary models decreases through time which results in credit spreads that are too small relative to empirical observation. This occures because the value of the firm's assets continues to trend upward while the default boundary remains fixed. Collin-Dresne and Goldstein (2001) suggest keeping the leverage ratio constant by allowing the firm to issue additional debt through time. They show that this model is more consistent with empirically observed credit spreads. This concept could potentially be incorporated into either the Klein and Inglis (1999) FDB model or the VDB model presented in this thesis. This additional feature would be most important for long-dated options.

Empirical testing will be required to support or reject the approach presented here. To date there has been no published empirical tests of any vulnerable option model, beyond their simplistic use to estimate the margin required to remove default risk in exchange traded index options. The development of a vulnerable option data base and testing of the various vulnerable option pricing model would make a strong contribution to the literature in this area.

Appendix A

Derivation of the Partial Differential Equation Satisfied by Vulnerable Options

In this appendix we derive the partial differential equation for pricing European options subject to both default risk and interest rate risk. Let F be the value of the vulnerable option, in units of the discount bond, such that $F = F(V,S,B,T)$. Applying Ito's lemma and using the stochastic processes for V, S and B gives the expression for the change in the value of the option:

$$\frac{dF}{F} = \beta dt + \gamma dZ_V + \delta dZ_S + \eta dZ_B$$

where:

$$\beta = \frac{\left\{ \begin{array}{c} F_t + F_V \mu_V V + F_S \mu_S S + F_B \mu_B B + \frac{1}{2} F_{VV} \sigma_V^2 V^2 + \frac{1}{2} F_{SS} \sigma_S^2 S^2 + \frac{1}{2} F_{BB} \sigma_B^2 B^2 + F_{VS} \rho_{VS} \sigma_V \sigma_S VS + \\ F_{VB} \rho_{VB} \sigma_V \sigma_B VB + F_{SB} \rho_{SB} \sigma_S \sigma_B SB \end{array} \right\}}{F}$$

$$\gamma = \frac{F_V \sigma_V V}{F}$$

$$\delta = \frac{F_S \sigma_S S}{F}$$

$$\eta = \frac{F_B \sigma_B B}{F}$$

Now consider forming a four security portfolio containing (1) the assets of the counterparty, (2) the security underlying the option, (3) the risk free bond and (4) the vulnerable option. Let W_1 be the instantaneous number of dollars invested in the counterparty, W_2 the number of dollars invested in the security underlying the option, W_3 the number of dollars invested in risk free debt and W_4 the number of dollars invested in the vulnerable option. We want to form a self financing portfolio with no risk to any of the four securities. For the portfolio to be self financing we require:

$$W_1 = -(W_2 + W_3 + W_4)$$

Now, if dQ is the instantaneous dollar return to the portfolio, then:

$$dQ = W_1 \frac{dV}{V} + W_2 \frac{dS}{S} + W_3 \frac{dB}{B} + W_4 \frac{dF}{F}$$

Substituting the stochastic processes for V, S, B and F into the expression for dQ gives:

$$dQ = \left[W_2(\mu_S - \mu_V) + W_3(\mu_B - \mu_V) + W_4(\beta - \mu_V)\right]dt + \left[-W_2\sigma_V - W_3\sigma_V + W_4(\gamma - \sigma_V)\right]dZ_V + \\ \left[W_2\sigma_S + W_4\delta\right]dZ_S + \left[W_3\sigma_B + W_4\eta\right]dZ_B$$

Now, if we choose W_2, W_3 and W_4 such that that coefficients of dZ_V, dZ_S and dZ_B are all zero, then we will have a risk free portfolio. Since the portfolio is now self financing and riskless, to avoid arbitrage, the portfolio's instantaneous return must be zero. This leads to the following 4 by 3 system of linear equations:

$$(\mu_S - \mu_V)W_2 + (\mu_B - \mu_V)W_3 + (\beta - \mu_V)W_4 = 0$$

$$-\sigma_V W_2 - \sigma_V W_3 + (\gamma - \sigma_V)W_4 = 0$$

$$\sigma_S W_2 + \delta W_4 = 0$$

$$\sigma_B W_3 + \eta W_4 = 0$$

A nontrivial solution (i.e. not all $W_i = 0$, $i = 2 \ldots 4$) exists if and only if:

$$\frac{\left[\beta - \mu_V + \dfrac{\mu_V}{\sigma_B}\eta - \dfrac{\mu_B}{\sigma_B}\eta\right]}{\mu_S - \mu_V} = \frac{\left[\gamma - \sigma_V + \dfrac{\sigma_V}{\sigma_B}\eta\right]}{\sigma_V} = \frac{\delta}{\sigma_S}$$

Substituting into the second equality leads to:

$$F = VF_V + BF_B + SF_S$$

Substituting into the first equality leads to the desired fundamental partial differential equation for the pricing of vulnerable European options subject to interest rate risk:

$$F_t + \frac{1}{2}\sigma_V^2 V^2 F_{VV} + \frac{1}{2}\sigma_S^2 S^2 F_{SS} + \frac{1}{2}\sigma_B^2 B^2 F_{BB} + \rho_{VS}\sigma_V\sigma_B VSF_{VS} + \\ \rho_{VB}\sigma_V\sigma_B VBF_{VB} + \rho_{SB}\sigma_S\sigma_B SBF_{SB} = 0$$

Appendix B

Derivation of Bivariate Normal Distribution of X and Y

Define the T-forward risk adjusted processes $\widetilde{V}(t)$ and $\widetilde{S}(t)$, using the risk free bond price *B(T-t)* as the numeriaire, as follows:

$$\widetilde{V}(t) = \frac{V(t)}{B(T-t)} \tag{B1}$$

and

$$\widetilde{S}(t) = \frac{S(t)}{B(T-t)} \tag{B2}$$

Using Ito's Lemma we can find the risk neutral processes for $\widetilde{V}(t)$ and $\widetilde{S}(t)$. Starting with $\widetilde{V}(t)$:

$$d\widetilde{V}(t) = \left(\frac{\partial \widetilde{V}}{\partial V}\mu_V V + \frac{\partial \widetilde{V}}{\partial B}\mu_B B + \frac{\partial \widetilde{V}}{\partial t} + \frac{1}{2}\frac{\partial^2 \widetilde{V}}{\partial V^2}\sigma_V^2 V^2 + \frac{1}{2}\frac{\partial^2 \widetilde{V}}{\partial B^2}\sigma_B^2 B^2 + \frac{\partial^2 \widetilde{V}}{\partial V \partial B}\sigma_V V \sigma_B B \rho_{VB} \right) dt$$
$$+ \frac{\partial \widetilde{V}}{\partial V}\sigma_V V dZ_V + \frac{\partial \widetilde{V}}{\partial B}\sigma_B B dZ_B \tag{B3}$$

Substituting in the appropriate derivatives:

$$d\widetilde{V}(t) = \left(\frac{1}{B}\mu_V V - \frac{V}{B^2}\mu_B B + \frac{V}{B^3}\sigma_B^2 B^2 - \frac{1}{B^2} V B \sigma_V \sigma_B \rho_{VB} \right) dt + \frac{1}{B}\sigma_V V dZ_V - \frac{V}{B^2}\sigma_B B dZ_B \tag{B4}$$

Simplifying gives:

$$\frac{d\widetilde{V}(t)}{\widetilde{V}(t)} = \hat{\mu}_V(T-t)dt + \sigma_V dZ_V + \sigma_B(T-t)dZ_V \tag{B5}$$

where:

$$\hat{\mu}_V(T-t) = \mu_V - \mu_B + \sigma_B^2(T-t) - \rho_{VB}\sigma_V\sigma_B(T-t)$$

Expression B5 can be further simplified by noting the following:

1. The expected value of expression B5 is zero in a risk neutral world, since we are using the risk-free bond as the numeriar.

$$E\left[\frac{d\tilde{V}(t)}{\tilde{V}(t)}\right] = 0 \qquad \text{i.e.} \qquad \hat{\mu}_{V}(T-t) = 0$$

2. The two sources of risk in expression B5 can be combined into a single source by noting that the variance of the expression in the next small increment of time is given by:

$$Var\left[\frac{d\tilde{V}(t)}{\tilde{V}(t)}\right] = E\left[\left(\frac{d\tilde{V}(t)}{\tilde{V}(t)}\right)^2\right] - E\left[\frac{d\tilde{V}(t)}{\tilde{V}(t)}\right]^2$$

$$= \mathrm{E}\left[\left(\sigma_V dZ_V - \sigma_B(T-t)dZ_B\right)^2\right]$$

$$= \left(\sigma_V^2 + \sigma_B^2(T-t) - 2\sigma_V\sigma_B\rho_{VB}\right)dt$$

$$= \hat{\sigma}_V^2(T-t)dt$$

Substituting into expression B5 gives:

$$d\tilde{V}(t) = \hat{\sigma}_V(T-t)dZ_{\tilde{V}} \qquad \text{(B6)}$$

Following the same procedure for $\tilde{S}(t)$ results in

$$d\tilde{S}(t) = \hat{\sigma}_S(T-t)\tilde{S}(t)dZ_{\tilde{S}} \qquad \text{(B7)}$$

where:

$$\hat{\sigma}_S^2(T-t) = \sigma_S^2 + \sigma_B^2(T-t) - 2\sigma_S\sigma_B(T-t)\rho_{SB}$$

The covariance between $\tilde{V}(t)$ and $\tilde{S}(t)$ can be found as follows:

$$Cov\left[\frac{d\tilde{V}(t)}{\tilde{V}(t)},\frac{d\tilde{S}(t)}{\tilde{S}(t)}\right]=E\left[\left(\frac{d\tilde{V}(t)}{\tilde{V}(t)}\right)\left(\frac{d\tilde{S}(t)}{\tilde{S}(t)}\right)\right]-E\left[\frac{d\tilde{V}(t)}{\tilde{V}(t)}\right]E\left[\frac{d\tilde{S}(t)}{\tilde{S}(t)}\right]$$

$$=\mathrm{E}\left[(\sigma_V dZ_V-\sigma_B(T-t)dZ_B)(\sigma_V dZ_V-\sigma_B(T-t)dZ_B)\right]$$

$$=\left(\sigma_V\sigma_S\rho_{VS}-\sigma_V\sigma_B(T-t)\rho_{VB}-\sigma_S\sigma_B(T-t)\rho_{SB}+\sigma_B^2(T-t)\right)dt$$

We can then define the correlation between $\tilde{V}(t)$ and $\tilde{S}(t)$ as:

$$\hat{\rho}_{VS}(T-t)=\frac{\sigma_V\sigma_S\rho_{VS}-\sigma_V\sigma_B(T-t)\rho_{VB}-\sigma_S\sigma_B(T-t)\rho_{SB}+\sigma_B^2(T-t)}{\hat{\sigma}_V(T-t)\hat{\sigma}_S(T-t)}$$

(B8)

We now need to find the risk neutral processes for $X=\ln(\tilde{V}(t))$ and $Y=\ln(\tilde{S}(t))$. Starting with *X(t)* and using Ito's Lemma again we get:

$$dX(t)=\left(\frac{\partial\tilde{V}}{\partial t}+\frac{1}{2}\frac{\partial^2 X}{\partial\tilde{V}^2}\hat{\sigma}_V^2\tilde{V}^2\right)dt+\frac{\partial X}{\partial\tilde{V}}\hat{\sigma}_V\tilde{V}dZ_{\tilde{V}}$$

(B9)

Substituting in the appropriate partial derivatives:

$$dX(t)=\left(-\frac{1}{2\tilde{V}^2}\hat{\sigma}_V^2\tilde{V}^2\right)dt+\frac{1}{\tilde{V}}\hat{\sigma}_V\tilde{V}dZ_{\tilde{V}} \qquad \text{(B10)}$$

Simplifying gives:

$$dX(t) = -\frac{\hat{\sigma}_{V}^{2}}{2}dt + \hat{\sigma}_{V}^{2}dZ_{\tilde{V}} \tag{B11}$$

Applying the same procedure to *Y(t)* results in:

$$dY(t) = -\frac{\hat{\sigma}_{S}^{2}}{2}dt + \hat{\sigma}_{S}^{2}dZ_{\tilde{S}} \tag{B12}$$

Therefore we known that X_T and Y_T are bivariate normally distributed. Since the drift and variance terms for $dX(t)$ and $dY(t)$ depend only on time we can integrate to generate the means, variances and covariance of X_T and Y_T. The means are given by:

$$E[X_T] = X_t - \int_t^T \frac{\hat{\sigma}_{V}^{2}(T-\tau)}{2}d\tau \qquad E[Y_T] = Y_t - \int_t^T \frac{\hat{\sigma}_{S}^{2}(T-\tau)}{2}d\tau$$

where the variances, s_V^2 and s_S^2, are given by:

$$s_V^2 = \int_t^T \hat{\sigma}_V^2(T-\tau)d\tau$$

$$s_S^2 = \int_t^T \hat{\sigma}_S^2(T-\tau)d\tau$$

The covariance, s_{VS} and correlation $\bar{\rho}_{VS}$ are given by:

$$s_{VS}(T-t) = \int_t^T \hat{\rho}_{VS}(T-\tau)\hat{\sigma}_V(T-\tau)\hat{\sigma}_S(T-\tau)d\tau$$

$$\bar{\rho}_{VS} = \frac{s_{VS}}{s_V s_S}$$

Therefore, X_T and Y_T are bivariate normally distributed as:

$$X_T, Y_T \sim N_2\left[X_t - \frac{s_V^2}{2}, Y_t - \frac{s_S^2}{2}, s_V, s_S, \bar{\rho}_{VS}\right] \tag{B13}$$

Appendix C

Proof of Proposition #1: Value of Vulnerable Call with Fixed Default Boundary

Using risk neutral pricing the value of the vulnerable call is given by:

$$c = \int_{\ln(K)}^{\infty} \int_{\ln(D^*)}^{\infty} \frac{(e^{Y_T} - K)}{2\pi\sqrt{1-\bar{\rho}_{VS}{}^2}\, s_V s_S} \exp\left\{ -\frac{1}{2(1-\bar{\rho}_{VS}{}^2)} \left[\left(\frac{Y_T - Y + \frac{s_S^2}{2}}{s_S} \right)^2 - 2\bar{\rho}_{VS} \left(\frac{Y_T - Y + \frac{s_S^2}{2}}{s_S} \right) \left(\frac{X_T - X + \frac{s_V^2}{2}}{s_V} \right) + \left(\frac{X_T - X + \frac{s_V^2}{2}}{s_V} \right)^2 \right] \right\} dX_T dY_T$$

$$+ \int_{\ln(K)}^{\infty} \int_{-\infty}^{\ln(D^*)} \left[\frac{(1-\alpha)e^{X_T}}{D^*} \right] \frac{(e^{Y_T} - K)}{2\pi\sqrt{1-\bar{\rho}_{VS}{}^2}\, s_V s_S} \exp\left\{ -\frac{1}{2(1-\bar{\rho}_{VS}{}^2)} \left[\left(\frac{Y_T - Y + \frac{s_S^2}{2}}{s_S} \right)^2 - 2\bar{\rho}_{VS} \left(\frac{Y_T - Y + \frac{s_S^2}{2}}{s_S} \right) \left(\frac{X_T - X + \frac{s_V^2}{2}}{s_V} \right) + \left(\frac{X_T - X + \frac{s_V^2}{2}}{s_V} \right)^2 \right] \right\} dX_T dY_T$$

Standardizing the normal distribution and substituting for Y_T *and* X_T, which are defined in Appendix B as,

$$Y_T = \log\left(\frac{S}{B}\right) \qquad X_T = \log\left(\frac{V}{B}\right)$$

results in:

$$c = \left[\int_{-a}^{\infty} \int_{-bb}^{\infty} \frac{S}{B} \exp\left(-\frac{s_S^2}{2} + s_S \widetilde{u} \right) \frac{1}{2\pi\sqrt{1-\overline{\rho}_{VS}^{\,2}}} \exp\left\{ -\frac{1}{2(1-\overline{\rho}_{VS}^{\,2})} \left[\widetilde{u}^2 - 2\overline{\rho}_{VS}\widetilde{u}\widetilde{v} + \widetilde{v}^{21} \right] \right\} d\widetilde{v}\, d\widetilde{u} \right.$$

$$- \int_{-a}^{\infty} \int_{-bb}^{\infty} \frac{K}{2\pi\sqrt{1-\overline{\rho}_{VS}^{\,2}}} \exp\left\{ -\frac{1}{2(1-\overline{\rho}_{VS}^{\,2})} \left[\widetilde{u}^2 - 2\overline{\rho}_{VS}\widetilde{u}\widetilde{v} + \widetilde{v}^{21} \right] \right\} d\widetilde{v}\, d\widetilde{u}$$

$$+ \int_{-a}^{\infty} \int_{-\infty}^{-bb} \frac{(1-\alpha)\frac{V}{B}\frac{S}{B} \exp\left(-\frac{s_S^2}{2} + s_S\widetilde{u} - \frac{s_V^2}{2} + s_V\widetilde{v} \right)}{D^{*}.} \frac{1}{2\pi\sqrt{1-\overline{\rho}_{VS}^{\,2}}}$$

$$\exp\left\{ -\frac{1}{2(1-\overline{\rho}_{VS}^{\,2})} \left[\widetilde{u}^2 - 2\overline{\rho}_{VS}\widetilde{u}\widetilde{v} + \widetilde{v}^{21} \right] \right\} d\widetilde{v}\, d\widetilde{u}$$

$$- \int_{-a}^{\infty} \int_{-\infty}^{-bb} \frac{(1-\alpha)K\frac{V}{B} \exp\left(-\frac{s_V^2}{2} + s_V\cdot\widetilde{v} \right)}{D^{*}.} \frac{1}{2\pi\sqrt{1-\overline{\rho}_{VS}^{\,2}}}$$

$$\left. \exp\left\{ -\frac{1}{2(1-\overline{\rho}_{VS}^{\,2})} \left[\widetilde{u}^2 - 2\overline{\rho}_{VS}\widetilde{u}\widetilde{v} + \widetilde{v}^{21} \right] \right\} d\widetilde{v}\, d\widetilde{u} \right]$$

where

$$a = \frac{\ln\left(\frac{S}{BK} \right) - \frac{s_S^2}{2}}{s_S} \quad \text{and} \quad bb = \frac{\ln\left(\frac{V}{BD^{*}} \right) - \frac{s_V^2}{2}}{s_V}$$

where : $\widetilde{u}$ and $\widetilde{v}$ are standard normal variates with zero mean and standard deviation equal to one.

Now, completing the square:

$$c=\left[\frac{S}{B}\int_{-a}^{\infty}\int_{-bb}^{\infty}\frac{1}{2\pi\sqrt{1-\bar{\rho}_{VS}^{\,2}}}\right.$$
$$\exp\left\{-\frac{1}{2(1-\bar{\rho}_{VS}^{\,2})}\left[(\tilde{u}-s_S)^2-2\bar{\rho}_{VS}(\tilde{u}-s_S)(\tilde{v}-\bar{\rho}_{VS}s_S)+(\tilde{v}-\bar{\rho}_{VS}s_S)^{21}\right]\right\}d\tilde{v}d\tilde{u}$$
$$-\int_{-a}^{\infty}\int_{-bb}^{\infty}\frac{K}{2\pi\sqrt{1-\bar{\rho}_{VS}^{\,2}}}\exp\left\{-\frac{\exp(\;)}{2(1-\bar{\rho}_{VS}^{\,2})}\left[\tilde{u}^2-2\bar{\rho}_{VS}\tilde{u}\tilde{v}+\tilde{v}^{21}\right]\right\}d\tilde{v}d\tilde{u}$$

$$+\frac{(1-\alpha)}{D^{*}}\frac{V}{B}\frac{S}{B}\exp(2\bar{\rho}_{VS}s_Vs_S)\int_{-a}^{\infty}\int_{-\infty}^{-bb}\frac{1}{2\pi\sqrt{1-\bar{\rho}_{VS}^{\,2}}}$$
$$\exp\left\{-\frac{1}{2(1-\bar{\rho}_{VS}^{\,2})}\begin{bmatrix}(\tilde{u}-s_S-\bar{\rho}_{VS}s_V)^2\\-2\bar{\rho}_{VS}(\tilde{u}-s_S-\bar{\rho}_{VS}s_V)(\tilde{v}-\bar{\rho}_{VS}s_S-s_V)\\+(\tilde{v}-\bar{\rho}_{VS}s_S-s_V)^{21}\end{bmatrix}\right\}d\tilde{v}d\tilde{u}$$

$$-\frac{(1-\alpha)}{D^{*}}K\frac{V}{B}\int_{-a}^{\infty}\int_{-\infty}^{-bb}\frac{1}{2\pi\sqrt{1-\bar{\rho}_{VS}^{\,2}}}$$
$$\left.\exp\left\{-\frac{1}{2(1-\bar{\rho}_{VS}^{\,2})}\begin{bmatrix}(\tilde{u}-\bar{\rho}_{VS}s_V)^2\\-2\bar{\rho}_{VS}(\tilde{u}-\bar{\rho}_{VS}s_V)(\tilde{v}-s_V)\\+(\tilde{v}-s_V)^{21}\end{bmatrix}\right\}d\tilde{v}d\tilde{u}\right]$$

A simple transformation then gives the desired result:

$$c = \left[\frac{S}{B} \int_{-a-s_S}^{\infty} \int_{-bb-\bar{\rho}_{VS}s_S}^{\infty} \frac{1}{2\pi\sqrt{1-\bar{\rho}^2}} \exp\left\{ -\frac{1}{2(1-\bar{\rho}_{VS}^{\ 2})} \left[\tilde{x}^2 - 2\bar{\rho}_{VS}\tilde{x}\tilde{y} + \tilde{y}^{21} \right] \right\} d\tilde{y}d\tilde{x} \right.$$

$$- \int_{-a}^{\infty} \int_{-bb}^{\infty} \frac{K}{2\pi\sqrt{1-\bar{\rho}_{VS}^{\ 2}}} \exp\left\{ -\frac{\exp(\ )}{2(1-\bar{\rho}_{VS}^{\ 2})} \left[\tilde{x}^2 - 2\bar{\rho}_{VS}\tilde{u}\tilde{y} + \tilde{y}^{21} \right] \right\} d\tilde{y}d\tilde{x}$$

$$+ \frac{(1-\alpha)}{D^*} \frac{V}{B} \frac{S}{B} \exp(2\bar{\rho}_{VS}s_V s_S) \int_{-a}^{\infty} \int_{-\infty}^{-bb} \frac{1}{2\pi\sqrt{1-\bar{\rho}_{VS}^{\ 2}}}$$

$$\exp\left\{ -\frac{1}{2(1-\bar{\rho}_{VS}^{\ 2})} \left[\tilde{x}^2 - 2\bar{\rho}_{VS}\tilde{x}\tilde{y} + \tilde{y}^{21} \right] \right\} d\tilde{y}d\tilde{x}$$

$$\left. - \frac{(1-\alpha)}{D^*} K \frac{V}{B} \int_{-a}^{\infty} \int_{-\infty}^{-bb} \frac{1}{2\pi\sqrt{1-\bar{\rho}_{VS}^{\ 2}}} \exp\left\{ -\frac{1}{2(1-\bar{\rho}_{VS}^{\ 2})} \left[\tilde{x}^2 - 2\bar{\rho}_{VS}\tilde{x}\tilde{y} + \tilde{y}^{21} \right] \right\} d\tilde{y}d\tilde{x} \right]$$

This can be rewritten as:

$$\begin{aligned} c = {} & \frac{S}{B} N_2(a + s_S, bb + \bar{\rho}_{VS}s_S, \bar{\rho}) - KN_2(a, bb, \bar{\rho}_{VS}) \\ & + \frac{(1-\alpha)}{D^*} \frac{V}{B} \frac{S}{B} N_2(a - s_S - \bar{\rho}_{VS}s_V, -bb + \bar{\rho}_{VS}s_S + s_V, -\bar{\rho}_{VS}) \\ & - \frac{(1-\alpha)}{D^*} K \frac{V}{B} \mathrm{N}_2(\mathrm{a} - \bar{\rho}_{VS}\mathrm{s}_V, -bb + s_V, -\bar{\rho}_{VS}) \end{aligned}$$

where N_2 represents the bivariate cumulative normal distribution.

The above expression represents the value of the call in units of the discount bond. To determine the nominal value of the call we multiply through by B.

$$\begin{aligned} c = {} & SN_2(a + s_S, bb + \bar{\rho}_{VS}s_S, \bar{\rho}) - KBN_2(a, bb, \bar{\rho}_{VS}) \\ & + \frac{(1-\alpha)}{D^*} V \frac{S}{B} N_2(a - s_S - \bar{\rho}_{VS}s_V, -bb + \bar{\rho}_{VS}s_S + s_V, -\bar{\rho}_{VS}) \\ & - \frac{(1-\alpha)}{D^*} KV\, \mathrm{N}_2(\mathrm{a} - \bar{\rho}_{VS}\mathrm{s}_V, -bb + s_V, -\bar{\rho}_{VS}) \end{aligned}$$

Appendix D

Proof of Proposition #2: Value of Vulnerable Call with Variable Default Boundary

Using risk neutral pricing the value of the vulnerable call is given by:

$$
\begin{aligned}
c = \int_{\ln(K)}^{\infty} \int_{\ln(D^{*}+e^{Y_T}-K)}^{\infty} \frac{(e^{Y_T}-K)}{2\pi\sqrt{1-\bar{\rho}_{VS}^{\,2}}\, s_V s_S} \exp\Bigg\{ -\frac{1}{2(1-\bar{\rho}_{VS}^{\,2})} \Bigg(\left(\frac{Y_T - Y + \frac{s_S^2}{2}}{s_S} \right)^2 \\
- 2\bar{\rho}_{VS} \left(\frac{Y_T - Y + \frac{s_S^2}{2}}{s_S} \right) \left(\frac{X_T - X + \frac{s_V^2}{2}}{s_V} \right) + \left(\frac{X_T - X + \frac{s_V^2}{2}}{s_V} \right)^2 \Bigg) \Bigg\} dX_T \, dY_T \\
+ \int_{\ln(K)}^{\infty} \int_{-\infty}^{\ln(D^{*}+e^{Y_T}-K)} \left[\frac{(1-\alpha)e^{X_T}}{D^{*}+e^{Y_T}-K} \right] \frac{(e^{Y_T}-K)}{2\pi\sqrt{1-\bar{\rho}_{VS}^{\,2}}\, s_V s_S} \exp\Bigg\{ -\frac{1}{2(1-\bar{\rho}_{VS}^{\,2})} \Bigg(\left(\frac{Y_T - Y + \frac{s_S^2}{2}}{s_S} \right)^2 \\
- 2\bar{\rho}_{VS} \left(\frac{Y_T - Y + \frac{s_S^2}{2}}{s_S} \right) \left(\frac{X_T - X + \frac{s_V^2}{2}}{s_V} \right) + \left(\frac{X_T - X + \frac{s_V^2}{2}}{s_V} \right)^2 \Bigg) \Bigg\} dX_T \, dY_T
\end{aligned}
$$

Standardizing the normal distribution and substituting for Y_T *and* X_T, which are defined in Appendix B as:

$$
Y_T = \log\left(\frac{S}{B}\right) \qquad X_T = \log\left(\frac{V}{B}\right)
$$

results in:

$$c=\left[\int_{-a}^{\infty}\int_{f(\tilde{u})}^{\infty}\frac{S}{B}\exp\left(-\frac{s_S^2}{2}+s_S\tilde{u}\right)\frac{1}{2\pi\sqrt{1-\bar{\rho}_{VS}^{\,2}}}\exp\left\{-\frac{1}{2(1-\bar{\rho}_{VS}^{\,2})}\left[\tilde{u}^2-2\bar{\rho}_{VS}\tilde{u}\tilde{v}+\tilde{v}^{2}\right]\right\}d\tilde{v}\,d\tilde{u}\right.$$

$$-\int_{-a}^{\infty}\int_{f(\tilde{u})}^{\infty}\frac{K}{2\pi\sqrt{1-\bar{\rho}_{VS}^{\,2}}}\exp\left\{-\frac{1}{2(1-\bar{\rho}_{VS}^{\,2})}\left[\tilde{u}^2-2\bar{\rho}_{VS}\tilde{u}\tilde{v}+\tilde{v}^{2}\right]\right\}d\tilde{v}\,d\tilde{u}$$

$$+\int_{-a}^{\infty}\int_{-\infty}^{f(\tilde{u})}\frac{(1-\alpha)\frac{V}{B}\frac{S}{B}\exp\left(-\frac{s_S^2}{2}+s_S\tilde{u}-\frac{s_V^2}{2}+s_V\tilde{v}\right)}{D^{*}-K+\frac{S}{B}\exp\left(-\frac{s_S^2}{2}+s_S\tilde{u}\right)}\frac{1}{2\pi\sqrt{1-\bar{\rho}_{VS}^{\,2}}}$$
$$\exp\left\{-\frac{1}{2(1-\bar{\rho}_{VS}^{\,2})}\left[\tilde{u}^2-2\bar{\rho}_{VS}\tilde{u}\tilde{v}+\tilde{v}^{2}\right]\right\}d\tilde{v}\,d\tilde{u}$$

$$-\int_{-a}^{\infty}\int_{-\infty}^{f(\tilde{u})}\frac{(1-\alpha)K\frac{V}{B}\exp\left(-\frac{s_V^2}{2}+s_V\tilde{v}\right)}{D^{*}-K+\frac{S}{B}\exp\left(-\frac{s_S^2}{2}+s_S\tilde{u}\right)}\frac{1}{2\pi\sqrt{1-\bar{\rho}_{VS}^{\,2}}}$$
$$\left.\exp\left\{-\frac{1}{2(1-\bar{\rho}_{VS}^{\,2})}\left[\tilde{u}^2-2\bar{\rho}_{VS}\tilde{u}\tilde{v}+\tilde{v}^{2}\right]\right\}d\tilde{v}\,d\tilde{u}\right]$$

where

$$a=\left[\frac{\ln\left(\frac{S}{KB}\right)-\frac{s_S^2}{2}}{s_S}\right] \quad \text{and} \quad f(\tilde{u})=\frac{\ln\left(\frac{D^{*}-K+\frac{S}{B}\exp\left(-\frac{s_S^2}{2}+s_S\tilde{u}\right)}{\frac{V}{B}}\right)+\frac{s_V^2}{2}}{s_V}$$

where : $\tilde{u}$ and $\tilde{v}$ are standard normal variates with zero mean and standard deviation equal to one.

Next we linerize the non-liner boundary $f(\tilde{u})$ in the second integral by taking a first order Taylor series expansion around the point 'p':

$$f(\tilde{u}) \approx f(p) + f'(p)(u-p) = b + m(u-p)$$

where:

$$b = f(p) = \frac{\ln\left(\dfrac{D^* - K + \dfrac{S}{B}\exp\left(-\dfrac{s_S^2}{2} + s_S p\right)}{\dfrac{V}{B}}\right) - \dfrac{s_V^2}{2}}{s_V}$$

and

$$m = f'(p) = \frac{\sigma_S}{\sigma_V}\left[\frac{\dfrac{S}{B}\exp\left(-\dfrac{s_S^2}{2} + s_S p\right)}{D^* - K + \dfrac{S}{B}\exp\left(-\dfrac{s_S^2}{2} + s_S p\right)}\right]$$

We also need to modify the denominator in the third and fourth integrals of the call valuation formula:

$$Let \quad F(\tilde{u}) = \frac{1}{D^* - K + \dfrac{S}{B}\exp\left(-\dfrac{s_S^2}{2} + s_S\tilde{u}\right)}$$

The problem occurs because of the constant term $D^* - K$. We want to make $F(\tilde{u})$ look like an exponential function. To do this define $G(\tilde{u})$ as follows:

$$G(\tilde{u}) = \ln(F(\tilde{u})) = \ln\left(\frac{1}{D^* - K + \dfrac{S}{B}\exp\left(-\dfrac{s_S^2}{2} + s_S\tilde{u}\right)}\right)$$

Now we can use another first order Taylor series expansion around the point 'q' to linerize $G(\tilde{u})$:

$$G(\tilde{u}) \approx G(q) + G'(p)(u-q) = f + g(u-q)$$

Where:

$$f = G(q) = \ln\left(\frac{1}{D^{*} - K + \frac{S}{B}\exp\left(-\frac{s_S^2}{2} + s_S q\right)}\right)$$

and

$$g = G'(q) = \frac{-s_S \frac{S}{B}\exp\left(-\frac{s_S^2}{2} + s_S q\right)}{D^{*} - K + \frac{S}{B}\exp\left(-\frac{s_S^2}{2} + s_S q\right)}$$

Therefore:

$$F(\tilde{u}) = \exp[G(\tilde{u})] \approx \exp[f + g(\tilde{u} - q)]$$

$$F(\tilde{u}) \approx \frac{\exp[g(\tilde{u} - q)]}{D^{*} - K + \frac{S}{B}\exp\left(-\frac{s_S^2}{2} + s_S q\right)}$$

Now substituting the two Taylor series approximations into the valuation equation:

$$c = \left[\frac{S}{B}\exp\left(\frac{s_S^2}{2}\right)\int_{-a}^{\infty}\int_{b+m(\tilde{u}-p)}^{\infty}\frac{\exp(s_S\tilde{u})}{2\pi\sqrt{1-\bar{\rho}_{VS}^{\,2}}}\exp\left\{-\frac{1}{2(1-\bar{\rho}_{VS}^{\,2})}\left[\tilde{u}^2 - 2\bar{\rho}_{VS}\tilde{u}\tilde{v} + \tilde{v}^{2}\right]\right\}d\tilde{v}\,d\tilde{u} \right.$$

$$-K\int_{-a}^{\infty}\int_{b+m(\tilde{u}-p)}^{\infty}\frac{1}{2\pi\sqrt{1-\bar{\rho}_{VS}^{\,2}}}\exp\left\{-\frac{1}{2(1-\bar{\rho}_{VS}^{\,2})}\left[\tilde{u}^2 - 2\bar{\rho}_{VS}\tilde{u}\tilde{v} + \tilde{v}^{2}\right]\right\}d\tilde{v}\,d\tilde{u}$$

$$+\frac{(1-\alpha)\frac{V}{B}\frac{S}{B}\exp\left(-\frac{s_V^2}{2}-\frac{s_S^2}{2}\right)\exp(-gq)}{D^* - K + \frac{S}{B}\exp\left(-\frac{s_S^2}{2}+s_S q\right)}\int_{-a}^{\infty}\int_{-\infty}^{b+m(\tilde{u}-p)}\frac{\exp[(g+s_S)\tilde{u}+s_V\,\tilde{v}]}{2\pi\sqrt{1-\bar{\rho}_{VS}^{\,2}}}$$

$$\exp\left\{-\frac{1}{2(1-\bar{\rho}_{VS}^{\,2})}\left[\tilde{u}^2 - 2\bar{\rho}_{VS}\tilde{u}\tilde{v} + \tilde{v}^{2}\right]\right\}d\tilde{v}\,d\tilde{u}$$

$$-\frac{(1-\alpha)K\frac{V}{B}\exp\left(-\frac{s_V^2}{2}\right)\exp(-gq)}{D^* - K + \frac{S}{B}\exp\left(-\frac{s_S^2}{2}+s_S q\right)}\int_{-a}^{\infty}\int_{-\infty}^{b+m(\tilde{u}-p)}\frac{\exp(g\tilde{u}+s_S\,\tilde{v})}{2\pi\sqrt{1-\bar{\rho}_{VS}^{\,2}}}$$

$$\left. \exp\left\{-\frac{1}{2(1-\bar{\rho}_{VS}^{\,2})}\left[\tilde{u}^2 - 2\bar{\rho}_{VS}\tilde{u}\tilde{v} + \tilde{v}^{2}\right]\right\}d\tilde{v}\,d\tilde{u} \right]$$

Next we need to rotate the default boundary (i.e. the limit of integration in the second integral of each of the above terms) to eliminate its dependence on the random variable $\tilde{u}$.

Consider the following transformation:

$$\tilde{u} = \frac{1}{\sqrt{1+m^2}}\tilde{x} \qquad \tilde{v} = \tilde{y} + \frac{m}{\sqrt{1+m^2}}\tilde{x}$$

The determinant of the Jacobian of this mapping is $|J| = \frac{1}{\sqrt{1+m^2}}$. Applying this transformation to the valuation equation gives:

This expressions can be rewritten:

$$c=\left[\frac{S}{B}\exp\left(-\frac{s_S^2}{2}\right)\left(\frac{\sqrt{1-\delta^2}}{\sqrt{1-\bar{\rho}_{VS}{}^2}\sqrt{1+m^2}}\right)\int_{-a\sqrt{1+m^2}}^{\infty}\int_{b-mp}^{\infty}\frac{\exp\left(\frac{s_S}{\sqrt{1+m^2}}\tilde{x}\right)}{2\pi\sqrt{1-\delta^2}}\Psi(\tilde{x},\tilde{y})d\tilde{y}d\tilde{x}\right.$$

$$-K\left(\frac{\sqrt{1-\delta^2}}{\sqrt{1-\bar{\rho}_{VS}{}^2}\sqrt{1+m^2}}\right)\int_{-a\sqrt{1+m^2}}^{\infty}\int_{b-mp}^{\infty}\frac{1}{2\pi\sqrt{1-\delta^2}}\Psi(\tilde{x},\tilde{y})d\tilde{y}d\tilde{x}$$

$$+\frac{(1-\alpha)\frac{V}{B}\frac{S}{B}\exp\left(-\frac{s_V^2}{2}-\frac{s_S^2}{2}\right)\exp(-gq)}{D^*-K+\frac{S}{B}\exp\left(-\frac{s_S^2}{2}+s_Sq\right)}\left(\frac{\sqrt{1-\delta^2}}{\sqrt{1-\bar{\rho}_{VS}{}^2}\sqrt{1+m^2}}\right)$$

$$\int_{-a\sqrt{1+m^2}}^{\infty}\int_{-\infty}^{b-mp}\frac{\exp\left[\left(\frac{g+s_S+ms_V}{\sqrt{1+m^2}}\right)\tilde{x}+s_V\,\tilde{y}\right]}{2\pi\sqrt{1-\delta^2}}\Psi(\tilde{x},\tilde{y})d\tilde{y}d\tilde{x}$$

$$-\frac{(1-\alpha)K\frac{V}{B}\exp\left(-\frac{s_V^2}{2}\right)\exp(-gq)}{D^*-K+\frac{S}{B}\exp\left(-\frac{s_S^2}{2}+s_Sq\right)}\left(\frac{\sqrt{1-\delta^2}}{\sqrt{1-\bar{\rho}_{VS}{}^2}\sqrt{1+m^2}}\right)$$

$$\left.\int_{-a\sqrt{1+m^2}}^{\infty}\int_{-\infty}^{b-mp}\frac{\exp\left[\left(\frac{g+ms_V}{\sqrt{1+m^2}}\right)\tilde{x}+s_V\,\tilde{y}\right]}{2\pi\sqrt{1-\delta^2}}\Psi(\tilde{x},\tilde{y})d\tilde{y}d\tilde{x}\right]$$

Where

$$\Psi(\tilde{x},\tilde{y})=\exp\left\{-\frac{1}{2(1-\delta^2)}\left[\left(\frac{\tilde{x}}{\sqrt{1+m^2}}\right)^2-2\delta\left(\frac{\tilde{x}}{\sqrt{1+m^2}}\right)\left(\frac{\tilde{y}}{\sqrt{1-2\bar{\rho}_{VS}m+m^2}}\right)+\left(\frac{\tilde{y}}{\sqrt{1-2\bar{\rho}_{VS}m+m^2}}\right)^2\right]\right\}$$

and

$$\delta=\frac{\bar{\rho}_{VS}-m}{\sqrt{1-2\bar{\rho}_{VS}m+m^2}}$$

Simplifying the exponential term $\Psi(\tilde{x},\tilde{y})$:

$$c = \Bigg[\frac{S}{B}\exp\left(-\frac{s_S^2}{2}\right)\int_{-a}^{\infty}\int_{\frac{b-mp}{\sqrt{1-2\bar{\rho}_{VS}m+m^2}}}^{\infty} \frac{\exp(s_S\tilde{q})}{2\pi\sqrt{1-\delta^2}}\Lambda(\tilde{q},\tilde{r})d\tilde{r}d\tilde{q}$$

$$-K\int_{-a}^{\infty}\int_{\frac{b-mp}{\sqrt{1-2\bar{\rho}_{VS}m+m^2}}}^{\infty} \frac{1}{2\pi\sqrt{1-\delta^2}}\Lambda(\tilde{q},\tilde{r})d\tilde{r}d\tilde{q}$$

$$+\frac{(1-\alpha)\frac{V}{B}\frac{S}{B}\exp\left(-\frac{s_V^2}{2}-\frac{s_S^2}{2}\right)\exp(-gq)}{D^{\bullet}-K+\frac{S}{B}\exp\left(-\frac{s_S^2}{2}+s_Sq\right)}$$

$$\int_{-a}^{\infty}\int_{-\infty}^{\frac{b-mp}{\sqrt{1-2\bar{\rho}_{VS}m+m^2}}} \frac{\exp\left[(g+s_S+ms_V)\tilde{q}+\sqrt{1-2\bar{\rho}_{VS}m+m^2}\,s_V\,\tilde{r}\right]}{2\pi\sqrt{1-\delta^2}}\Lambda(\tilde{q},\tilde{r})d\tilde{r}d\tilde{q}$$

$$-\frac{(1-\alpha)K\frac{V}{B}\exp\left(-\frac{s_V^2}{2}\right)\exp(-gq)}{D^{\bullet}-K+\frac{S}{B}\exp\left(-\frac{s_S^2}{2}+s_Sq\right)}$$

$$\int_{-a}^{\infty}\int_{-\infty}^{\frac{b-mp}{\sqrt{1-2\rho m+m^2}}} \frac{\exp\left[(g+ms_V)\tilde{q}+\sqrt{1-2\bar{\rho}_{VS}m+m^2}\,s_V\,\tilde{r}\right]}{2\pi\sqrt{1-\delta^2}}\Lambda(\tilde{q},\tilde{r})d\tilde{r}d\tilde{q} \Bigg]$$

Where:

$$\Lambda(\tilde{q},\tilde{r}) = \exp\left\{-\frac{1}{2(1-\delta^2)}\left[\tilde{q}^2-2\delta\tilde{q}\tilde{r}+\tilde{r}^2\right]\right\}$$

The next step is to complete the square in the integrand. This is shown on the next page.

$$c=\left[\frac{S}{B}\int_{-a}^{\infty}\int_{\frac{b-mp}{\sqrt{1-2\bar{\rho}_{VS}m+m^2}}}^{\infty}\frac{1}{2\pi\sqrt{1-\delta^2}}\right.$$

$$\exp\left\{-\frac{1}{2(1-\delta^2)}\left[(\tilde{q}-s_S)^2-2\delta(\tilde{q}-s_S)(\tilde{r}-\delta s_S)+(\tilde{r}-\delta s_S)^2\right]\right\}d\tilde{r}\,d\tilde{q}$$

$$-K\int_{-a}^{\infty}\int_{\frac{b-mp}{\sqrt{1-2\bar{\rho}_{VS}m+m^2}}}^{\infty}\frac{1}{2\pi\sqrt{1-\delta^2}}\exp\left\{-\frac{1}{2(1-\delta^2)}\left[\tilde{q}^2-2\delta\tilde{q}\tilde{r}+\tilde{r}^2\right]\right\}d\tilde{r}\,d\tilde{q}$$

$$+\frac{(1-\alpha)\frac{V}{B}\frac{S}{B}\exp\left(-\frac{s_V^2}{2}-\frac{s_S^2}{2}\right)\exp(-gq)}{D^{*}-K+\frac{S}{B}\exp\left(-\frac{s_S^2}{2}+s_Sq\right)}$$

$$\exp\left(\frac{(g+s_S+ms_V)^2+2\delta(g+s_S+ms_V)\left(\sqrt{1-2\bar{\rho}_{VS}m+m^2}\,s_V\right)+\left(\sqrt{1-2\bar{\rho}_{VS}m+m^2}\,s_V\right)^2}{2}\right)$$

$$\int_{-a}^{\infty}\int_{-\infty}^{\frac{b-mp}{\sqrt{1-2\bar{\rho}_{VS}m+m^2}}}\frac{1}{2\pi\sqrt{1-\delta^2}}\exp\left\{-\frac{1}{2(1-\delta^2)}\left[\begin{array}{l}\left\{\tilde{q}-(g+s_S+ms_V)-\delta\left(\sqrt{1-2\bar{\rho}_{VS}m+m^2}\,s_V\right)\right\}^2\\-2\delta\left\{\tilde{q}-(g+s_S+ms_V)-\delta\left(\sqrt{1-2\bar{\rho}_{VS}m+m^2}\,s_V\right)\right\}\\\left\{\tilde{r}-\delta(g+s_S+ms_V)-\left(\sqrt{1-2\bar{\rho}_{VS}m+m^2}\,s_V\right)\right\}\\+\left\{\tilde{r}-\delta(g+s_S+ms_V)-\left(\sqrt{1-2\bar{\rho}_{VS}m+m^2}\,s_V\right)\right\}^2\end{array}\right]\right\}d\tilde{r}\,d\tilde{q}$$

$$-\frac{(1-\alpha)K\frac{V}{B}\exp\left(-\frac{s_V^2}{2}\right)\exp(-gq)}{D^{*}-K+\frac{S}{B}\exp\left(-\frac{s_S^2}{2}+s_Sq\right)}$$

$$\exp\left(\frac{\begin{array}{r}(g+s_S+ms_V)^2+2\delta(g+s_S+ms_V)\left(\sqrt{1-2\bar{\rho}_{VS}m+m^2}\,s_V\right)\\+\left(\sqrt{1-2\bar{\rho}_{VS}m+m^2}\,s_V\right)^2\end{array}}{2}\right)\int_{-a}^{\infty}\int_{-\infty}^{\frac{b-mp}{\sqrt{1-2\bar{\rho}_{VS}m+m^2}}}\frac{1}{2\pi\sqrt{1-\delta^2}}$$

$$\exp\left\{-\frac{1}{2(1-\delta^2)}\begin{bmatrix}\left\{\tilde{q}-(g+s_S+ms_V)-\delta\left(\sqrt{1-2\bar{\rho}_{VS}m+m^2}\,s_V\right)\right\}^2\\ -2\delta\left\{\tilde{q}-(g+s_S+ms_V)-\delta\left(\sqrt{1-2\bar{\rho}_{VS}m+m^2}\,s_V\right)\right\}\\ \left\{\tilde{r}-\delta(g+s_S+ms_V)-\left(\sqrt{1-2\bar{\rho}_{VS}m+m^2}\,s_V\right)\right\}\\ +\left\{\tilde{r}-\delta(g+s_S+ms_V)-\left(\sqrt{1-2\bar{\rho}_{VS}m+m^2}\,s_V\right)\right\}\end{bmatrix}^2\right\}d\tilde{r}\,d\tilde{q}\;\Bigg]$$

A simple change of variables results in:

$$c=\Bigg[\frac{S}{B}\int_{-a-\sigma_S\sqrt{T-t}}^{\infty}\int_{\frac{h-mp}{\sqrt{1-2\bar{\rho}_{VS}m+m^2}}-\delta\sigma_S\sqrt{T-t}}^{\infty}\frac{1}{2\pi\sqrt{1-\delta^2}}\exp\left\{-\frac{1}{2(1-\delta^2)}\left[\tilde{w}^2-2\delta\tilde{w}\tilde{z}+\tilde{z}^2\right]\right\}d\tilde{z}\,d\tilde{w}$$

$$-K\int_{-a}^{\infty}\int_{\frac{h-mp}{\sqrt{1-2\bar{\rho}_{VS}m+m^2}}}^{\infty}\frac{1}{2\pi\sqrt{1-\delta^2}}\exp\left\{-\frac{1}{2(1-\delta^2)}\left[\tilde{w}^2-2\delta\tilde{w}\tilde{z}+\tilde{z}^2\right]\right\}d\tilde{z}\,d\tilde{w}$$

$$+\frac{(1-\alpha)\frac{V}{B}\frac{S}{B}\exp\left(-\frac{s_V^2}{2}-\frac{s_S^2}{2}\right)\exp(-gq)}{D^{*}-K+\frac{S}{B}\exp\left(-\frac{s_S^2}{2}+s_Sq\right)}$$

$$\exp\left(\frac{(g+s_S+ms_V)^2+2\delta(g+s_S+ms_V)\left(\sqrt{1-2\bar{\rho}_{VS}m+m^2}\,s_V\right)+\left(\sqrt{1-2\bar{\rho}_{VS}m+m^2}\,s_V\right)^2}{2}\right)$$

$$\int_{-a-(g+s_S+ms_V)-\delta\left(\sqrt{1-2\bar{\rho}_{VS}m+m^2}\,s_V\right)}^{\infty}\int_{-\infty}^{\frac{h-mp}{\sqrt{1-2\bar{\rho}_{VS}m+m^2}}-\delta(g+s_S+ms_V)-\left(\sqrt{1-2\bar{\rho}_{VS}m+m^2}\,s_V\right)}\frac{1}{2\pi\sqrt{1-\delta^2}}$$

$$\exp\left\{-\frac{1}{2(1-\delta^2)}\left[\tilde{w}^2-2\delta\tilde{w}\tilde{z}+\tilde{z}^2\right]\right\}d\tilde{z}\,d\tilde{w}$$

$$
-\frac{(1-\alpha)K\frac{V}{B}\exp\left(-\frac{s_V^2}{2}\right)\exp(-gq)}{D^{*}-K+\frac{S}{B}\exp\left(-\frac{s_S^2}{2}+s_S q\right)}
$$

$$
\exp\left(\frac{(g+s_S+ms_V)^2+2\delta(g+s_S+ms_V)\left(\sqrt{1-2\bar{\rho}_{VS}m+m^2}s_V\right)+\left(\sqrt{1-2\bar{\rho}_{VS}m+m^2}s_V\right)^2}{2}\right)
$$

$$
\int_{-a-(g+ms_V)-\delta\left(\sqrt{1-2\bar{\rho}_{VS}m+m^2}s_V\right)}^{\infty}\quad\int_{-\infty}^{\frac{b-mp}{\sqrt{1-2\bar{\rho}_{VS}m+m^2}}-\delta(g+ms_V)-\left(\sqrt{1-2\bar{\rho}_{VS}m+m^2}s_V\right)}\quad\frac{1}{2\pi\sqrt{1-\delta^2}}e
$$

$$
\left.\exp\left\{-\frac{1}{2(1-\delta^2)}\left[\tilde{w}^2-2\delta\tilde{w}\tilde{z}+\tilde{z}^2\right]\right\}d\tilde{z}\,d\tilde{w}\right]
$$

Thus, the value of the vulnerable call can be expressed as shown on the next page.

$$c = \frac{S}{B} N\left(a + s_S, -\frac{b - mp}{\sqrt{1 - 2\bar{\rho}_{VS} m + m^2}} + \delta s_S, \delta \right) - KN\left(a, -\frac{b - mp}{\sqrt{1 - 2\bar{\rho}_{VS} m + m^2}}, \delta \right)$$

$$+ \frac{(1-\alpha)\dfrac{V}{B}\dfrac{S}{B}\exp\left(-\dfrac{s_V^2}{2} - \dfrac{s_S^2}{2} \right)\exp(-gq)}{D^{*} - K + \dfrac{S}{B}\exp\left(-\dfrac{s_S^2}{2} + s_S q \right)}$$

$$\exp\left(\frac{(g + s_S + ms_V)^2 + 2\delta(g + s_S + ms_V)\left(\sqrt{1 - 2\bar{\rho}_{VS} m + m^2}\, s_V\right) + \left(\sqrt{1 - 2\bar{\rho}_{VS} m + m^2}\, s_V\right)^2}{2} \right)$$

$$N\left(a + (g + s_S + ms_V) + \delta\left(\sqrt{1 - 2\bar{\rho}_{VS} m + m^2}\, s_V\right), \frac{b - mp}{\sqrt{1 - 2\bar{\rho}_{VS} m + m^2}} - \delta(g + s_S + ms_V) - \left(\sqrt{1 - 2\bar{\rho}_{VS} m + m^2}\, s_V\right), -\delta \right)$$

$$- \frac{(1-\alpha)K\dfrac{V}{B}\exp\left(-\dfrac{s_V^2}{2} \right)\exp(-gq)}{D^{*} - K + \dfrac{S}{B}\exp\left(-\dfrac{s_S^2}{2} + s_S q \right)}$$

$$\exp\left(\frac{(g + s_S + ms_V)^2 + 2\delta(g + s_S + ms_V)\left(\sqrt{1 - 2\bar{\rho}_{VS} m + m^2}\, s_V\right) + \left(\sqrt{1 - 2\bar{\rho}_{VS} m + m^2}\, s_V\right)^2}{2} \right)$$

$$N\left(a + (g + ms_V) + \delta\left(\sqrt{1 - 2\bar{\rho}_{VS} m + m^2}\, s_V\right), \frac{b - mp}{\sqrt{1 - 2\bar{\rho}_{VS} m + m^2}} - \delta(g + ms_V) - \left(\sqrt{1 - 2\bar{\rho}_{VS} m + m^2}\, s_V\right), -\delta \right)$$

Where, $N(-,-,-)$ represents a bivariate cumulative normal distribution function.

The above expression represents the value of the call in units of the discount bond. To determine the nominal value of the call we multiply through by *B*.

$$c = SN\left(a + s_S, -\frac{b - mp}{\sqrt{1 - 2\bar{\rho}_{VS} m + m^2}} + \delta s_S, \delta\right) - KBN\left(a, -\frac{b - mp}{\sqrt{1 - 2\bar{\rho}_{VS} m + m^2}}, \delta\right)$$

$$+ \frac{(1-\alpha)V\frac{S}{B}\exp\left(-\frac{s_V^2}{2} - \frac{s_S^2}{2}\right)\exp(-gq)}{D^* - K + \frac{S}{B}\exp\left(-\frac{s_S^2}{2} + s_S q\right)}$$

$$\exp\left(\frac{(g + s_S + ms_V)^2 + 2\delta(g + s_S + ms_V)\left(\sqrt{1 - 2\bar{\rho}_{VS} m + m^2}\, s_V\right) + \left(\sqrt{1 - 2\bar{\rho}_{VS} m + m^2}\, s_V\right)^2}{2}\right)$$

$$N\left(a + (g + s_S + ms_V) + \delta\left(\sqrt{1 - 2\bar{\rho}_{VS} m + m^2}\, s_V\right), \frac{b - mp}{\sqrt{1 - 2\bar{\rho}_{VS} m + m^2}} - \delta(g + s_S + ms_V) - \left(\sqrt{1 - 2\bar{\rho}_{VS} m + m^2}\, s_V\right), -\delta\right)$$

$$- \frac{(1-\alpha)KV\exp\left(-\frac{s_V^2}{2}\right)\exp(-gq)}{D^* - K + \frac{S}{B}\exp\left(-\frac{s_S^2}{2} + s_S q\right)}$$

$$\exp\left(\frac{(g + s_S + ms_V)^2 + 2\delta(g + s_S + ms_V)\left(\sqrt{1 - 2\bar{\rho}_{VS} m + m^2}\, s_V\right) + \left(\sqrt{1 - 2\bar{\rho}_{VS} m + m^2}\, s_V\right)^2}{2}\right)$$

$$N\left(a + (g + ms_V) + \delta\left(\sqrt{1 - 2\bar{\rho}_{VS} m + m^2}\, s_V\right), \frac{b - mp}{\sqrt{1 - 2\bar{\rho}_{VS} m + m^2}} - \delta(g + ms_V) - \left(\sqrt{1 - 2\bar{\rho}_{VS} m + m^2}\, s_V\right), -\delta\right)$$

Appendix E

Proof of Proposition 3: Value of Option Writer's Liabilities with Variable Default Boundary

Using risk neutral pricing the value of the liabilities is given by:

$$L = \int_{-\infty}^{\ln(K)} \int_{-\infty}^{\ln(D^*)} \frac{(1-\alpha)e^{X_T}}{2\pi\sqrt{1-\bar{\rho}_{VS}^{\,2}}\, s_V s_S} dX_T dY_T + \int_{-\infty}^{\ln(K)} \int_{\ln(D^*)}^{\infty} \frac{D^*}{2\pi\sqrt{1-\bar{\rho}_{VS}^{\,2}}\, s_V s_S} \Omega(Y_T, X_T) dX_T dY_T$$

$$+ \int_{\ln(K)}^{\infty} \int_{-\infty}^{\ln(D^*+e^{Y_T}-K)} \frac{(1-\alpha)e^{X_T}}{2\pi\sqrt{1-\bar{\rho}_{VS}^{\,2}}\, s_V s_S} dX_T dY_T$$

$$+ \int_{\ln(K)}^{\infty} \int_{\ln(D^*+e^{Y_T}-K)}^{\infty} \frac{D^* + e^{Y_T} - K}{2\pi\sqrt{1-\bar{\rho}_{VS}^{\,2}}\, s_V s_S} \Omega(Y_T, X_T) dX_T dY_T$$

Where :

$$\Omega(Y_T, X_T) = \exp\left\{ -\frac{1}{2(1-\bar{\rho}_{VS}^{\,2})} \left(\frac{Y_T - Y + \frac{s_S^2}{2}}{s_S} \right)^2 - 2\bar{\rho}_{VS} \left(\frac{Y_T - Y + \frac{s_S^2}{2}}{s_S} \right) \left(\frac{X_T - X + \frac{s_V^2}{2}}{s_V} \right) + \left(\frac{X_T - X + \frac{s_V^2}{2}}{s_V} \right)^2 \right\}$$

Standardizing the normal distribution and substituting for Y_T *and* X_T, which are defined in Appendix B as:

$$Y_T = \log\left(\frac{S}{B}\right) \qquad X_T = \log\left(\frac{V}{B}\right)$$

results in:

$$L=\left[\int_{-\infty}^{-a}\int_{-\infty}^{-bb}(1-\alpha)\frac{V}{B}\exp\left(-\frac{s_V^2}{2}+s_V\tilde{v}\right)\frac{1}{2\pi\sqrt{1-\bar{\rho}_{VS}^{\;2}}}\right.$$

$$\exp\left\{-\frac{1}{2(1-\bar{\rho}_{VS}^{\;2})}\left[\tilde{u}^2-2\bar{\rho}_{VS}\tilde{u}\tilde{v}+\tilde{v}^2\right]\right\}d\tilde{v}d\tilde{u}$$

$$+\int_{-\infty}^{-a}\int_{-bb}^{\infty}\frac{D^*}{2\pi\sqrt{1-\bar{\rho}_{VS}^{\;2}}}\exp\left\{-\frac{1}{2(1-\bar{\rho}_{VS}^{\;2})}\left[\tilde{u}^2-2\bar{\rho}_{VS}\tilde{u}\tilde{v}+\tilde{v}^2\right]\right\}d\tilde{v}d\tilde{u}$$

$$+\int_{-a}^{\infty}\int_{-\infty}^{f(\tilde{u})}(1-\alpha)\frac{V}{B}\exp\left(-\frac{s_V^2}{2}+s_V\tilde{v}\right)\frac{1}{2\pi\sqrt{1-\bar{\rho}_{VS}^{\;2}}}$$

$$\exp\left\{-\frac{1}{2(1-\bar{\rho}_{VS}^{\;2})}\left[\tilde{u}^2-2\bar{\rho}_{VS}\tilde{u}\tilde{v}+\tilde{v}^2\right]\right\}d\tilde{v}d\tilde{u}$$

$$+\int_{-a}^{\infty}\int_{f(\tilde{u})}^{\infty}\frac{S}{B}\exp\left(-\frac{s_S^2}{2}+s_S\tilde{u}\right)\frac{1}{2\pi\sqrt{1-\bar{\rho}_{VS}^{\;2}}}$$

$$\exp\left\{-\frac{1}{2(1-\bar{\rho}_{VS}^{\;2})}\left[\tilde{u}^2-2\bar{\rho}_{VS}\tilde{u}\tilde{v}+\tilde{v}^2\right]\right\}d\tilde{v}d\tilde{u}$$

$$\left.+\int_{-a}^{\infty}\int_{f(\tilde{u})}^{\infty}\frac{D^*-K}{2\pi\sqrt{1-\bar{\rho}_{VS}^{\;2}}}\exp\left\{-\frac{1}{2(1-\bar{\rho}_{VS}^{\;2})}\left[\tilde{u}^2-2\bar{\rho}_{VS}\tilde{u}\tilde{v}+\tilde{v}^2\right]\right\}d\tilde{v}d\tilde{u}\right]$$

where

$$a=\left[\frac{\ln(\frac{S}{BK})-\frac{s_S^2}{2}}{s_S}\right]\quad;\quad \mathrm{bb}=\left[\frac{\ln(\frac{V}{BD^*})-\frac{s_V^2}{2}}{s_V}\right]\quad \text{and}$$

$$f(\tilde{u})=\frac{\ln\left(\dfrac{D^*-K+\frac{S}{B}\exp\left(-\frac{s_S^2}{2}+s_S\tilde{u}\right)}{\frac{V}{B}}\right)+\frac{s_V^2}{2}}{s_V}$$

where : $\tilde{u}$ and $\tilde{v}$ are standard normal variates with zero mean and standard deviation equal to one.

Next we linerize the non-liner boundary $f(\tilde{u})$ by taking a first order Taylor series expansion around the point 'p':

$$f(\tilde{u}) \approx f(p) + f'(p)(u-p) = b + m(u-p)$$

where:

$$b = f(p) = \frac{\ln\left(\dfrac{D^{*} - K + \dfrac{S}{B}\exp\left(-\dfrac{s_S^2}{2} + s_S p\right)}{\dfrac{V}{B}}\right) - \dfrac{s_V^2}{2}}{s_V}$$

and

$$m = f'(p) = \frac{\sigma_S}{\sigma_V}\left[\frac{\dfrac{S}{B}\exp\left(-\dfrac{s_S^2}{2} + s_S p\right)}{D^{*} - K + \dfrac{S}{B}\exp\left(-\dfrac{s_S^2}{2} + s_S p\right)}\right]$$

Now substituting the Taylor series approximation into the valuation equation:

$$L = \Bigg[(1-\alpha)\frac{V}{B}\exp\left(-\frac{s_V^2}{2}\right)\int_{-\infty}^{-a}\int_{-\infty}^{-bb}\frac{\exp(s_V\tilde{v})}{2\pi\sqrt{1-\bar{\rho}_{VS}^{\,2}}}\exp\left\{-\frac{1}{2(1-\bar{\rho}_{VS}^{\,2})}\left[\tilde{u}^2 - 2\bar{\rho}_{VS}\tilde{u}\tilde{v} + \tilde{v}^{2}\right]\right\}d\tilde{v}\,d\tilde{u}$$

$$+ D^*\int_{-\infty}^{-a}\int_{-bb}^{\infty}\frac{1}{2\pi\sqrt{1-\bar{\rho}_{VS}^{\,2}}}\exp\left\{-\frac{1}{2(1-\bar{\rho}_{VS}^{\,2})}\left[\tilde{u}^2 - 2\bar{\rho}_{VS}\tilde{u}\tilde{v} + \tilde{v}^{2}\right]\right\}d\tilde{v}\,d\tilde{u}$$

$$+ (1-\alpha)\frac{V}{B}\exp\left(-\frac{s_V^2}{2}\right)\int_{-a}^{\infty}\int_{-\infty}^{b+m(\tilde{u}-p)}\frac{\exp(s_V\tilde{v})}{2\pi\sqrt{1-\bar{\rho}_{VS}^{\,2}}}$$
$$\exp\left\{-\frac{1}{2(1-\bar{\rho}_{VS}^{\,2})}\left[\tilde{u}^2 - 2\bar{\rho}_{VS}\tilde{u}\tilde{v} + \tilde{v}^{2}\right]\right\}d\tilde{v}\,d\tilde{u}$$

$$+ \frac{S}{B}\exp\left(-\frac{s_S^2}{2}\right)\int_{-a}^{\infty}\int_{b+m(\tilde{u}-p)}^{\infty}\frac{\exp(s_S\tilde{u})}{2\pi\sqrt{1-\bar{\rho}_{VS}^{\,2}}}$$
$$\exp\left\{-\frac{1}{2(1-\bar{\rho}_{VS}^{\,2})}\left[\tilde{u}^2 - 2\bar{\rho}_{VS}\tilde{u}\tilde{v} + \tilde{v}^{2}\right]\right\}d\tilde{v}\,d\tilde{u}$$

$$+ (D^* - K)\int_{-a}^{\infty}\int_{b+m(\tilde{u}-p)}^{\infty}\frac{1}{2\pi\sqrt{1-\bar{\rho}_{VS}^{\,2}}}$$
$$\exp\left\{-\frac{1}{2(1-\bar{\rho}_{VS}^{\,2})}\left[\tilde{u}^2 - 2\bar{\rho}_{VS}\tilde{u}\tilde{v} + \tilde{v}^{2}\right]\right\}d\tilde{v}\,d\tilde{u}\Bigg]$$

Next we need to rotate the default boundary (i.e. the limit of integration in the second integral of the third and fourth terms) to eliminate its dependence on the random variable $\tilde{u}$.

Consider the following transformation:

$$\tilde{u} = \frac{1}{\sqrt{1+m^2}}\tilde{x} \qquad \tilde{v} = \tilde{y} + \frac{m}{\sqrt{1+m^2}}\tilde{x}$$

The determinant of the Jacobian of this mapping is $|J| = \frac{1}{\sqrt{1+m^2}}$. Applying this transformation to the valuation equation gives:

$$L=\Bigg[(1-\alpha)\frac{V}{B}\exp\left(-\frac{s_V^2}{2}\right)\int_{-\infty}^{-a}\int_{-\infty}^{-bb}\frac{\exp(s_V\tilde{v})}{2\pi\sqrt{1-\bar{\rho}_{VS}^{\,2}}}\exp\left\{-\frac{1}{2(1-\bar{\rho}_{VS}^{\,2})}\left[\tilde{u}^2-2\bar{\rho}_{VS}\tilde{u}\tilde{v}+\tilde{v}^{2}\right]\right\}d\tilde{v}\,d\tilde{u}$$

$$+D^{*}\int_{-\infty}^{-a}\int_{-bb}^{\infty}\frac{1}{2\pi\sqrt{1-\bar{\rho}_{VS}^{\,2}}}\exp\left\{-\frac{1}{2(1-\bar{\rho}_{VS}^{\,2})}\left[\tilde{u}^2-2\bar{\rho}_{VS}\tilde{u}\tilde{v}+\tilde{v}^{2}\right]\right\}d\tilde{v}\,d\tilde{u}$$

$$+(1-\alpha)\frac{V}{B}\exp\left(-\frac{s_V^2}{2}\right)\int_{-a\sqrt{1+m^2}}^{\infty}\int_{-\infty}^{b-mp}\frac{\exp\left(\frac{ms_V}{\sqrt{1+m^2}}\tilde{x}+s_V\tilde{y}\right)}{2\pi\sqrt{1-\bar{\rho}_{VS}^{\,2}}}\Omega(\tilde{x},\tilde{y})\,d\tilde{y}\,d\tilde{x}$$

$$+\frac{S}{B}\exp\left(-\frac{s_S^2}{2}\right)\int_{-a\sqrt{1+m^2}}^{\infty}\int_{b-mp}^{\infty}\frac{\exp\left(\frac{s_S}{\sqrt{1+m^2}}\tilde{x}\right)}{2\pi\sqrt{1-\bar{\rho}_{VS}^{\,2}}}\Omega(\tilde{x},\tilde{y})\,d\tilde{y}\,d\tilde{x}$$

$$+(D^{*}-K)\int_{-a\sqrt{1+m^2}}^{\infty}\int_{b-mp}^{\infty}\frac{1}{2\pi\sqrt{1-\bar{\rho}_{VS}^{\,2}}}\Omega(\tilde{x},\tilde{y})\,d\tilde{y}\,d\tilde{x}\Bigg]$$

where :

$$\Omega(\tilde{x},\tilde{y})=\exp\left\{-\frac{1}{2(1-\bar{\rho}_{VS}^{\,2})}\left[\left(\frac{\tilde{x}}{\sqrt{1+m^2}}\right)^2-2\bar{\rho}_{VS}\left(\frac{\tilde{x}}{\sqrt{1+m^2}}\right)\left(y+\frac{m}{\sqrt{1+m^2}}\tilde{x}\right)+\left(y+\frac{m}{\sqrt{1+m^2}}\tilde{x}\right)^2\right]\right\}$$

This expression can be rewritten as:

$$L=\left[(1-\alpha)\frac{V}{B}\exp\left(-\frac{s_V^2}{2}\right)\int_{-\infty}^{-a}\int_{-\infty}^{-bb}\frac{\exp(s_V\tilde{v})}{2\pi\sqrt{1-\bar{\rho}_{VS}{}^2}}\exp\left\{-\frac{1}{2(1-\bar{\rho}_{VS}{}^2)}\left[\tilde{u}^2-2\bar{\rho}_{VS}\tilde{u}\tilde{v}+\tilde{v}^{21}\right]\right\}d\tilde{v}d\tilde{u}\right.$$

$$+D^*\int_{-\infty}^{-a}\int_{-bb}^{\infty}\frac{1}{2\pi\sqrt{1-\bar{\rho}_{VS}{}^2}}\exp\left\{-\frac{1}{2(1-\bar{\rho}_{VS}{}^2)}\left[\tilde{u}^2-2\bar{\rho}_{VS}\tilde{u}\tilde{v}+\tilde{v}^{21}\right]\right\}d\tilde{v}d\tilde{u}$$

$$+(1-\alpha)\frac{V}{B}\exp\left(-\frac{s_V^2}{2}\right)\left(\frac{\sqrt{1-\delta^2}}{\sqrt{1-\bar{\rho}_{VS}{}^2}\sqrt{1+m^2}}\right)\int_{-a\sqrt{1+m^2}}^{\infty}\int_{-\infty}^{b-mp}\frac{\exp\left(\frac{ms_V}{\sqrt{1+m^2}}\tilde{x}+s_V\tilde{y}\right)}{2\pi\sqrt{1-\delta^2}}\Psi(\tilde{x},\tilde{y})d\tilde{y}d\tilde{x}$$

$$+\frac{S}{B}\exp\left(-\frac{s_S^2}{2}\right)\left(\frac{\sqrt{1-\delta^2}}{\sqrt{1-\bar{\rho}_{VS}{}^2}\sqrt{1+m^2}}\right)\int_{-a\sqrt{1+m^2}}^{\infty}\int_{b-mp}^{\infty}\frac{\exp\left(\frac{s_S}{\sqrt{1+m^2}}\tilde{x}\right)}{2\pi\sqrt{1-\delta^2}}\Psi(\tilde{x},\tilde{y})d\tilde{y}d\tilde{x}$$

$$\left.+(D^*-K)\left(\frac{\sqrt{1-\delta^2}}{\sqrt{1-\bar{\rho}_{VS}{}^2}\sqrt{1+m^2}}\right)\int_{-a\sqrt{1+m^2}}^{\infty}\int_{b-mp}^{\infty}\frac{1}{2\pi\sqrt{1-\delta^2}}\Psi(\tilde{x},\tilde{y})d\tilde{y}d\tilde{x}\right]$$

where :

$$\Psi(\tilde{x},\tilde{y})=\exp\left\{-\frac{1}{2(1-\delta^2)}\left[\left(\frac{\tilde{x}}{\sqrt{1+m^2}}\right)^2-2\delta\left(\frac{\tilde{x}}{\sqrt{1+m^2}}\right)\left(\frac{\tilde{y}}{\sqrt{1-2\bar{\rho}_{VS}m+m^2}}\right)+\left(\frac{\tilde{y}}{\sqrt{1-2\bar{\rho}_{VS}m+m^2}}\right)^2\right]\right\}$$

and

$$\delta=\frac{\bar{\rho}_{VS}-m}{\sqrt{1-2\bar{\rho}_{VS}m+m^2}}$$

Simplifying the exponential term $\Psi(\widetilde{x},\widetilde{y})$:

$$L=\Bigg[(1-\alpha)\frac{V}{B}\exp\left(-\frac{s_V^2}{2}\right)\int_{-\infty}^{-u}\int_{-\infty}^{-bb}\frac{\exp(s_V\widetilde{v})}{2\pi\sqrt{1-\overline{\rho}_{VS}^{\,2}}}\exp\left\{-\frac{1}{2(1-\overline{\rho}_{VS}^{\,2})}\left[\widetilde{u}^2-2\overline{\rho}_{VS}\widetilde{u}\widetilde{v}+\widetilde{v}^2\right]\right\}d\widetilde{v}\,d\widetilde{u}$$

$$+D^*\int_{-\infty}^{-u}\int_{-bb}^{\infty}\frac{1}{2\pi\sqrt{1-\overline{\rho}_{VS}^{\,2}}}\exp\left\{-\frac{1}{2(1-\overline{\rho}_{VS}^{\,2})}\left[\widetilde{u}^2-2\overline{\rho}_{VS}\widetilde{u}\widetilde{v}+\widetilde{v}^2\right]\right\}d\widetilde{v}\,d\widetilde{u}$$

$$+(1-\alpha)\frac{V}{B}\exp\left(-\frac{s_V^2}{2}\right)\int_{-u}^{\infty}\int_{-\infty}^{\frac{b-mp}{\sqrt{1-2\overline{\rho}_{VS}m+m^2}}}\frac{\exp\left(ms_V\widetilde{q}+\sqrt{1-2\overline{\rho}_{VS}m+m^2}\,s_V\widetilde{r}\right)}{2\pi\sqrt{1-\delta^2}}$$

$$\exp\left\{-\frac{1}{2(1-\delta^2)}\left[\widetilde{q}^2-2\delta\widetilde{q}\widetilde{r}+\widetilde{r}^2\right]\right\}d\widetilde{r}\,d\widetilde{q}$$

$$+\frac{S}{B}\exp\left(-\frac{s_S^2}{2}\right)\int_{-u}^{\infty}\int_{\frac{b-mp}{\sqrt{1-2\overline{\rho}_{VS}m+m^2}}}^{\infty}\frac{\exp(s_S\widetilde{q})}{2\pi\sqrt{1-\delta^2}}\exp\left\{-\frac{1}{2(1-\delta^2)}\left[\widetilde{q}^2-2\delta\widetilde{q}\widetilde{r}+\widetilde{r}^2\right]\right\}d\widetilde{r}\,d\widetilde{q}$$

$$+(D^*-K)\int_{-u}^{\infty}\int_{\frac{b-mp}{\sqrt{1-2\overline{\rho}_{VS}m+m^2}}}^{\infty}\frac{1}{2\pi\sqrt{1-\delta^2}}\exp\left\{-\frac{1}{2(1-\delta^2)}\left[\widetilde{q}^2-2\delta\widetilde{q}\widetilde{r}+\widetilde{r}^2\right]\right\}d\widetilde{r}\,d\widetilde{q}\Bigg]$$

The next step is to complete the square in the integrand in the first, third and fourth terms:

$$L = \Bigg[(1-\alpha)\frac{V}{B}\int_{-\infty}^{-a}\int_{-\infty}^{-bb}\frac{1}{2\pi\sqrt{1-\bar{\rho}_{VS}{}^2}}$$

$$\exp\left\{-\frac{1}{2(1-\bar{\rho}_{VS}{}^2)}\left[(\tilde{u}-\bar{\rho}_{VS}s_V)^2 - 2\bar{\rho}_{VS}(\tilde{u}-\bar{\rho}_{VS}s_V)(\tilde{v}-s_V)+(\tilde{v}-s_V)^2\right]\right\}d\tilde{v}\,d\tilde{u}$$

$$+ D^*\int_{-\infty}^{-a}\int_{-bb}^{\infty}\frac{1}{2\pi\sqrt{1-\bar{\rho}_{VS}{}^2}}\exp\left\{-\frac{1}{2(1-\bar{\rho}_{VS}{}^2)}\left[\tilde{u}^2 - 2\bar{\rho}_{VS}\tilde{u}\tilde{v}+\tilde{v}^2\right]\right\}d\tilde{v}\,d\tilde{u}$$

$$+(1-\alpha)\frac{V}{B}\exp\left(-\frac{s_V^2}{2}\right)\exp\left(\frac{(ms_V)^2 + 2\delta(ms_V)\left(\sqrt{1-2\bar{\rho}_{VS}m+m^2}\,s_V\right)+\left(\sqrt{1-2\bar{\rho}_{VS}m+m^2}\,s_V\right)^2}{2}\right)$$

$$\int_{-a}^{\infty}\int_{-\infty}^{\frac{b-mp}{\sqrt{1-2\bar{\rho}_{VS}m+m^2}}}\exp\left\{-\frac{1}{2(1-\delta^2)}\left[\begin{array}{l}\left\{\tilde{q}-(ms_V)-\delta\left(\sqrt{1-2\bar{\rho}_{VS}m+m^2}\,s_V\right)\right\}^2\\ -2\delta\left\{\tilde{q}-(ms_V)-\delta\left(\sqrt{1-2\bar{\rho}_{VS}m+m^2}\,s_V\right)\right\}\\ \left\{\tilde{r}-\delta(ms_V)-\left(\sqrt{1-2\bar{\rho}_{VS}m+m^2}\,s_V\right)\right\}\\ +\left\{\tilde{r}-\delta(ms_V)-\left(\sqrt{1-2\bar{\rho}_{VS}m+m^2}\,s_V\right)\right\}^2\end{array}\right]\right\}d\tilde{r}\,d\tilde{q}$$

$$+\frac{S}{B}\int_{-a}^{\infty}\int_{\frac{b-mp}{\sqrt{1-2\bar{\rho}_{VS}m+m^2}}}^{\infty}\frac{1}{2\pi\sqrt{1-\delta^2}}$$

$$\exp\left\{-\frac{1}{2(1-\delta^2)}\left[(\tilde{q}-s_S)^2 - 2\delta(\tilde{q}-s_S)(\tilde{r}-\delta s s_S)+(\tilde{r}-\delta s s_S)^2\right]\right\}d\tilde{r}\,d\tilde{q}$$

$$+(D^*-K)\int_{-a}^{\infty}\int_{\frac{b-mp}{\sqrt{1-2\bar{\rho}_{VS}m+m^2}}}^{\infty}\frac{1}{2\pi\sqrt{1-\delta^2}}\exp\left\{-\frac{1}{2(1-\delta^2)}\left[\tilde{q}^2 - 2\delta\tilde{q}\tilde{r}+\tilde{r}^2\right]\right\}d\tilde{r}\,d\tilde{q}$$

A simple change of variables results in:

$$L = \Bigg[(1-\alpha)\frac{V}{B} \int_{-\infty}^{-a-\bar{\rho}_{VS}s_{1'}} \int_{-\infty}^{-bb-s_{1'}} \frac{1}{2\pi\sqrt{1-\bar{\rho}_{VS}{}^{2}}} \exp\left\{ -\frac{1}{2(1-\bar{\rho}_{VS}{}^{2})}\left[\tilde{w}^{2} - 2\bar{\rho}_{VS}\tilde{w}\tilde{z} + \tilde{z}^{2}\right]\right\} d\tilde{z}\, d\tilde{w}$$

$$+ D^{*} \int_{-\infty}^{-a} \int_{-bh}^{\infty} \frac{1}{2\pi\sqrt{1-\bar{\rho}_{VS}{}^{2}}} \exp\left\{ -\frac{1}{2(1-\bar{\rho}_{VS}{}^{2})}\left[\tilde{u}^{2} - 2\bar{\rho}_{VS}\tilde{u}\tilde{v} + \tilde{v}^{2}\right]\right\} d\tilde{v}\, d\tilde{u}$$

$$+ (1-\alpha)\frac{V}{B} \int_{-a-ms_{1'}-(\bar{\rho}_{VS}-m)s_{1'}}^{\infty} \int_{-\infty}^{\frac{b-mp}{\sqrt{1-2\bar{\rho}_{VS}m+m^{2}}}-\delta ms_{1'}-\sqrt{1-2\bar{\rho}_{VS}m+m^{2}}\,s_{1'}} \exp\left\{ -\frac{1}{2(1-\delta^{2})}\left[\tilde{w}^{2} - 2\delta\tilde{w}\tilde{z} + \tilde{z}^{2}\right]\right\} d\tilde{z}\, d\tilde{w}$$

$$+ \frac{S}{B} \int_{-a-s_{S}}^{\infty} \int_{\frac{b-mp}{\sqrt{1-2\bar{\rho}_{VS}m+m^{2}}}-\delta s_{S}}^{\infty} \frac{1}{2\pi\sqrt{1-\delta^{2}}} \exp\left\{ -\frac{1}{2(1-\delta^{2})}\left[\tilde{w}^{2} - 2\delta\tilde{w}\tilde{z} + \tilde{z}^{2}\right]\right\} d\tilde{z}\, d\tilde{w}$$

$$+ (D^{*}-K) \int_{-a}^{\infty} \int_{\frac{b-mp}{\sqrt{1-2\bar{\rho}_{VS}m+m^{2}}}}^{\infty} \frac{1}{2\pi\sqrt{1-\delta^{2}}} \exp\left\{ -\frac{1}{2(1-\delta^{2})}\left[\tilde{q}^{2} - 2\delta\tilde{q}\tilde{r} + \tilde{r}^{2}\right]\right\} d\tilde{r}\, d\tilde{q}$$

Thus, the value of the liabilities can be expressed as:

$$L = \Bigg[(1-\alpha)\frac{V}{B}N_2\left(-a-\bar{\rho}_{VS}s_V,-bb-s_V,\bar{\rho}_{VS}\right)+D^{*}N_2\left(-a,bb,-\bar{\rho}_{VS}\right)$$

$$+(1-\alpha)\frac{V}{B}N_2\left(a+\bar{\rho}_{VS}s_V,\frac{b-mp}{\sqrt{1-2\bar{\rho}_{VS}m+m^2}}-\delta m s_V-\sqrt{1-2\bar{\rho}_{VS}m+m^2}\,s_V,-\delta\right)$$

$$+\frac{S}{B}N_2\left(a+s_{SV},-\frac{b-mp}{\sqrt{1-2\bar{\rho}_{VS}m+m^2}}+\delta s_S,\delta\right)+(D^{*}-K)N_2\left(a,-\frac{b-mp}{\sqrt{1-2\bar{\rho}_{VS}m+m^2}},\delta\right)$$

Where, $N(-,-,-)$ represents a bivariate cumulative normal distribution function.

The above expression represents the value of the call in units of the discount bond. To determine the nominal value of the call we multiply through by B.

$$L = \left[(1-\alpha)VN_2\left(-a-\bar{\rho}_{VS}s_V,-bb-s_V,\bar{\rho}_{VS}\right)+D^{*}BN_2\left(-a,bb,-\bar{\rho}_{VS}\right)\right.$$

$$+(1-\alpha)V\,N_2\left(a+\bar{\rho}_{VS}s_V,\frac{b-mp}{\sqrt{1-2\bar{\rho}_{VS}m+m^2}}-\delta m s_V-\sqrt{1-2\bar{\rho}_{VS}m+m^2}\,s_V,-\delta\right)$$

$$+SN_2\left(a+s_{SV},-\frac{b-mp}{\sqrt{1-2\bar{\rho}_{VS}m+m^2}}+\delta s_S,\delta\right)+(D^{*}-K)BN_2\left(a,-\frac{b-mp}{\sqrt{1-2\bar{\rho}_{VS}m+m^2}},\delta\right)$$

Appendix F

Proof that the Actual VDB is Greater than or Equal to the Approximate VDB when $D^* > K$ and Less then or Equal when $D^* < K$

The Variable Default Boundary is given in Appendix D as:

$$f(\tilde{u}) = \frac{\ln\left(\dfrac{D^* - K + \dfrac{S}{B}\exp\left(-\dfrac{s_S^2}{2} + s_S\tilde{u}\right)}{V}\right) + \dfrac{s_V^2}{2}}{s_V}$$

The first derivative of $f(\tilde{u})$ is:

$$f'(\tilde{u}) = \frac{s_S}{s_V}\left[\frac{\dfrac{S}{B}\exp\left(-\dfrac{s_S^2}{2} + s_S\tilde{u}\right)}{D^* - K + \dfrac{S}{B}\exp\left(-\dfrac{s_S^2}{2} + s_S\tilde{u}\right)}\right]$$

The second derivative of $f(\tilde{u})$ is:

$$f''(\tilde{u}) = \frac{s_S^2}{s_V}\left[\frac{\dfrac{S}{B}\exp\left(-\dfrac{s_S^2}{2} + s_S\tilde{u}\right)}{D^* - K + \dfrac{S}{B}\exp\left(-\dfrac{s_S^2}{2} + s_S\tilde{u}\right)}\right]\left[1 - \frac{\dfrac{S}{B}\exp\left(-\dfrac{s_S^2}{2} + s_S\tilde{u}\right)}{D^* - K + \dfrac{S}{B}\exp\left(-\dfrac{s_S^2}{2} + s_S\tilde{u}\right)}\right]$$

This is a function of the form

$$f''(\tilde{u}) = \frac{s_S^2}{s_V}\left[X(1-X)\right]$$

Where:

$$X = \frac{\frac{S}{B}\exp\left(-\frac{s_S^2}{2}+s_S\tilde{u}\right)}{D^* - K + \frac{S}{B}\exp\left(-\frac{s_S^2}{2}+s_S\tilde{u}\right)}$$

The second derivative is positive if $0 < X < 1$ and negative otherwise. Note that if the second derivative is positive then the Actual VDB will reach a minimum and the Actual VDB is always greater then or equal to the Approximate VDB. The Approximate VDB is the tangent to the Actual VDB at some point "p".

For illustrational purposes, Figure 33 shows the impact of the relative size of D^* and K on the shape of X as a function of $\tilde{u}$. We now consider three cases: $D^* > K$, $D^* = K$ and $D^* < K$.

Case #1: $D^* > K$

Figure 32 shows that X is a continuous function of $\tilde{u}$ that is trapped between 0 and 1. More formally:

$$\lim_{\tilde{u}\to-\infty}[X] = \lim_{\tilde{u}\to-\infty}\left[\frac{1}{\frac{D^*-K}{\frac{S}{B}\exp\left(-\frac{s_S^2}{2}+s_S\tilde{u}\right)}+1}\right] = 0^+$$

$$\lim_{\tilde{u}\to\infty}[X] = \lim_{\tilde{u}\to\infty}\left[\frac{1}{\frac{D^*-K}{\frac{S}{B}\exp\left(-\frac{s_S^2}{2}+s_S\tilde{u}\right)}+1}\right] = 1^-$$

Also, there are no singularities in the function. The denominator of X is always positive since all of the individual terms in the denominator are positive.

Therefore, if $D^* > K$ then:

$X \in (0,1),\ \forall\ \tilde{u} \in (-\infty,+\infty) \Rightarrow f''(\tilde{u}) > 0 \Rightarrow f(\tilde{u}) \geq$ Approximate VDB (i.e. any tangent line)

Case #2: $D^{*} = K$

In this case, $D^{*} = K \;\Rightarrow\; X = 1,\;\; \forall\ \tilde{u} \in (-\infty, +\infty) \Rightarrow f''(\tilde{u}) = 0 \Rightarrow f(\tilde{u})$ is linear

Case #3: $D^{*} < K$

Figure 33 shows that X is a discontinuous function of $\tilde{u}$ with a one singularity that occurs when

$$\tilde{u} = \frac{\ln\left(\frac{B(K - D^{*})}{S}\right) + \frac{s_S^2}{2}}{s_S} = \theta$$

Now:

$$\lim_{\tilde{u} \to -\infty} [X] = \lim_{\tilde{u} \to -\infty} \left[\frac{1}{\frac{D^{*} - K}{\frac{S}{B}\exp\left(-\frac{s_S^2}{2} + s_S\tilde{u}\right)} + 1} \right] = 0^{+}$$

$$\lim_{\tilde{u} \to \theta^{-}} [X] = \lim_{\tilde{u} \to \theta^{-}} \left[\frac{1}{\frac{D^{*} - K}{\frac{S}{B}\exp\left(-\frac{s_S^2}{2} + s_S\tilde{u}\right)} + 1} \right] = -\infty$$

$$\lim_{\tilde{u} \to \theta^{+}} [X] = \lim_{\tilde{u} \to \theta^{+}} \left[\frac{1}{\frac{D^{*} - K}{\frac{S}{B}\exp\left(-\frac{s_S^2}{2} + s_S\tilde{u}\right)} + 1} \right] = +\infty$$

$$\lim_{\tilde{u} \to \infty} [X] = \lim_{\tilde{u} \to \infty} \left[\frac{1}{\frac{D^{*} - K}{\frac{S}{B}\exp\left(-\frac{s_S^2}{2} + s_S\tilde{u}\right)} + 1} \right] = 1^{-}$$

In Summary, if $D^{*} < K$ then:

$$\left. \begin{cases} X \in (-\infty, 0), & \forall\ \tilde{u} \in (-\infty, \theta) \\ X \in (1, \infty), & \forall\ \tilde{u} \in (\theta, \infty) \end{cases} \right\} \Rightarrow f''(\tilde{u}) < 0 \Rightarrow f(\tilde{u}) \leq \textit{Approximate}\ \text{VDB}$$

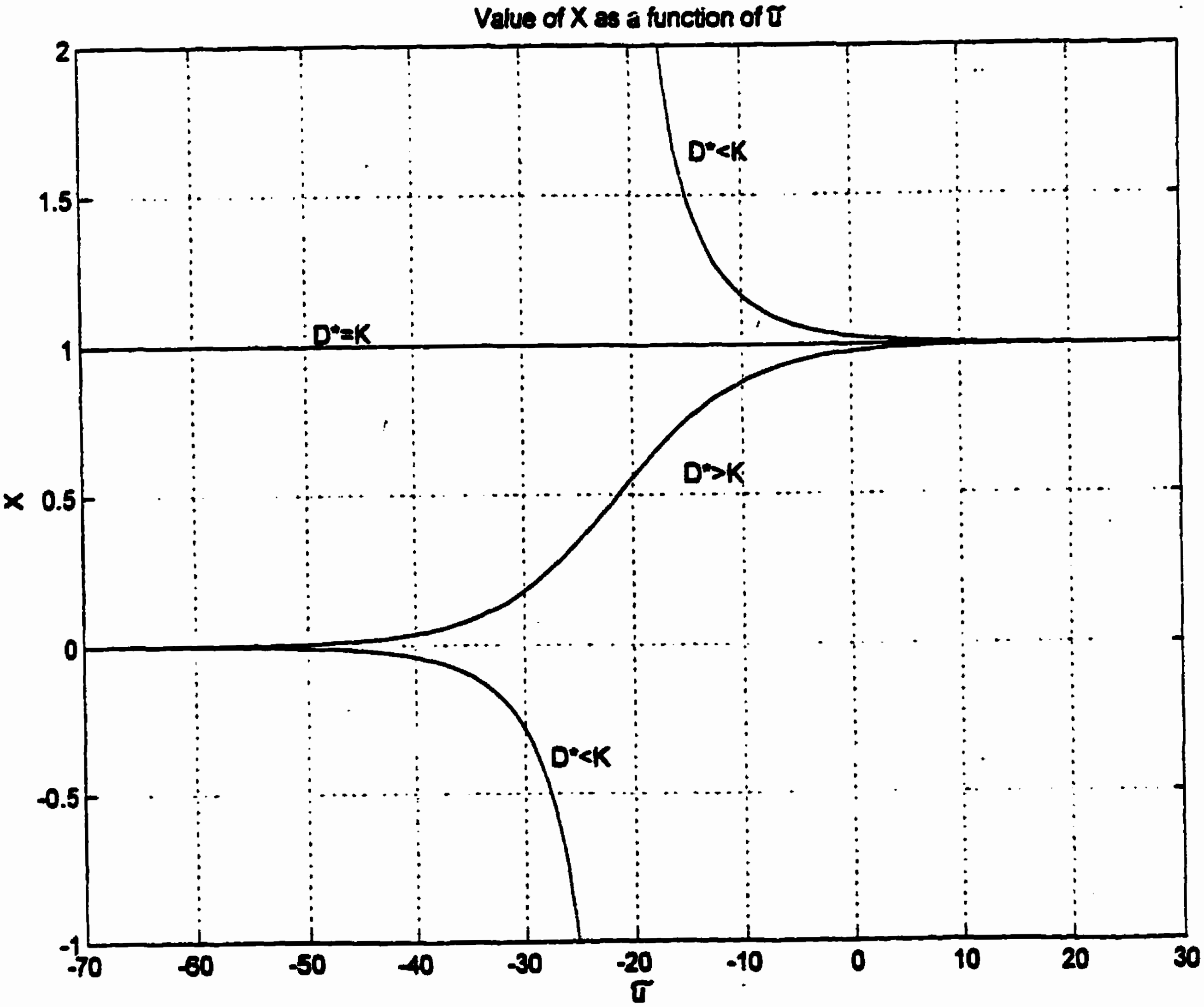
Figure 33

Value of X as a Function of the Random Variable u

Calculations of vulnerable call option prices are based on the following parameter values: $S = 50$, $K = 40$, $V = 50$, $T = 1, \alpha = 0$, $\sigma_V = 0.1$, $\sigma_S = 0.3$, $\rho_{VS} = 0$, $r = 0.05$, $a = 0.5, b = 0.08$, $\sigma_r = 0.03, \rho_{Vr} = 0$, $\rho_{Sr} = 0.0$. The function X is given by:

$$X = \frac{\frac{S}{B}\exp\left(-\frac{s_S^2}{2} + s_S\tilde{u}\right)}{D^* - K + \frac{S}{B}\exp\left(-\frac{s_S^2}{2} + s_S\tilde{u}\right)}$$

The figure shows the impact on X of the relationship between V and D^*. The function X is shown for three different values of D^*: $D^* = 30$, $D^* = 40$ and $D^* = 50$.



Appendix G

Proof that the Actual Hyperbolic Function (Equation 4.4.4.1) is Less than or Equal to its Approximation (Equation 4.4.4.2) when $D^{*} > K$ and Greater Then or Equal to its Approximation when $D^{*} < K$

The actual hyperbolic function that appears in the third and fourth terms of the valuation equation for the VDB vulnerable call option is given by equation 4.4.4.1:

$$F(\tilde{u}) = \frac{1}{D^{*} - K + \frac{S}{B}\exp\left(-\frac{s_S^2}{2} + s_S\tilde{u}\right)}$$

The exponential approximation of this function is derived in appendix D and is given by equation 4.4.4.2:

$$F_{approx}(\tilde{u}) \approx \frac{\exp[g(\tilde{u} - q)]}{D^{*} - K + \frac{S}{B}\exp\left(-\frac{s_S^2}{2} + s_S q\right)}$$

The parameter g is defined in appendix D and given in section 4.4.4.

We wish to show that the actual hyperbolic function is always less than or equal to its exponential approximation whenever $D^{*} > K$ and greater than or equal to its approximation whenever $D^{*} < K$.

This is equivalent to showing that $G(\tilde{u}) = \ln(F(\tilde{u}))$ is less than or equal to $\ln(F_{approx}(\tilde{u}))$ when $D^{*} > K$ and greater than or equal to $\ln(F_{approx}(\tilde{u}))$ when $D^{*} < K$. $G(\tilde{u})$ is given by:

$$G(\tilde{u}) = \ln(F(\tilde{u})) = \ln\left(\frac{1}{D^{*} - K + \frac{S}{B}\exp\left(-\frac{s_S^2}{2} + s_S\tilde{u}\right)}\right)$$

The first derivative of $G(\tilde{u})$ is:

$$G'(\tilde{u}) = s_S \left[\frac{\frac{S}{B}\exp\left(-\frac{s_S^2}{2} + s_S\tilde{u}\right)}{D^* - K + \frac{S}{B}\exp\left(-\frac{s_S^2}{2} + s_S\tilde{u}\right)} \right]$$

The second derivative of $G(\tilde{u})$ is:

$$G''(\tilde{u}) = s_S^2 \left[\frac{\frac{S}{B}\exp\left(-\frac{s_S^2}{2} + s_S\tilde{u}\right)}{D^* - K + \frac{S}{B}\exp\left(-\frac{s_S^2}{2} + s_S\tilde{u}\right)} \right] \left[\frac{\frac{S}{B}\exp\left(-\frac{s_S^2}{2} + s_S\tilde{u}\right)}{D^* - K + \frac{S}{B}\exp\left(-\frac{s_S^2}{2} + s_S\tilde{u}\right)} - 1 \right]$$

This is a function of the form:

$$G''(\tilde{u}) = \frac{s_S^2}{s_V}\left[X(X-1)\right]$$

Where:

$$X = \frac{\frac{S}{B}\exp\left(-\frac{s_S^2}{2} + s_S\tilde{u}\right)}{D^* - K + \frac{S}{B}\exp\left(-\frac{s_S^2}{2} + s_S\tilde{u}\right)}$$

The second derivative is negative if $0 < X < 1$ and positive otherwise. . Note that if the second derivative is negative then $G(\tilde{u})$ will reach a maximum and is always less then or equal to the approximation to $G(\tilde{u})$. The approximate function $G(\tilde{u})$ is the tangent to the actual function $G(\tilde{u})$ at some point "q". Note that the proof from this point forward is exactly the opposite of the proof presented in appendix F

Case #1: If $D^{*} > K$ then:

$$\left.\begin{cases} X \in (-\infty, 0), & \forall\ \tilde{u} \in (-\infty, \theta) \\ X \in (1, \infty), & \forall\ \tilde{u} \in (\theta, \infty) \end{cases}\right\} \Rightarrow G''(\tilde{u}) > 0 \Rightarrow F(\tilde{u}) \text{ the actual hyperbolic}$$

function is less than or equal to its approximation.

Case #2: If $D^{*} = K$ then:

$X \in (0,1),\ \ \forall\ \tilde{u} \in (-\infty, +\infty) \Rightarrow G''(\tilde{u}) = 0 \Rightarrow \mathrm{F}(\tilde{u})$ the actual hyperbolic function is equal to its approximation.

Case #3: If $D^{*} < K$ then:

$X \in (0,1),\ \ \forall\ \tilde{u} \in (-\infty, +\infty) \Rightarrow G''(\tilde{u}) < 0 \Rightarrow F(\tilde{u})$ the actual hyperbolic function is greater than or equal to its approximation.

Appendix H

Probability of Default under FDB Model:

Using risk neutral pricing the risk-neutral probability of default is given by:

$$\Pr[Default] = \Pr[V_T \leq D^*] = \Pr[X_T \leq \ln(D^*)]$$

$$= \int_{-\infty}^{\infty} \int_{\infty}^{\ln(D^*)} \frac{1}{2\pi\sqrt{1-\bar{\rho}_{VS}^{\,2}}\, s_V s_S} \exp\left\{ -\frac{1}{2(1-\bar{\rho}_{VS}^{\,2})} \left[\left(\frac{Y_T - Y + \frac{s_S^2}{2}}{s_S} \right)^2 - \right.\right.$$

$$\left.\left. 2\bar{\rho}_{VS} \left(\frac{Y_T - Y + \frac{s_S^2}{2}}{s_S} \right)\left(\frac{X_T - X + \frac{s_V^2}{2}}{s_V} \right) + \left(\frac{X_T - X + \frac{s_V^2}{2}}{s_V} \right)^2 \right] \right\} dX_T\, dY_T$$

Standardizing the normal distribution and substituting for Y_T *and* X_T, which are defined in Appendix B as,

$$Y_T = \log\left(\frac{S_T}{B_T}\right) \qquad X_T = \log\left(\frac{V_T}{B_T}\right)$$

results in:

$$\Pr[Default] = \int_{-\infty}^{\infty} \int_{-\infty}^{-b_1} \frac{1}{2\pi\sqrt{1-\bar{\rho}_{VS}^{\,2}}} \exp\left\{ -\frac{1}{2(1-\bar{\rho}_{VS}^{\,2})} \left[\tilde{u}^2 - 2\bar{\rho}_{VS}\tilde{u}\tilde{v} + \tilde{v}^2 \right] \right\} d\tilde{v}\, d\tilde{u}$$

where $$b_1 = \frac{\ln\left(\frac{V}{BD^*}\right) - \frac{s_V^2}{2}}{s_V}$$

This can be rewritten as:

$$\Pr[Default] = N_2(\infty, -b_1, -\bar{\rho}) = N_1(-b_1)$$

Where N_2 represents the cumulative bi-variate normal distribution and N_1 represents the cumulative uni-variate normal distribution.

Appendix I

Probability of Default under VDB Model:

Using risk neutral pricing the risk-neutral probability of default is given by:

$$\begin{aligned}\Pr[Default] &= \Pr\left[V_T \le D^* + \max(S_T - K)\right] \\ &= \Pr\left[X_T \le \ln(D^*) \mid e^{Y_T} < K\right] + \Pr\left[X_T \le \ln\left(D^* + e^{Y_T} - K\right) \mid e^{Y_T} \ge K\right] \\ &= \int_{-\infty}^{\ln(K)} \int_{-\infty}^{\ln(D^*)} \frac{1}{2\pi\sqrt{1-\bar{\rho}_{VS}^{\,2}}\, s_V s_S} \exp\{\Omega(Y_T, X_T)\} dY_T dX_T \, + \\ &\qquad \int_{\ln(K)}^{\infty} \int_{-\infty}^{\ln(D^* + e^{Y_T} - K)} \frac{1}{2\pi\sqrt{1-\bar{\rho}_{VS}^{\,2}}\, s_V s_S} \exp\{\Omega(Y_T, X_T)\} dY_T dX_T\end{aligned}$$

Where:

$$\Omega(Y_T, X_T) = -\frac{1}{2(1-\bar{\rho}_{VS}^{\,2})}\left[\left(\frac{Y_T - Y + \frac{s_S^2}{2}}{s_S}\right)^2 - 2\bar{\rho}_{VS}\left(\frac{Y_T - Y + \frac{s_S^2}{2}}{s_S}\right)\left(\frac{X_T - X + \frac{s_V^2}{2}}{s_V}\right) + \left(\frac{X_T - X + \frac{s_V^2}{2}}{s_V}\right)^2\right]$$

Standardizing the normal distribution and substituting for Y_T *and* X_T, which are defined in Appendix B as,

$$Y_T = \log\left(\frac{S_T}{B_T}\right) \qquad X_T = \log\left(\frac{V_T}{B_T}\right)$$

results in:

$$\Pr[Default] = \int_{-\infty}^{-a} \int_{-\infty}^{-b_1} \frac{1}{2\pi\sqrt{1-\bar{\rho}_{VS}^{\,2}}} \exp\left\{-\frac{1}{2(1-\bar{\rho}_{VS}^{\,2})}\left[\tilde{u}^2 - 2\bar{\rho}_{VS}\tilde{u}\tilde{v} + \tilde{v}^{21}\right]\right\} d\tilde{v}\, d\tilde{u}$$

$$+ \int_{-a}^{\infty} \int_{f(\tilde{u})}^{\infty} \frac{1}{2\pi\sqrt{1-\bar{\rho}_{VS}^{\,2}}} \exp\left\{-\frac{1}{2(1-\bar{\rho}_{VS}^{\,2})}\left[\tilde{u}^2 - 2\bar{\rho}_{VS}\tilde{u}\tilde{v} + \tilde{v}^{21}\right]\right\} d\tilde{v}\, d\tilde{u}$$

where

$$b_1 = \frac{\ln\left(\frac{V}{BD^*}\right) - \frac{s_V^2}{2}}{s_V}$$

$$f(\tilde{u}) = \frac{\ln\left(\frac{D^* - K + \frac{S}{B}\exp\left(-\frac{s_S^2}{2} + s_S\tilde{u}\right)}{\frac{V}{B}}\right) + \frac{s_V^2}{2}}{s_V}$$

Next we linerize the non-liner boundary $f(\tilde{u})$ in the second integral by taking a first order Taylor series expansion around the point 'p':

$$f(\tilde{u}) \approx f(p) + f'(p)(u-p) = b + m(u-p)$$

where :

$$b = f(p) = \frac{\ln\left(\frac{D^* - K + \frac{S}{B}\exp\left(-\frac{s_S^2}{2} + s_S p\right)}{\frac{V}{B}}\right) - \frac{s_V^2}{2}}{s_V}$$

and

$$m = f'(p) = \frac{\sigma_S}{\sigma_V}\left[\frac{\frac{S}{B}\exp\left(-\frac{s_S^2}{2}+s_S p\right)}{D^{*}-K+\frac{S}{B}\exp\left(-\frac{s_S^2}{2}+s_S p\right)}\right]$$

Substituting in gives:

$$\Pr[\mathit{Default}] = \int_{-\infty}^{-a}\int_{-\infty}^{-h_1} \frac{1}{2\pi\sqrt{1-\bar{\rho}_{VS}^{\,2}}}\exp\left\{-\frac{1}{2(1-\bar{\rho}_{VS}^{\,2})}\left[\tilde{u}^2 - 2\bar{\rho}_{VS}\tilde{u}\tilde{v} + \tilde{v}^{2}\right]\right\}d\tilde{v}\,d\tilde{u}$$

$$+\int_{-a}^{\infty}\int_{-\infty}^{b+m(\tilde{u}-p)} \frac{1}{2\pi\sqrt{1-\bar{\rho}_{VS}^{\,2}}}\exp\left\{-\frac{1}{2(1-\bar{\rho}_{VS}^{\,2})}\left[\tilde{u}^2 - 2\bar{\rho}_{VS}\tilde{u}\tilde{v} + \tilde{v}^{2}\right]\right\}d\tilde{v}\,d\tilde{u}$$

Next we need to rotate the default boundary (i.e. the limit of integration in the second integral) to eliminate its dependence on the random variable $\tilde{u}$.

Consider the following transformation:

$$\tilde{u} = \frac{1}{\sqrt{1+m^2}}\tilde{x} \qquad \tilde{v} = \tilde{y} + \frac{m}{\sqrt{1+m^2}}\tilde{x}$$

The determinant of the Jacobian of this mapping is $|J| = \frac{1}{\sqrt{1+m^2}}$. Applying this transformation gives:

$$\Pr[Default] = \int_{-\infty}^{-a}\int_{-\infty}^{-b_1} \frac{1}{2\pi\sqrt{1-\bar{\rho}_{VS}^{\,2}}} \exp\left\{-\frac{1}{2(1-\bar{\rho}_{VS}^{\,2})}\left[\tilde{u}^2 - 2\bar{\rho}_{VS}\tilde{u}\tilde{v} + \tilde{v}^{21}\right]\right\} d\tilde{v}\,d\tilde{u}$$

$$+ \int_{-a}^{\infty}\int_{-\infty}^{b-mp} \frac{1}{2\pi\sqrt{1-\bar{\rho}_{VS}^{\,2}}} \Omega(\tilde{x},\tilde{y})\,d\tilde{y}\,d\tilde{x}$$

where:

$$\Omega(\tilde{x},\tilde{y}) = \exp\left\{-\frac{1}{2(1-\bar{\rho}_{VS}^{\,2})}\left[\left(\frac{\tilde{x}}{\sqrt{1+m^2}}\right)^2 - 2\bar{\rho}_{VS}\left(\frac{\tilde{x}}{\sqrt{1+m^2}}\right)\left(y + \frac{m}{\sqrt{1+m^2}}\tilde{x}\right) + \left(y + \frac{m}{\sqrt{1+m^2}}\tilde{x}\right)^2\right]\right\}$$

This expression can be rewritten:

$$\Pr[Default] = \int_{-\infty}^{-a}\int_{-\infty}^{-b_1} \frac{1}{2\pi\sqrt{1-\bar{\rho}_{VS}^{\,2}}} \exp\left\{-\frac{1}{2(1-\bar{\rho}_{VS}^{\,2})}\left[\tilde{u}^2 - 2\bar{\rho}_{VS}\tilde{u}\tilde{v} + \tilde{v}^{21}\right]\right\} d\tilde{v}\,d\tilde{u}$$

$$+ \left(\frac{\sqrt{1-\delta^2}}{\sqrt{1-\bar{\rho}_{VS}^{\,2}}\sqrt{1+m^2}}\right) \int_{-\infty}^{\infty}\int_{b-mp}^{\infty} \frac{1}{2\pi\sqrt{1-\delta^2}} \Psi(\tilde{x},\tilde{y})\,d\tilde{y}\,d\tilde{x}$$

where:

$$\Psi(\tilde{x},\tilde{y}) = \exp\left\{-\frac{1}{2(1-\delta^2)}\left[\left(\frac{\tilde{x}}{\sqrt{1+m^2}}\right)^2 - 2\delta\left(\frac{\tilde{x}}{\sqrt{1+m^2}}\right)\left(\frac{\tilde{y}}{\sqrt{1-2\bar{\rho}_{VS}m+m^2}}\right) + \left(\frac{\tilde{y}}{\sqrt{1-2\bar{\rho}_{VS}m+m^2}}\right)^2\right]\right\}$$

and

$$\delta = \frac{\bar{\rho}_{VS} - m}{\sqrt{1-2\bar{\rho}_{VS}m+m^2}}$$

Using a simple change of variables:

$$\Pr[Default] = \int_{-\infty}^{-a}\int_{-\infty}^{-b_1} \frac{1}{2\pi\sqrt{1-\bar{\rho}_{VS}^{\,2}}}\exp\left\{-\frac{1}{2(1-\bar{\rho}_{VS}^{\,2})}\left[\tilde{u}^2 - 2\bar{\rho}_{VS}\tilde{u}\tilde{v} + \tilde{v}^{21}\right]\right\}d\tilde{v}\,d\tilde{u}$$

$$+\int_{-a}^{\infty}\int_{\frac{b-mp}{\sqrt{1-2\bar{\rho}_{VS}m+m^2}}}^{\infty} \frac{1}{2\pi\sqrt{1-\delta^2}}\exp\left\{-\frac{1}{2(1-\delta^2)}\left[\tilde{w}^2 - 2\delta\tilde{w}\tilde{z} + \tilde{z}^2\right]\right\}d\tilde{z}\,d\tilde{w}$$

This can be rewritten as:

$$\Pr\left[Default\right] = N_2\left(-a,-b_1,-\bar{\rho}_{VS}\right) + N_2\left(a,\frac{-(b-mp)}{\sqrt{1-2\bar{\rho}_{VS}m+m^2}},-\delta\right)$$

Where N_2 represents the cumulative bi-variate normal distribution and N_1 represents the cumulative uni-variate normal distribution.

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