University of Alberta

The Generic Black Hole Singularity

by

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A thesis submitted to the Faculty of Graduate Studies and Research in partial fulfillment of the requirements for the degree of **Doctor of Philosophy**

in

Theoretical Physics

Department of Physics

Edmonton, Alberta
Spring 1997

0-612-21606-3

Preface

During my years as a graduate student at the University of Alberta I have benefitted from collaborations with my supervisor, Werner Israel, and my fellow graduate students, Warren Anderson, Alfio Bonanno, Patrick Brady and Serge Droz. In order to reflect the collaborative nature of the research I have written the thesis in the first person plural.

Chapter 2 is based on

A. Bonanno, S. Droz, W. Israel and S.M. Morsink, *Structure of the inner singularity of a spherical black hole*, Phys. Rev. **D50**, 7372 (1994)

and

A. Bonanno, S. Droz, W. Israel and S.M. Morsink, *Structure of the charged spherical black hole interior*, Proc. R. Soc. Lond. **A 450**, 553 (1995).

Chapter 3 is based on

P.R. Brady, S. Droz, W. Israel and S.M. Morsink, *Covariant double-null dynamics: (2+2)-splitting of the Einstein equations*, Class. Quant. Grav. **13**, 2211 (1996).

Chapter 4 is based on

S.M. Morsink, *Gravitational radiation, Cauchy horizons and the dynamics of the mass function*, to be submitted to Phys. Rev. D.

Chapter 5 is based on

P.R. Brady, S. Droz and S.M. Morsink, *Mass inflation in non-spherical black holes*. to be submitted to Phys. Rev. D.

Chapter 6 is based on

W.G. Anderson, P.R. Brady, W. Israel, S.M. Morsink, *Quantum effects in black hole interiors*, Phys. Rev. Lett. **70**, 1041 (1993).

Figures 1, 2, 7, 8 and 9 were drawn by Serge Droz, who has kindly given me permission to use these figures with some alterations in this thesis.

Abstract

The gravitational collapse of a rotating star to a black hole generically produces a weakly decaying gravitational wave tail which partially falls into the black hole. Although the influx of gravitational radiation is weak, its backreaction onto the black hole's interior geometry becomes significant near the black hole's inner horizon. The inner horizon of Kerr is a Cauchy horizon (CH), a lightlike hypersurface of infinite blueshift, behind which lies a tunnel to other universes. At CH the influx of gravitational radiation is infinitely amplified by the geometry indicating that CH is unstable. Past work on a spherically symmetric model of the interior has revealed a relatively simple scenario [1]. When the influx is combined with any nominal outflux, the effect of backreaction is to cause the black hole's local mass function to diverge exponentially at CH. Since the Weyl curvature is completely determined by the mass function in spherical symmetry, this signals the appearance of a lightlike observer-independent curvature singularity at CH, which effectively seals off the CH tunnel.

In this thesis we investigate whether the mass inflation picture can be extended to the non-spherical black hole interior. To do so, we solve the characteristic initial value problem for a general metric in the region near CH. Given initial conditions which correspond to the scattering of the incoming gravitational wave tail by the interior Kerr geometry, we show that generically a lightlike singularity forms at CH. The general solution exhibits some features similar to the spherically symmetric solution, for the leading order divergences of Hawking's quasi-local mass function and the Kretschmann invariant are the same. However, in the general solution, all components of the Weyl tensor diverge, and the solution is not algebraically special. To leading order, the solution is very closely approximated by a colliding plane gravitational wave metric. This leads to the following heuristic picture: the infalling gravitational

radiation is scattered by the black hole's interior curvature into two cross-flowing streams of gravitational radiation which are approximately plane symmetric. The interaction of the two streams in the region of high blueshift strongly focuses light rays and a curvature singularity results.

Acknowledgements

The fact that my five years of research at the University of Alberta have been a rewarding and enjoyable experience is due to the warm and friendly atmosphere in the Department of Physics and to my supervisor, Werner Israel. Werner, it has been a great pleasure to have had the opportunity to work with, and learn from you. Your enthusiasm for physics has been inspiring and contagious. I hope that you and Inge have many decades of health and happiness in your new home in Victoria.

During my years in the department I have had the opportunity to collaborate on many research projects with Werner and with my fellow graduate students Warren Anderson, Alfio Bonanno, Pat Brady and Serge Droz. Working with all of you has been great learning experience and a lot of fun as well! I would also like to thank all the graduate students, post-docs, professors and visitors who have discussed physics with me. An incomplete list includes Warren Anderson, Herbert Balasin, Alfio Bonanno, Pat Brady, Steve Braham, Bruce Campbell, Bahman Darian, Sacha Davidson, Serge Droz, Valery Frolov, Shaun Hendy, Viqar Husain, Tomas Kopf, Pavel Krtous, Robb Mann, Erik Martinez, Dario Nunez, Don Page, Eric Poisson, Tom Steele, Robert Thacker, and Andrei Zelnikov. Discussions with all of you have helped shape my knowledge of physics. I would especially like to thank Robb Mann for encouraging me to work in the field of general relativity as an undergraduate and for suggesting the problems in lower dimensional gravity which gave me the confidence to pursue a graduate degree.

I am most grateful to June Bhatia for initiating the Avadh Bhatia fellowship and for agreeing to allow me to be the first person to hold the fellowship. Thank you for your interest in supporting the research of women physicists.

I would like to thank my family for their love and support. Dad, thank you for all your questions about physics and the interest that you showed in my research. Mom, thanks for insisting that I take physics in high school. To my "little" sister, Coreen, thanks for being a great friend and correspondent. To Bubby Toby and (belatedly) to Zeidy Moe thanks for caring about me throughout my life. And, since I like to save the best for the end, thanks to Erik Talvila for making the last three years of my life so happy.

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Chapter 1

Introduction

It is embarrassing that a direct consequence of the laws of general relativity is a prediction that it is inevitable for regions to form which general relativity is unable to describe: the complete gravitational collapse of a star produces the ultimate physical blemish, a spacetime singularity. It was once thought that the appearance of a singularity was an artifact of idealized boundary conditions, such as the assumption of spherical symmetry. However, the singularity theorems of classical general relativity [2] prove that the formation of a singularity is a generic feature of gravitational collapse. This has dire consequences for the notion of predictability, since there is no unique prescription for placing initial data on a singularity.

The situation for physics may not be as bad as we have made it sound if Penrose's cosmic censorship conjecture [3] is true. The strong cosmic censorship conjecture (SCCC) states: it is impossible for any observer to view the singularity formed from gravitational collapse. Since all observers are out of causal contact with the singularity, predictability is preserved. However, SCCC is most certainly not true, as several counter-examples, such as the shell-crossing singularities [4] have been found. A more plausible view is that a weaker conjecture which places a physical restriction on the collapsing body is probably true. An example of weak cosmic censorship (WCC) is the hoop conjecture [5] (HC): an event horizon forms if and only if the circumference, in all directions, of a collapsing body is less than $4\pi Gm/c^2$ (where m is the mass of the body). While HC hasn't been proved, numerical experiments [6] strongly suggest that it is true.

If some form of WCC is true then any reasonable collapse will result in a black hole. A black hole is a region of spacetime from which no information can escape.

The black hole necessarily contains a singularity, but observers outside the hole are shielded from the singularity by an event horizon. However, inside the Kerr black hole there are regions where it is possible to view the singularity. Clearly the assumption of WCC has just swept the problem of singularities under the rug. It is interesting to ask, does the assumption of WCC imply the validity of SCCC inside the black hole. once a realistic model of collapse is considered? The main goal of this thesis is to answer this question, by providing a description of the singularity which generically forms when a rotating star collapses to a black hole.

1.1 Spherically symmetric gravitational collapse

It is constructive to begin with a discussion of spherically symmetric collapse without perturbations [7]. The gravitational collapse of a star appears to take an infinite amount of time, as viewed by a static observer far from the star. However, observers freely falling in with the star's surface measure the time taken to collapse to be finite. The disparity in the observers' view of the collapse is due to the presence of the event horizon, a limiting null hypersurface after which the gravitational field is so strong that no emitted light can escape its surface. Light emitted by the star when the surface is at a radius larger than the event horizon can be observed by the static observer. But the light must first climb out of the steep gravitational well. The time needed for light to travel through the distance is much longer than if the potential well were not present. In fact, the time approaches infinity when the light is emitted at the event horizon. The light, of course, still travels at the same speed, but the strong gravitational field has altered the geometry of the spacetime and effectively "slows down" the light seen by static external observers. The slowing down also has the effect that the frequency of the light is exponentially redshifted, so that at late times, the only light observed from the collapsing star is of very low frequency.

The causal structure of the spherical black hole's interior depends on two parameters: its mass (m) and electric charge (e). The simplest black hole is the spherically symmetric Schwarzschild solution, which has $e = 0$. The Schwarzschild solution (see figure 1) has a strong all-encompassing spacelike singularity hidden behind its event horizon. A traveler (grey world line) inside the black hole finds the journey to be relatively tame (except for tidal forces) until a certain time (the zig-zag line at $r = 0$) at which the singularity appears at all spatial points. The traveler has no news of the existence of the singularity lying to her future and is unable to avoid running into it. Since she never views the singularity, the Cauchy (or initial) value problem is well posed and predictability is preserved in the Schwarzschild interior.

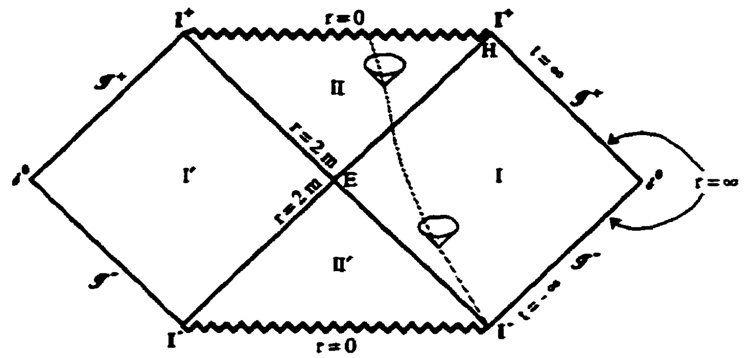

Figure 1: Conformal diagram of the Schwarzschild black hole. On a conformal spacetime diagram, time increases upwards, the world-lines of light are inclined 45° to the vertical, and each point corresponds to a 2-sphere of radius r. Region I represents an asymptotically flat spacetime exterior to a black hole, where light rays can escape to infinity. The line segment EH at $r = 2m$ is the event horizon of the black hole. Region II represents the black hole interior, $0 < r < 2m$. All light signals emitted in this region are trapped: they must move to smaller r and ultimately intersect the singularity. Region I' is another asymptotically flat universe similar to, but causally disjoint from region I. Region II' is the interior of a white hole. In the diagram of a black hole formed from the collapse of a star, regions I' and II' are omitted.

The causal situation is not as straightforward if the black hole has even the smallest amount of electric charge. Consider the charged, spherically symmetric Reissner-Nordström solution. The energy of the electric field acts as a source for Einstein's equations and effectively causes gravity to be repulsive at small radii. The result is an inner horizon (see figure 2), called a Cauchy horizon (CH), behind which lies the black hole's singularity and a tunnel to another universe (region V). The gravitational repulsion causes the singularity to be timelike, so it is possible for the traveler to avoid running into the singularity and successfully navigate through the tunnel. However, the singularity can causally influence every point in the tunnel (regions III' and IV) after the Cauchy horizon. We have no way to predict what will come out of the singularity and so the predictive power has been lost in the region after the Cauchy horizon.

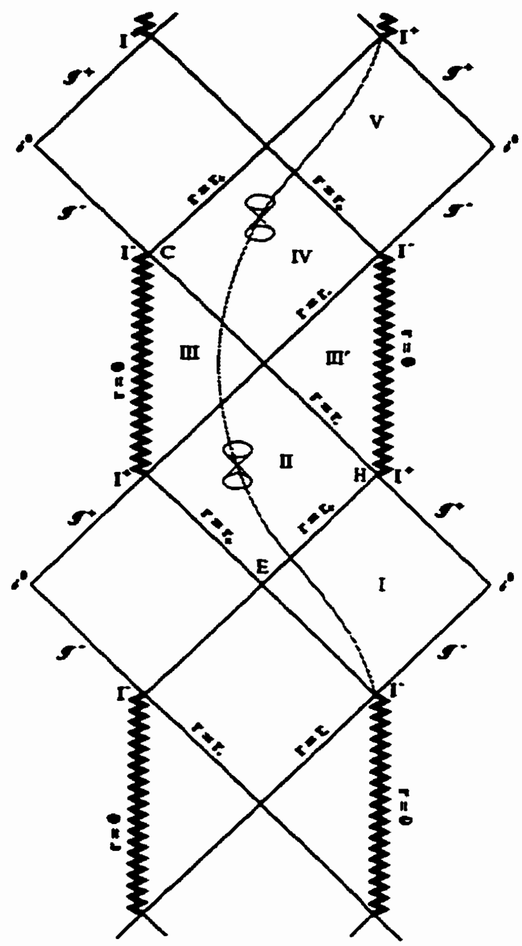

Figure 2: Conformal diagram of the Reissner-Nordström black hole. The analytic extension of the Reissner-Nordström solution corresponds to an infinite chain of universes linked by charged black holes. The diagram to the left of the grey curve should be omitted to describe the collapse of a charged star.

An observation due to Penrose [8] may provide a mechanism for a restoration of predictability to the interior. The journey from the event to Cauchy horizon takes a finite amount of the traveler's proper time (on the order of 30 minutes for a $10^8 M_\odot$

black hole). During this time she will receive signals from the outside world originating over an infinite period of external time. Thus the incoming signals will be blueshifted infinitely. It is possible that the backreaction of the blueshifted radiation could cause a curvature singularity to form at CH, sealing off the tunnel and the locally naked singularity from view. Our aim is to investigate the effect of perturbations on the black hole interior formed by a generic collapse and to determine whether CH is stable.

1.2 Gravitational collapse with perturbations

The analysis of the black hole's interior region, hidden from view by an event horizon, is a simpler proposition than the study of a star's interior. A complete description of a star requires a knowledge of the complicated stellar physics occurring deep in the central region which influences the outer regions. Causality simplifies the analogous problem for black holes, since the radius (r) is a timelike coordinate inside the event horizon. Thus layers lying at small radius actually occur in the future of the outer layers. The region of strongest gravity near the centre, where an unknown quantum theory of gravity is needed for a full description, can't causally influence regions at larger values of r. The study of the black hole's interior is essentially reduced to an evolutionary problem, with "initial" data placed on the event horizon. Remarkably, the initial data which correspond to the isolated collapse of a star are known with great precision, so the evolutionary problem is well defined.

The initial data in which we are interested, effectively amounts to the information about the initial gravitational field of the star. The star's field, while predominantly spherical, contains contributions from higher multipole moments with $\ell \geq 2$. However the existence of the black hole no-hair theorems [9, 10] leads to the conclusion that the higher multipole moments must be radiated away when a black hole forms. But the gravitational radiation emitted near the event horizon is, like light, highly redshifted and can't propagate freely on the black hole's geometry. The outgoing gravitational radiation is partially reflected and transmitted by the black hole's effective gravitational potential barrier. The final result first shown by Price [11], is a weak flux, decaying in time as an inverse power law, which is transmitted through the barrier and can be detected by external observers. The flux of radiation which is backscattered by the barrier has a similar power law decay and falls into the event horizon. It is the consequences of this weak influx on the interior geometry which will be investigated in this dissertation.

The effect of the gravitational wave influx on the interior geometry depends on the black hole's parameters. The weakness of the influx guarantees that in the Schwarzschild interior, its effect is negligible, leading to an asymptotic ($v \to \infty$) no-hair property for the Schwarzschild interior [12, 13]. On the other hand, inside the Reissner-Nordström black hole, the weak gravitational wave tail is amplified by an exponential blueshift factor at CH. The first backreaction calculation, performed on the background of a spherical charged black hole [14], showed that certain components of the Ricci curvature tensor diverge at CH, although all curvature invariants remain finite. The result is an observer dependent singularity. The mildness of this singularity is unstable, for if any small amount of radiation crosses the Cauchy horizon (such as light emitted from the surface of a collapsing star) a lightlike scalar curvature singularity forms [1]. The lightlike nature of the singularity guarantees that the black hole interior will be completely predictable up to the singularity by the laws of physics. One of the more dramatic features of the solution is the prediction that the locally measured mass of the black hole inflates exponentially as CH is approached. For this reason the solution has been dubbed "mass inflation". No news of the increase of mass can escape from the event horizon to the outside world, where the black hole's mass is measured to be slightly smaller than the original star's.

The spherically symmetric mass inflation solution is only a toy model of the singularity formed by the collapse of a rotating star. But, it seems likely that it is a good model of a realistic black hole interior, since the causal structure of the charged spherical black hole is similar to the stationary Kerr black hole. (The conformal diagram for Kerr, restricted to the axis of symmetry, is similar to figure 2, except that it is possible to extend the spacetime in regions III and III' through $r = 0$ to a negative mass universe (not pictured).) In this dissertation, the backreaction of gravitational perturbations on the non-spherical black hole interior will be calculated. We find that generically, a lightlike scalar curvature singularity forms at CH. While we can show that Hawking's quasi-local mass diverges, the mass no longer completely determines the curvature in a non-spherical spacetime. More geometrically, we show that all five of the Weyl scalars ($\Psi_0, ..., \Psi_4$) diverge at CH in the tetrad which we introduce in chapter 3. Close to the singularity, the dominant behaviour of the spacetime is similar to that of a colliding plane gravitational wave metric. The singularity is of an integrable form, similar to the spherically symmetric solution, which implies that tidal distortions will be finite at CH. We find no evidence of a stronger spacelike oscillatory BKL [15] singularity forming at an earlier time. The classical description given here is not complete, since near CH, when the Weyl curvature approaches Planckian levels, quantum effects will become important. We calculate the one-loop expectation value of the stress tensor for quantized fields propagating on the spherical mass inflation background, in order to make an estimate of the backreaction.

1.3 Overview of the dissertation

Chapter 2: The spherical black hole interior

In this chapter, we present an overview of the physics of the charged spherical black hole interior. The chapter is partially a review of past work, focusing on the spherically symmetric mass inflation model [1], in which gravitational radiation is modelled by a null fluid. In the solution of the backreaction model, the black hole's mass function diverges as

$$m \sim f(U)|\ln(-V)|^{-p}/(-V)\ ,V \to 0_-\ ,$$

where V is the Kruskal advanced time for the inner horizon, and takes the value $V = 0$ at CH. The function f is arbitrary, and p is a positive constant, which for gravitational radiation is $p = 12$. In the presence of a cross-flowing null fluid on a charged spherically symmetric background, the only non-zero component of the Weyl tensor is

$$\Psi_2 = \frac{1}{r^2}\left(\frac{m(U,V)}{r} - \frac{e^2}{2r^2}\right),$$

so it follows that the divergence of the mass function signifies the formation of an observer independent curvature singularity. Since the first integral of Ψ_2 is finite, the tidal distortions at the singularity are finite. In this sense the singularity is weak.

We review this work since it will serve as a qualitative guide for the physics of the non-spherical black hole interior. As well, we close up a loop-hole present in the original spherical analysis, by presenting a model of the scattering of radiation within the black hole's interior.

Chapter 3: Double-null dynamics

We wish to find a general solution to the Einstein field equations in a region close to the black hole's Cauchy horizon. In order to meet this goal we need to make use of a formalism which is ideally suited for calculations involving lightlike hypersurfaces. In this chapter we will present a new formalism developed for a foliation of spacetime by two intersecting families of null hypersurfaces. The intersection of the families forms a two parameter collection of two dimensional spacelike surfaces, so this can be pictured as an evolution of the geometry on a surface in either of the lightlike directions normal to the surface. In our notation, the general line element takes the form

$$ds^2 = -2e^{\lambda}du^0du^1 + g_{ab}(d\theta^a + s^a_A du^A)(d\theta^b + s^b_B du^B)$$

where capital Latin indices take values (0,1), while lower case Latin indices range over (2,3). The coordinates u^0 and u^1 are null while the two coordinates θ^a are spacelike. We assume in our formulae that the families of lightlike hypersurfaces are hypersurface orthogonal, so that their generators are proportional to gradients of the null coordinates u^A. This allows us to write the Einstein field equations in a concise manner in terms of two dimensionally covariant quantities. The equations presented in this chapter will be used extensively in the next two chapters to present a general picture of the generic black hole singularity.

Chapter 4: Dynamics of the mass function

In spherical symmetry, the black hole's mass function plays a special role, since the mass and the circumferential radius function completely determine the curvature of the spacetime. In a non-spherical spacetime the equivalence principle forbids the measurement of a local gravitational mass. It is possible, though, to define a quasi-local mass (such as Hawking's definition) which is an average of a local mass aspect function m_H over a spacelike surface. Hawking's quasi-local mass has many qualities which are similar to energy, so it is interesting to investigate its properties inside the black hole. In particular, we would like to know if there is an analogous mass inflation phenomenon inside a non-spherical black hole. We derive formulae describing the dynamics of Hawking's mass aspect function including the following integral equation which holds near CH:

$$m_H(U,V,\theta^a) = \int dU \int dV \left(\frac{1}{2}(\det g)^{3/4} e^{-\lambda} |\sigma_V|^2 |\sigma_U|^2 + m_H(U,V,\theta^a)\sigma_{Uab}\sigma_V^{ab}\right) + O(1) .$$

where U, V are Kruskal coordinates for the Cauchy horizon, defined so that $V = 0$ at CH and the functions λ and $\det g$ are the metric functions defined in chapter 3. The shear tensors σ_{Aab} represent the gravitational perturbations. Without any knowledge of the rest of the thesis, it is possible to deduce qualitatively the existence of mass inflation from this equation. Consider metric perturbations of the Kerr black hole. In Kruskal coordinates the metric functions e^λ and $\det g$ are non-zero constants at CH. The gravitational perturbations are expected to have the Price form $|\sigma_V|^2 \sim |\ln(-V)|^{-p}/V^2$, so that if the scattered perturbations $|\sigma_U|^2$ are non-zero, and the first term on the right-hand side of the equation dominates over the second term, the mass aspect takes the form

$$m_H \sim f(U,\theta^a)|\ln(-V)|^{-p}/(-V)\ , V \to 0_-\ ,$$

where f is an arbitrary function. The form of the divergence is identical to the divergence of the mass function found in spherical symmetry. We leave the details of proving that this is the behaviour for the mass function to the succeeding chapter, where the characteristic initial value problem for the black hole interior will be solved.

Chapter 5: The generic black hole interior

Our aim is to describe the general black hole singularity formed by the gravitational collapse of a rotating star. Before doing so, we present a simple model of colliding plane gravitational waves which captures the essential features of the general solution. We prove a key theorem for the plane wave metric which states that if the initial data for the gravitational perturbations are of the Price power law form, the evolution of the perturbations through the Einstein equations preserves the power law form at later times. The results of the theorem are also applicable in the general black hole spacetime.

The main result of this chapter is an approximate general solution to the Einstein field equations (using the metric of chapter 2) near the Cauchy horizon. At leading order (in our expansion) the solution is very well approximated by the simple plane wave model. We find that a lightlike curvature singularity forms at CH. The square of the Weyl tensor diverges in a similar fashion as in the spherical model, but now all of its components diverge, so the solution is not algebraically special. However, the components are integrable, so the tidal distortions are finite, and the volume is non-zero, so the singularity is not as strong as the spacelike Schwarzschild or BKL singularities.

Chapter 6: Quantum effects in the black hole interior

The classical picture of the black hole singularity presented in the previous chapters is not a complete description. When curvatures grow large, we expect that, through the uncertainty principle, elementary particles will be produced and vacuum polarization will contribute to the stress tensor. In this chapter we investigate the backreaction of one-loop quantum effects on the classical black hole interior. The calculation of the renormalized quantum stress tensor is very difficult so, to simplify the problem, we have assumed spherical symmetry. We have noted in chapter 5 that the general black hole interior is qualitatively similar to the charged spherical black hole, so our results should be indicative of the behaviour of quantum fields in the non-spherical black hole. We show that the Ori model [16] of mass inflation (section 2.3) can be approximated by a simpler metric conformal to a spacetime which is a linear perturbation of Minkowski space. This allows us to use Horowitz's formula [17] for the renormalized stress tensor for massless quantum fields in a classical background described by linearized gravity. The result is then conformally transformed, using Page's formula [18], to the physical spacetime. Our calculations show that the quantum corrections to the stress tensor diverge faster than the classical contributions. The sign of the corrections depends on the renormalization mass scale which is not fixed. Hence we can not predict whether the quantum effects tend to strengthen or weaken the mass inflation singularity. The strength of the correction terms leads us to conclude that the classical picture presented in the preceding chapters is only accurate down to Kruskal advanced times of the order $V \sim \hbar/r_-$, at which time the quantum effects will become dominant.

Chapter 2

The spherical black hole interior

At first glance the analysis of the interior geometry of a black hole formed from the collapse of a rotating star may seem a formidable challenge. However, accumulating evidence [19, 20] suggests that after a rotating star collapses, the exterior spacetime asymptotically relaxes to a stationary Kerr-Newman state in a manner similar to the relaxation to a static case after a non-rotating but aspherical collapse. Although there is no no-hair theorem for the interior of a rotating black hole, it seems likely that the Kerr-Newman solution may act as a rough model for the general black hole interior. The causal structure of the Kerr-Newman interior is similar to the spherically symmetric Reissner-Nordström interior, so it is pedagogically useful to first understant the simpler spherically symmetric problem. In this chapter we will review the spherically symmetric solution of the black hole interior so that it may act as a guide to the general situation where no symmetries are assumed.

The uniqueness theorems [9, 10] state that the exterior geometry of an isolated black hole is described by the three parameter Kerr-Newman solution. The causal structure of the rotating Kerr-Newman solution is essentially the same as that of the simple spherical charged Reissner-Nordström black hole. When both solutions are analytically extended past the event horizon into the interior, the same causal features appear: a timelike singularity which is preceded by an unstable Cauchy horizon. For this reason the assumption of spherical symmetry can be used as a crude model of the physics of the more general problem. Since the physics in the exterior of a spherically symmetric black hole is known with great precision [11] it is possible to calculate the effects of the backreaction of perturbations onto the geometry. In a non-rotating but aspherical gravitational collapse to form a spherical charged black hole, the backscattered gravitational wave tails enter the black hole [21, 22] and are blueshifted at the Cauchy horizon. This has a catastrophic effect if combined with an outflux crossing the Cauchy horizon: a null scalar curvature singularity develops and the effective mass inflates [1].

This chapter is organised as follows. In the first section, the static Reissner-Nordström solution is reviewed. The aspherical collapse of a non-rotating star is discussed in section 2. Ori's model [16] of the black hole interior is presented in section 3. In section 4 the field equations for spherical symmetry are presented. The Poisson-Israel model [1] of the interior is reviewed in section 5 along with a discussion of problems with the initial conditions. The resolution [23, 24] of the problem of the initial conditions is presented in sections 6, 7 and 8. Section 6 is concerned with scattering in the interior. Sections 7 and 8 present an analytic approximation for lightlike crossflowing dust and a massless scalar field respectively. In the concluding section some speculations are made about the later evolution of the singularity, with reference to numerical results.

2.1 The Reissner-Nordström black hole

The Reissner-Nordström metric is the unique spherically symmetric solution to the Einstein equations coupled to an electric field and describes the exterior region of a charged star or of a black hole. The metric is

$$ds^2 = -f_s(r_s)dt^2 + \frac{1}{f_s(r_s)}dr_s^2 + r_s^2 d\Omega^2 \ , \tag{2.1}$$

where $d\Omega^2$ is the line element of the unit sphere, and the function f_s is defined by

$$f_s(r) = 1 - \frac{2m_0}{r} + \frac{e^2}{r^2} \, . \tag{2.2}$$

The subscript 's' refers to static. The constant m_0 is the mass of the black hole, which can be measured by using Kepler's third law outside the hole, and e is the electric charge of the hole. Setting $e = 0$ results in the uncharged Schwarzschild solution. The roots of f_s,

$$r_\pm = m_0 \pm \sqrt{m_0^2 - e^2}$$

correspond to the event horizon ($r_s = r_+$) and the Cauchy horizon ($r_s = r_-$) of the black hole. The positive surface gravities of the horizons are

$$\kappa_\pm = \frac{r_+ - r_-}{2r_\pm^2} .$$

The tortoise coordinate r_*, defined by

$$r_* = \int \frac{dr}{f_s} = r_s + \frac{1}{2\kappa_+} \ln|r_s - r_+| - \frac{1}{2\kappa_-} \ln(r_s - r_-) ,$$

is used to define the Eddington-Finkelstein null coordinates

$$v_{ex} = t + r_* \ , \ u_{ex} = t - r_*$$

which reduce to the usual advanced and retarded times far from the black hole. At the event horizon, there is a coordinate singularity, where $u_{ex} \to \infty$. Clearly these coordinates are only good in the exterior (region I of figure 2) of the black hole where the metric is

$$ds_{ex}^2 = -f_s du_{ex} dv_{ex} + r_s^2 d\Omega^2 .$$

Observers near the event horizon of the black hole do not notice anything particularly special about the event horizon. In the coordinate system used by the freely falling observers the metric components are finite and non-zero. The retarded time used by freely falling observers, U_{ff} is related to observers at infinity by

$$U_{ff} = e^{-\kappa_+ u_{ex}} .$$

Thus, observers far from the black hole measure signals sent by freely falling observers just outside the event horizon to be infinitely redshifted.

If we wish to extend the Reissner-Nordström solution into the interior of the black hole (region II of figure 2), we need to introduce a new coordinate patch, defined by

$$v = t + r_* \ , \ u = r_* - t . \tag{2.3}$$

The new coordinate u ranges from $[-\infty, \infty]$ as r_s ranges from (r_+, r_-). The metric in the interior is

$$ds_{in}^2 = f_s du dv + r_s^2 d\Omega^2 . \tag{2.4}$$

There are coordinate singularities on the two horizons, which can be removed locally by using Kruskal coordinates which are well behaved near each horizon. It should be noted that it is impossible to find coordinates which are well behaved everywhere in the interior. Instead it is only possible to use coordinates that are good in a region close to either horizon. Since the horizons are physically separated, this is not a problem. Kruskal coordinates, U_+ and V_+ for the event horizon are defined by

$$\kappa_+ U_+ = e^{\kappa_+ u} \ , \ \kappa_+ V_+ = e^{\kappa_+ v} \ . \tag{2.5}$$

The event horizon is the hypersurface $U_+ = 0$. Near the event horizon,

$$f_s(r) = -2e^{\kappa_+(u+v)}$$

so the metric near the event horizon is

$$ds^2 = -2dU_+ dV_+ + r_s^2 d\Omega^2$$

which is finite.

Similarly, Kruskal coordinates for the Cauchy horizon are

$$\kappa_- U = -e^{-\kappa_- u} \ , \ \kappa_- V = -e^{-\kappa_- v} \ . \tag{2.6}$$

Near the Cauchy horizon

$$f_s = -2e^{-\kappa_-(u+v)} \tag{2.7}$$

and the metric is

$$ds^2 = -2dUdV + r_s^2 d\Omega^2. \tag{2.8}$$

Consider radiation entering the black hole as in figure 3. The duration of the radiation is Δv as measured by observers far from the black hole. Inside the hole near the Cauchy horizon, local observers use the well behaved Kruskal coordinate to measure the duration of the influx, ΔV. From equation (2.6), the differential relation

$$dV = e^{-\kappa_- v} dv$$

corresponds to an infinite blueshift of the radiation as $v \rightarrow \infty$.

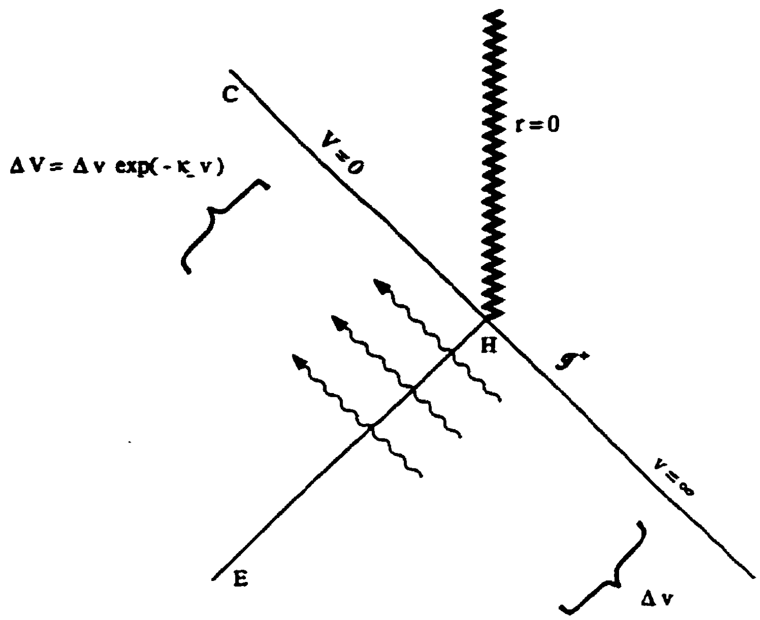

Figure 3: Radiation entering the Reissner-Nordström black hole. This diagram is a close-up of figure 2 showing radiation entering EH and traveling parallel to CH. The light is emitted during a period of external advanced time Δv. An observer inside the hole receives the signals during a period of internal advanced time ΔV.

Similarly, the metric can be extended beyond the Cauchy horizon, into region III' of figure 2. In this region, lies a timelike curvature singularity at $r = 0$. The singularity is locally naked: for any point in III', the singularity intersects its past lightcone. this is related to the breakdown of the Cauchy problem. Initial data placed on a Cauchy surface outside the black hole can only be evolved into the interior as far as the Cauchy horizon (CH). Events in region III' can only be predicted if initial data are placed on the singularity. Since no unique prescription for placing initial data on a singularity exists, it seems that this picture is seriously flawed. Penrose's observation that CH is a surface of infinite blueshift [8] suggests that region III' is unphysical: small perturbations outside the black hole will appear enormous to observers at CH, who will measure an infinite energy density[25, 26, 27]. As a result, it is expected that when the backreaction onto the geometry is taken into account perturbations will cause CH to be unstable. As a result either a null singularity will form at CH or a spacelike singularity will form before CH.

2.2 Aspherical collapse of a non-rotating star

Stars which are unable to produce an outward pressure are unstable due to gravity and collapse to form a singularity. If we assume the unproven, but plausible, cosmic censorship conjecture [3], then this singularity is hidden behind the event horizon of a black hole. The earliest description of the gravitational collapse to a black hole assumed that the collapsing star is spherical [7]. Real stars are, of course, not perfectly spherical: rotation will tend to flatten them into spheroids and the complicated stellar physics creates flares and other density perturbations. However, a slowly rotating star can be roughly approximated by a multipole expansion in spherical harmonics.

One might guess that the collapse of an isolated aspherical star would produce an aspherical static black hole, but this is not the case! The no hair theorems state that no information about the distribution of matter in the original star can be obtained after a black hole has formed and settled down to a stationary state. The only information available is its mass, electric charge and angular momentum. In particular, a static black hole must be spherically symmetric [9]. From Birkhoff's theorem, a static black hole must then be described by the Schwarzschild metric if it is in vacuum, or by the Reissner-Nordström metric if a static electric field is present. We are now left with the puzzle of describing how a non-rotating but aspherical star collapses to form a smooth perfectly spherical black hole.

A direct result of Birkhoff's theorem is that there can be no spherically symmetric gravitational radiation. The theorem states that the only spherically symmetric vacuum solution to Einstein's equations is the Schwarzschild solution. Since the Schwarzschild solution is static, and gravitational radiation is a vacuum phenomenon, it follows that a purely gravitational system can not radiate away the monopole component of its field. Gravitational fields have no dipole component, so this leads us to the conclusion that gravitational radiation is quadrupolar. A general physical principle is that anything that can be radiated must be radiated. As a result, if the star's original gravitational field was of the form

$$\Phi(t,r,\theta,\phi) = \Sigma_{l,m} \int_{-\infty}^{\infty} d\omega e^{-i\omega t} Y_{lm}(\theta,\phi)\psi_{lm\omega}(r)/r, \tag{2.9}$$

where Y_{lm} are the spin-weighted spherical harmonics, then as the star collapses, the perturbations with $l \geq 2$ will be radiated away [28]. The perturbations are not able to propagate freely, however, for they feel the gravity of the collapsing star. The effect of the gravity is studied by examining the wave equation of the perturbation. The wave equation is separable, so that each l-pole is described by a one dimensional wave equation, which for a scalar field is

$$\begin{aligned} \partial_u \partial_v \psi_{lm\omega} &= V_l(r)\psi_{lm\omega} \\ V_l(r) &= -\frac{(r_+ - r)(r - r_-)}{r^2}\left(\frac{l(l+1)}{r^2} + \frac{2m_0}{r^3} - \frac{2e^2}{r^4}\right), \end{aligned} \tag{2.10}$$

with the effect of gravity represented by the potential barrier $V_l(r)$. The potential for higher spin fields is similar to (2.10) and only differs by terms of order $1/r^5$. This is now a problem which can be treated with standard techniques of scattering theory. The important feature of the potential is that it dies off exponentially in $r_* = \int dr/f_s(r)$ near the horizon and as $1/r^2$ at infinity. A calculation of the reflection and transmission coefficients shows [11] that at late times the transmitted flux decays as $t^{-2(l+P+1)}$, where $P = 1$ if an l-pole moment is present before the collapse, and $P = 2$ otherwise. The result is that long after the black hole has formed the radiated perturbations are very weak. The barrier also partially backscatters the outgoing radiation forming a similar flux, of the form $v_{ex}^{-2(l+P+1)}$, which enters the black hole's event horizon [28, 21]. It has recently been verified numerically [22] that the original linear perturbation analysis agrees with a full non-linear analysis. We will refer to the inverse power law gravitational wave tail as the Price tail.

2.3 A simple model of the black hole interior

The first model of the black hole interior which captured the essential physics is the Poisson-Israel model [1]. Before we look at this model, it is useful to consider a simpler model, introduced by Ori [16]. There are two key features which should be included in a reasonable model of the interior. When a black hole is formed by the collapse of a star, an influx of gravitational radiation streams into the hole. In spherical symmetry, there are no gravitational waves, so we need to introduce a crude model which mimics the effect of the waves. When gravitational radiation is highly blueshifted, as it is near the Cauchy horizon, the Isaacson effective stress tensor [29] of gravitational radiation is a very good approximation. In this approximation, the effect of the gravitational radiation is modelled by lightlike dust.

The other key feature is that the gravitational radiation will be scattered by the curvature inside the black hole, forming an outflux transversely crossing the Cauchy horizon. We will refer to the scattered radiation as an outflux, although it doesn't escape from the hole.

In Ori's model [16], outflux is modelled by a thin shell. This allows the matching of two ingoing Vaidya solutions along the outgoing lightlike shell Σ, a finite Kruskal time after the event horizon (see figure 4):

$$ds^2 = dv_\pm(f_\pm dv_\pm - 2dr) + r^2 d\Omega^2, \qquad (2.11)$$
$$f_\pm = 1 - 2m_\pm(v_\pm)/r + e^2/r^2,$$

where the subscript + (-) refers to the region after (before) the shell. The Einstein equations link the mass with the influx

$$L_\pm = dm_\pm/dv_\pm,\ T^\pm_{\alpha\beta} = \frac{L_\pm(v_\pm)}{4\pi r^2}\partial_\alpha v_\pm \partial_\beta v_\pm. \qquad (2.12)$$

Continuity of the line element and the radial coordinate, r, yields the equations

$$f_+ dv_+ = f_- dv_- = 2dr \qquad (2.13)$$

along the shell. Continuity of the influx across the shell gives the equation

$$\frac{1}{f_+^2}\frac{dm_+}{dv_+} = \frac{1}{f_-^2}\frac{dm_-}{dv_-}. \qquad (2.14)$$

These two equations yield the simple equation

$$\frac{dm_+}{f_+} = \frac{dm_-}{f_-} \qquad (2.15)$$

in which mass inflation is evident. The metric function f_- goes to zero as the Cauchy horizon is approached, causing the right hand side of the equation to diverge. The presence of the outgoing shell displaces the apparent horizon to smaller radii so that $f_+ \neq 0$ at the Cauchy horizon. This equation implies that beyond the shell, the mass will diverge at CH.

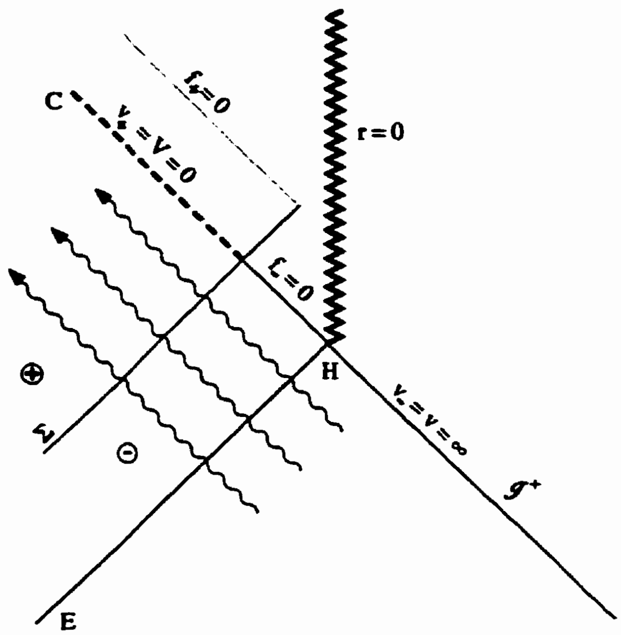

Figure 4: The Ori model of the interior. Radiation enters the black hole as in figure 3. Outgoing radiation is schematically represented by a thin shell Σ which crosses CH. A singularity forms in region + on the Cauchy horizon. The singularity is represented by a bold dashed line, as it is much weaker than the $r = 0$ singularity.

This model can be solved asymptotically close to the Cauchy horizon. The mass function prior to the shell must reproduce the Price power law tail: $m_- = m_0 - \frac{\beta}{\kappa_-(p-1)}(\kappa_- v_-)^{-(p-1)}$. Here v_- is the usual Eddington-Finkelstein advanced time coordinate (2.3) which is infinite on the Cauchy horizon, β is a dimensionless constant of order unity and κ_- is the surface gravity of the inner horizon. Equation (2.13) is then integrated for r along Σ as $v_- \to \infty$

$$r_\Sigma(v_-) \;=\; r_- + \frac{\beta}{r_-\kappa_-^2(p-1)}(\kappa_- v_-)^{-(p-1)}\left(1 + \frac{(p-1)}{\kappa_- v_-} + \ldots\right). \qquad (2.16)$$

Equation (2.15) can now be integrated using (2.16) to show that the mass function diverges exponentially

$$m_+(v_-) = m_0 e^{\kappa_- v_-}(\kappa_- v_-)^{-p}, \; v_- \to \infty \; . \qquad (2.17)$$

Equation (2.13) can be integrated to solve for v_+, by substituting the solution (2.17) for the mass function after the shell. The result is that the coordinate v_+ after the shell is

$$v_+ = \frac{\beta}{\kappa_- m_0} \frac{1}{-\kappa_-} e^{-\kappa_- v_-} = \frac{\beta}{\kappa_- m_0} V \,, \tag{2.18}$$

where V is the Kruskal coordinate defined by (2.6). This phenomenon has been dubbed mass inflation [1]. Indeed this is a scalar curvature singularity since the Weyl curvature invariant diverges exponentially, $\Psi_2 \sim e^{\kappa_- v_-}/r_-^2$ as the Cauchy horizon is approached.

The scalar curvature singularity is weak in the sense that the metric can be written in coordinates in which it is finite at the singularity. Near the singularity, where the mass diverges, the function f_+ can be approximated by $f_+ \sim -2m_+/r$, so that the line element is approximately,

$$ds^2 = 2\frac{dv_+}{r}(rdr + m_+(v_+)dv_+) + r^2 d\Omega^2 \,. \tag{2.19}$$

It is easily checked that the coordinate u, defined by

$$du = rdr + m_+(v_+)dv_+,$$

is bounded at CH. The metric is

$$ds^2 = 2\frac{dv_+ du}{r} + r^2 d\Omega^2. \tag{2.20}$$

The mass inflation singularity, though much stronger than a whimper singularity. is still very weak in this sense. Since tidal deformations of freely falling observers are roughly proportional to components of the metric, it is clear that in the coordinate system (2.20) the deformations are finite.

2.4 Field equations for spherical symmetry

In this section we will present the field equations for a spherically symmetric spacetime. The metric can be written using null coordinates u^0 and u^1,

$$ds^2 = -2e^{\lambda}\, du^0\, du^1 + r^2 \Omega_{ab} d\theta^a d\theta^b \,, \tag{2.21}$$

where Ω_{ab} is the metric on the unit sphere,

$$\Omega_{ab} d\theta^a d\theta^b = d\theta^2 + \sin^2\theta d\phi^2$$

and λ and r^2 are functions of u^A.

Notation: Our conventions are: Greek indices $\alpha, \beta, \ldots$ run from 0 to 3; upper-case Latin indices $A, B, \ldots$ take values $(0, 1)$; and lower-case Latin indices $a, b, \ldots$ take values $(2, 3)$.

Upper case Latin indices are raised and lowered with the constant matrix

$$\eta^{AB} = \eta_{AB} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \; \eta^{AB}\eta_{BC} = \delta^A_C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.22}$$

The Ricci tensor components for the metric (2.21) are

$$R^a{}_b = \frac{1}{r^2}\delta^a_b\left(1 - e^{-\lambda}(\partial^A r\partial_A r + r\partial_A\partial^A r)\right) \tag{2.23}$$

$$R_{AB} = \frac{1}{r}\left(-2\partial_A\partial_B \ln r + 2\partial_{(A}r\partial_{B)}\lambda - \frac{1}{2}\eta_{AB}(r\partial_E\partial^E\lambda + 2\partial^E\lambda\partial_E r)\right), \tag{2.24}$$

where we have introduced the notation

$$R_{AB} = R_{\alpha\beta}\frac{\partial x^\alpha}{\partial u^A}\frac{\partial x^\beta}{\partial u^B}.$$

The contracted Bianchi identities produce a differential relation between components of the stress tensor,

$$\partial_B(r^2 T_A{}^B) = e^\lambda(P\partial_A r^2 + \frac{1}{2}r^2\, T\partial_A\lambda) \tag{2.25}$$

$$2P := T_{ab}g^{ab}, \; T := T_{AB}g^{AB} = e^{-\lambda}T_{AB}\eta^{AB} \tag{2.26}$$

where $T_{\alpha\beta}$ is the non-Maxwellian component of the stress tensor. Since we will be focussing on solutions in the presence of a static electric field, the Einstein field equations take the form

$$G_{\alpha\beta} = 8\pi(T_{\alpha\beta} + E_{\alpha\beta}),$$

where E is the stress tensor for a static spherically symmetric electric field

$$E^\alpha_\beta = \frac{e^2}{8\pi r^4}\left(-\frac{\partial u^A}{\partial x^\beta}\frac{\partial x^\alpha}{\partial u^A} + \frac{\partial\theta^a}{\partial x^\beta}\frac{\partial x^\alpha}{\partial\theta^a}\right).$$

In spherical symmetry, only one component of the Weyl tensor is non-zero

$$\Psi_2 = \frac{1}{2r^2}(1 - \frac{e^2}{r^2} + 2e^{-\lambda}\partial_0 r\partial_1 r) + \frac{4\pi}{3}(T - P).$$

In spherical symmetry there is a unique definition of the mass $m(u^A)$,

$$f(u^A) := 1 - \frac{2m}{r} + \frac{e^2}{r^2} = g^{\alpha\beta}r_{,\alpha}r_{,\beta} = e^{-\lambda}r_{,A}r^{,A}. \tag{2.27}$$

It is also useful to introduce a function $\kappa(u^A)$, defined by

$$\kappa = -\frac{1}{2}\frac{\partial f}{\partial r}\,|_m = -\frac{m}{r^2} + \frac{e^2}{r^3}\,. \qquad (2.28)$$

The Coulomb component of the Weyl tensor can be rewritten as a function of the mass

$$\Psi_2 = \frac{1}{2r^2}(\frac{2m}{r} - \frac{e^2}{r^2}) + \frac{4\pi}{3}(T - P)\,. \qquad (2.29)$$

so that in vacuum the mass function and charge uniquely determine the curvature at any radius r.

It can be useful to promote the mass function to the level of a dynamical variable. This can be done by rewriting the Einstein field equations. Since $G_{AB} = R_{AB} - \frac{1}{2}g_{AB}(R^D_D + R^a_a)$, the Einstein field equations and the Ricci tensor (2.23) and (2.24) can be rearranged to

$$\begin{aligned}\partial_A\partial_B r &= -4\pi r(T_{AB} - g_{AB}T) + \partial_{(A}r\partial_{B)}\lambda \\ &\quad -\frac{1}{2}\eta_{AB}(2\kappa e^\lambda + \partial^E r\partial_E\lambda)\,. \end{aligned} \qquad (2.30)$$

Multiplying (2.30) by $\partial^B r$, we find that

$$\partial^B r\partial_A\partial_B r = -4\pi\partial^B r(T_{AB} - g_{AB}T) + \frac{1}{2}\partial_A\lambda e^\lambda f - \partial_A r\kappa e^\lambda\,. \qquad (2.31)$$

From the definition (2.27) of f, we can easily derive

$$\partial_A f = -\partial_A\lambda f + 2e^{-\lambda}\partial^B r\partial_A\partial_B r$$

and the equivalent definition, using the definition of κ,

$$\partial_A f = -2\kappa\partial_A r - \frac{2}{r}\partial_A m\,.$$

Equating both definitions of $\partial_A f$, it can be shown that the derivative of m is

$$\begin{aligned}\partial_A m &= -re^{-\lambda}\partial^B r\partial_A\partial_B r + \frac{r}{2}(\partial_A\lambda)f - r\kappa\partial_A r \\ &= 4\pi r^2 e^{-\lambda}(\partial^B r)(T_{AB} - g_{AB}T)\,. \end{aligned} \qquad (2.32)$$

Applying the derivative ∂^A to equation (2.32) and making use of equations (2.25) and (2.30), we find the following wave equation for the mass:

$$\begin{aligned}\partial^A\partial_A m &= -(4\pi)^2 r^3 e^{-\lambda}T_{AB}T^{AB} - 8\pi r f e^\lambda(P - T) - 4\pi r^2\kappa e^\lambda T \\ &\quad +4\pi r^2(\partial^B r)(\partial_B T)\,. \end{aligned} \qquad (2.33)$$

(This corrects an error in equation (19) of reference [24].)

2.5 The Poisson-Israel model

In the original mass inflation analysis [1], a null crossflow stress tensor was used to model the gravitational radiation. The form of the outflux is kept arbitrary. The stress tensor for null crossflowing radiation can be written as

$$T_{\alpha\beta} = \frac{L_{in}(V)}{4\pi r^2}\partial_\alpha V \partial_\beta V + \frac{L_{out}(U)}{4\pi r^2}\partial_\alpha U \partial_\beta U \tag{2.34}$$

which satisfies the conservation equations (2.25) and has $P = T = 0$. The conservation equations force L_{in} (L_{out}) to be a function only of V (U).

In the Kruskal coordinate V, the Price power-law tail has the form

$$L_{in}(V) = \frac{dm_{in}}{dv}(\frac{dv}{dV})^2 = \frac{\beta}{(-\kappa_- V)^2}(-\ln(-\kappa_- V))^{-p}. \tag{2.35}$$

As the Cauchy horizon is approached, in the limit $V \to 0_-$, L_{in} diverges and the source term in the wave equation for m diverges as well. The integral solution for the mass function is [1]

$$\begin{aligned} m(U,V) = \quad & \int_{U_1}^{U}\int_{V_1}^{V} r'^{-1}e^{-\lambda'}L_{in}(V')L_{out}(U')dU'dV' \\ & + m_{in}(V) + m_{out}(U) - m_1. \end{aligned} \tag{2.36}$$

The gravitational wave tail influx is turned on at advanced time V_1 and the outflux is assumed to be switched on at the advanced time U_1, which is behind the event horizon (see figure 5). The divergence of $L_{in}(V')dV'$ leads to mass inflation with the mass function behaving as $m \sim 1/V$. Of course, this is only true if the combination $r^{-1}e^{-\lambda}$ does not go to zero, but this has been proved by Poisson and Israel [1].

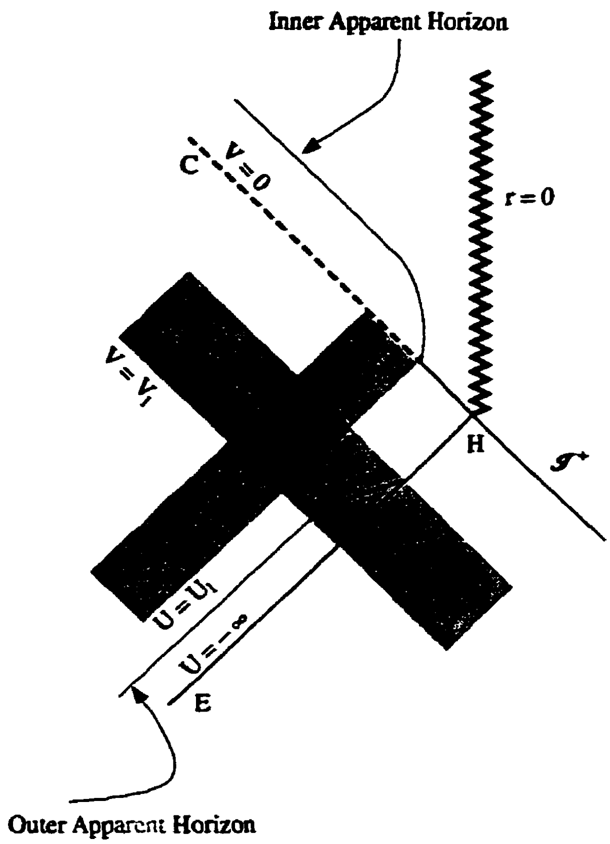

Figure 5: The Poisson-Israel model of the interior. In this diagram, regions where radiation is propagating are shaded grey. The influx is turned on at advanced time V_1, while the outflux is turned on at retarded time $U_1 > -\infty$. A mild lightlike curvature singularity forms on the portion of CH with $U > U_1$.

There have been suggestions that the picture presented by the Poisson-Israel and Ori models is not generic. Yurtsever [30] pointed out that null singularities in plane wave spacetimes are not generic: when perturbed, a stronger spacelike singularity forms before the Cauchy horizon. Yurtsever has suggested that something similar may happen when non-spherical perturbations of the PI model are considered. His unproven view is that the generic black hole interior would look similar to the Schwarzschild interior. We shall discuss this hypothesis in more detail in chapter 5.

This hypothesis is supported by a numerical evolution [31] of a spherically symmetric massless scalar field in a charged black hole. In this integration, the singularity was inferred to be spacelike (see figure 6*a*). However, a different simulation has shown that the singularity is null [32] (see figure 6*b*). What can we say analytically to clear up the confusion about the nature of the singularity?

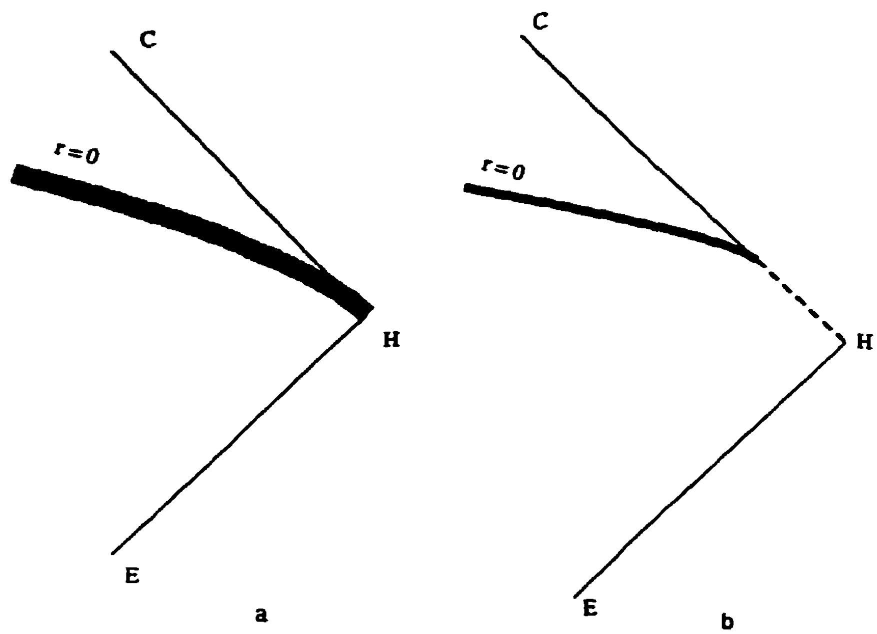

Figure 6: Two numerical evolutions of the spherical charged interior. In the Gnedin-Gnedin integration (6a) it is inferred that a strong $r = 0$ singularity forms asymptotic to CH. Since it occurs earlier than CH all light signals end at $r = 0$. (See figure 5 of reference [31].) In the Brady-Smith integration (6b) the $r = 0$ singularity crosses CH and a milder lightlike singularity exists prior to $r = 0$. (See figure 1 of reference [32].)

The previous mass inflation analyses suffer from some limitations. In the picture presented [1, 16] it is always assumed that the outflux is turned on abruptly after some finite time behind the event horizon. Essentially, this amounts to the assumption that a null portion of the Cauchy horizon exists, because the solution before the outflux begins is the Vaidya solution. This segment's existence depends on the form of the outflux crossing it. If the outflux at early retarded times ($U \to -\infty$) is too strong, a spacelike singularity will form. The effect of an outflux crossing a null ray is described by Raychaudhuri's equation

$$\frac{d^2r}{d\mu^2} = -4\pi r T_{\mu\mu} \tag{2.37}$$

where μ is an affine parameter on the null hypersurface, $T_{\mu\mu} = T_{\alpha\beta}\frac{dx^\alpha}{d\mu}\frac{dx^\beta}{d\mu}$ is the transverse flux and $dx^\beta/d\mu$ is tangent to the generators of the null hypersurface. In the case of interest, the null hypersurface is the Cauchy horizon and $\mu \to -\infty$ corresponds to its "meeting" with the event horizon at H in figure 7.

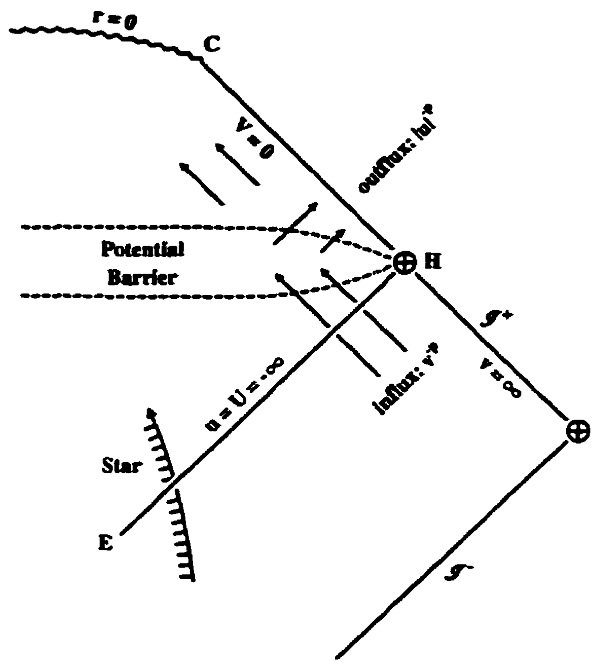

Figure 7: Conformal diagram of the spherical charged interior. The influx of radiation is scattered by the black hole's interior potential barrier. The transmitted flux travels parallel to CH and is infinitely blueshifted. The refracted outflux crosses CH transversely, causing CH to decrease in radius towards the point C where it is inferred that $r = 0$.

Examination of (2.37) shows that in order for r to be finite as $\mu \to -\infty$, $T_{\mu\mu}$ must satisfy

$$\mu^2 T_{\mu\mu} \to 0 \quad \text{as} \quad \mu \to -\infty. \tag{2.38}$$

To test this condition we need to relate the affine parameter μ to the null coordinate U by

$$\frac{dU}{d\mu} = -g^{UV} = e^{-\lambda}. \tag{2.39}$$

If $e^{-\lambda}$ diverges as $V \to 0$ and $U \to -\infty$, then depending on T_{UU}, the Cauchy horizon may not survive.

The earlier models of mass inflation do not address this issue since the outflux is turned on after the event horizon. The corner region, $V \to 0, -\infty < U < U_1$ is described by a Vaidya solution. (The metric function $e^{-\lambda}$ is finite in this region for Vaidya.) For more general models which include the corner region, the behaviour of λ must be found. To do this, we need to specify the appropriate initial conditions for T_{UU} which are physically reasonable.

2.6 The Outflux

A star collapsing through its event horizon provides two sources of outflux. First, the star shines as it collapses and will irradiate the Cauchy horizon after the event horizon is passed. While we will not attempt to describe the actual form of the the star's radiation, we do know that in a freely falling frame at the event horizon, the radiation must be measured to be bounded. Kruskal coordinates for the event horizon, $U_+ = e^{\kappa_+ u}$ are appropriate for freely falling observers. These observers measure $T_{U_+U_+} <$ constant. Transforming this to the Kruskal coordinate U appropriate to the inner horizon the outflux across CH is

$$(T_{UU})_{star} = T_{U_+U_+}\left(\frac{dU_+}{dU}\right)^2 \sim (-U)^{-2(1+\kappa_+/\kappa_-)} \quad , \quad U \to -\infty.$$

As we shall see, the outflux due to the star has a negligible effect compared to the backscattering of the incoming radiation.

Consider the evolution of a massless spherically symmetric scalar field in the black hole interior. The characteristic initial value problem is completely specified by data given on the the event horizon. The physical initial data are determined by the Price power law wave tail v^{-p}.

For reference purposes, consider the evolution of a scalar field in a fixed Reissner-Nordström background with mass m_0. The far right hand side of figure 8 (with all fluxes turned off), describes the static Reissner-Nordström solution. It is distinguished by an outer layer where the gravitational potential barrier is weak and perturbations can propagate without impediment. The potential is peaked around the radius $r = e^2/m_0$. This is where most of the perturbation will be scattered. Much further in, near $r \sim r_-$, the Cauchy horizon is approached and infalling radiation is strongly blueshifted. It is important to note that the radiation is scattered long before it reaches the large blueshift zone. This motivates our treatment of the evolution of fields as a scattering problem on a static Reissner-Nordström background.

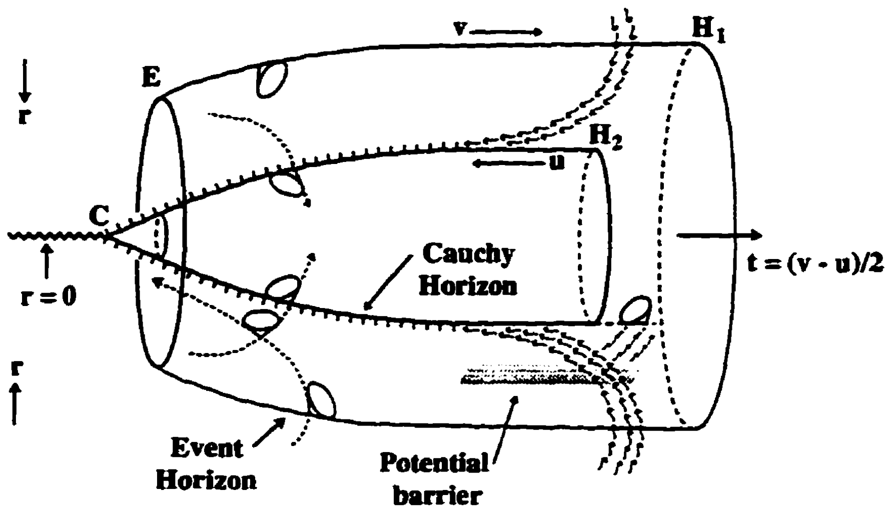

Figure 8: Spacetime diagram of the spherical charged black hole interior with one angular coordinate suppressed. (The content of this diagram is identical to the conformal diagram of figure 7, but now the point H is represented by two points H_1 and H_2.) The time coordinate t increases to the right, while the radius decreases towards the centre. The event horizon is represented by a tube which expands in radius as radiation enters the hole, asymptotically reaching its final radius at H_1, where $v = \infty$. In the region contained within the tube EH_1, the coordinate t is spacelike and r is timelike, so that decreasing r corresponds to increasing time. Radiation propagating in the interior is scattered by the potential barrier forming an outflux which crosses the Cauchy horizon, causing it to increase in size asymptotically to its maximum size at H_2, where $u = -\infty$. The transmitted flux accumulates along the Cauchy horizon where a weak singularity forms. As u increases the singular tube CH_2 evolves to a stronger zero volume singularity.

Mathematically, scattering of a massless field is given by $\Box \Psi = 0$ which, using the usual advanced/retarded coordinates, is for the dominant monopole term ($l = 0$)

$$\begin{aligned} \phi_{uv} &= V(r)\phi \ , \quad \Psi = \phi/r \\ V(r) &= \frac{f_s(r)}{4r}\frac{d}{dr}f_s(r) \end{aligned} \tag{2.40}$$

where f has been defined in (2.27) and the subscript s denotes the static Reissner-Nordström case with mass m_0.

This is a one-dimensional scattering problem. It is greatly simplified by the fact that the potential $V(r)$ is highly localized near e^2/m_0. It falls off exponentially [27] in the tortoise coordinate defined by $dx = dr/f_s(r)$ near the event horizon $x = -\infty$ and the Cauchy horizon $x = \infty$. A scalar field will propagate freely near the event and Cauchy horizons and will only strongly interact with the curvature in a thin belt around e^2/m_0. At the horizons the scalar field solutions will be of the form of ingoing and outgoing waves $e^{-i\omega v}$ and $e^{-i\omega u}$. The effect of the potential will be to alter the amplitudes by the reflection and transmission coefficients, $r(\omega)$ and $t(\omega)$.

If the initial value on the event horizon is $\phi_0(v)$ then its Fourier transform [27, 33, 34]

$$\tilde{\phi}_0(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi_0(v)e^{i\omega v}dv$$

allows us to write the form of the scattered waves as $X(v) + Y(u)$ where

$$\begin{aligned} X(v) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\tilde{\phi}_0(\omega)t(\omega)e^{-i\omega v}d\omega \\ Y(u) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\tilde{\phi}_0(\omega)r(\omega)e^{i\omega u}d\omega. \end{aligned} \tag{2.41}$$

The initial conditions are $\phi_0(v) = (\kappa_- v)^{-p/2}\Theta(v - v_1)$ where the influx is assumed to start after v_1. The Fourier transform behaves as [27, 33, 34]

$$\tilde{\phi}_0(\omega) \sim \omega^{p/2-1}. \tag{2.42}$$

This can be used to calculate the transmitted and reflected flux. The stationary phase approximation can be used to evaluate the integrals (2.41). For large v the transmitted flux has the form

$$\begin{aligned} X(v) &\sim t(\omega_0)(k_0 v)^{-p/2}, \omega_0 = -ip/2v \\ t(\omega_0) &\sim 1/v \end{aligned} \tag{2.43}$$

and the reflected outflux is for large negative u

$$\begin{aligned} Y(u) &\sim r(\omega_0)(-\kappa_- u)^{-p/2}, \omega_0 = -ip/2u \\ r(\omega_0) &\sim \text{constant}. \end{aligned} \tag{2.44}$$

We calculated the reflection and transmission coefficients shown in (2.43) and (2.44) using a simple model for the scattering potential: a rectangular well adjacent to a rectangular barrier. For low energy scattering it is expected that the perturbations will be strongly influenced by the potential so that there will be an almost total reflection. A computer simulation of the scattering [35] has shown that the rectangular potential approximates the actual behaviour very closely. Actually, it is more appropriate to use the term refraction here since the reflected beam continues on to smaller radii.

The general effect of the Reissner-Nordström curvature is to scatter the influx $T_{vv} \sim (\kappa_- v)^{-p}$ into a reflected flux $T_{uu} \sim \alpha\ (-\kappa_- u)^{-p}$ and a transmitted flux $T_{vv} \sim \beta(\kappa_- v)^{-p-2}$ near the Cauchy horizon, where α and β are the reflection and transmission coefficients and are $O(1)$. Is this linearized scattering theory useful for our problem?

Consider the initial layers just beneath the event horizon $r = r_+ - \epsilon$. This region is far above the strongly blueshifted region, so the flux of energy is only that of the Price gravitational wave tail v^{-p}. For late times this is very weak and will only be a small perturbation from the electro-vacuum Reissner-Nordström solution. The effect of the Reissner-Nordström geometry on the wave tail influx will be negligible and the influx from the event horizon will freely propagate to the Reissner-Nordström potential barrier.

The black hole interior can be approximated from the event horizon to the scattering potential as Reissner-Nordström . Our method will be to solve the Einstein equations in the interior region after the scattering potential. Initial conditions can be set just after the potential barrier, given by the Reissner-Nordström scattering problem. In fact, just after the potential barrier the fluxes are not particularly large since the blueshift region has not been approached yet. Until the Cauchy horizon is reached the perturbations to Reissner-Nordström are small. Our approach is to find an approximate solution which is good close to the Cauchy horizon (where perturbations are large) and which satisfies the initial conditions given by scattering from the potential barrier.

As a first step toward the analytic scalar field approximation, we shall first introduce a null cross flow solution which incorporates the boundary conditions discussed here.

2.7 Analytic approximation for lightlike crossflow

Our aim is to construct an analytic model for the black hole interior after the potential barrier. We shall first start with a null crossflow stress tensor and introduce some approximations. This can then be used as a model for what we expect to happen in the scalar field evolution.

The null crossflow stress tensor is of the form of equation (2.34) with the luminosity functions given by scattering

$$\begin{aligned} L_{in}(V) &= \beta(-\kappa_- V)^{-2}(-\ln(-\kappa_- V))^{-q} \\ L_{out}(U) &= \alpha(-\kappa_- U)^{-2}(\ln(-\kappa_- U))^{-p} \end{aligned} \tag{2.45}$$

where α and β are dimensionless positive numbers, corresponding to the reflection and transmission coefficients respectively and $q = p + 2$.

Introduce the functions $A(U), B(V)$ defined by

$$\begin{aligned} L_{out}(U) &= A''(U) \\ L_{in}(V) &= B''(V) \end{aligned}$$

where $'$ denotes ordinary differentiation. For $V \to 0$

$$\begin{aligned} B(V) &= -\frac{\beta}{\kappa_-^2(q-1)}(-\ln(-\kappa_- V))^{-q+1}\left(1 + \frac{q-1}{-\ln(-\kappa_- V)} + \dots\right) \\ B'(V) &= \frac{\beta}{\kappa_-^2(-V)}(-\ln(-\kappa_- V))^{-q}\left(1 + \frac{q}{-\ln(-\kappa_- V)} + \dots\right) \end{aligned} \tag{2.46}$$

and for $U \to -\infty$

$$\begin{aligned} A(U) &= \frac{\alpha}{\kappa_-^2(p-1)}(\ln(-\kappa_- U))^{-p+1}\left(1 - \frac{p-1}{\ln(-\kappa_- U)} + \dots\right) \\ A'(U) &= \frac{\alpha}{\kappa_-^2(-U)}(\ln(-\kappa_- U))^{-p}\left(1 - \frac{p}{\ln(-\kappa_- U)} + \dots\right). \end{aligned} \tag{2.47}$$

In the corner region $V \to 0, U \to -\infty$, the functions A and B are small, but derivatives of B with respect to V diverge.

We wish to concentrate on the region after the potential barrier at early times. Before the potential barrier we expect Reissner-Nordström to be a good model. In the innermost regions we must model the effect of the infinite blueshift of the inflowing radiation.

Using the metric (2.21) we note that the Einstein equations allow us to write wave equations for two combinations of the metric functions, which do not include the mass function as a source term:

$$\begin{aligned}(\ln(r^{-1}e^{-\lambda}))_{,UV} &= -\frac{e^{\lambda}}{2r^2}\left(1-\frac{3e^2}{r^2}\right)\\ (r^2)_{,UV} &= -e^{\lambda}\left(1-\frac{e^2}{r^2}\right).\end{aligned} \tag{2.48}$$

In order to solve the evolutionary problem, we need to find the solution in the intermediate region after the potential barrier and before the region where r goes to zero. This region will be defined by $r \neq 0$. When this stipulation is made it is impossible for $r^{-1}e^{-\lambda}$ to go to zero [1]. This means that both wave equations (2.48) do not have any potentially diverging source terms and both will have finite solutions. For conciseness, introduce the bounded and non-zero variables

$$\chi = r^{-1}e^{-\lambda}\ ,\ \rho = \frac{1}{2}r^2. \tag{2.49}$$

The Einstein equations can then be written as equations (2.48) and the null hypersurface constraint equations:

$$\begin{aligned}\partial_U(\chi\rho_{,U}) &= -\chi A''\\ \partial_V(\chi\rho_{,V}) &= -\chi B''.\end{aligned} \tag{2.50}$$

The mass function obeys the wave equation

$$m_{,UV} = \chi A''B''. \tag{2.51}$$

As long as $r \neq 0$, we can write a solution with χ and ρ being close to their Reissner-Nordström values plus perturbations which are small in this region. The metric functions for static Reissner-Nordström with a mass m_0 will be denoted with a subscript "s", so that $f_s(r_s)$ and $\kappa_s(r_s)$ are defined by equation (2.27) and (2.28). The functions ρ_s and χ_s and their derivatives are

$$\rho_s = \frac{1}{2}r_s^2\ ,\ \rho_{s,U} = -\frac{1}{2}\frac{r_s f_s}{\kappa_- U}\ ,\ \rho_{s,V} = -\frac{1}{2}\frac{r_s f_s}{\kappa_- V} \tag{2.52}$$

$$\begin{aligned}\chi_s &= \frac{-2UV}{f_s}\frac{\kappa_-^2}{r_s}\\ \chi_{s,U} &= -\frac{\kappa_- V}{r_s^2}\left(1+\frac{2r_s}{f_s}(\kappa_- - \kappa_s)\right)\\ \chi_{s,V} &= -\frac{\kappa_- U}{r_s^2}\left(1+\frac{2r_s}{f_s}(\kappa_- - \kappa_s)\right).\end{aligned} \tag{2.53}$$

In the limit of the Cauchy horizon, ($UV \to 0, r_s \to r_-$) these functions take on the limiting value

$$f_s \to -2UV\kappa_-^2 \tag{2.54}$$

$$\rho_s \to \frac{1}{2}r_-^2,\ \rho_{s,U} \to r_-V\kappa_-,\ \rho_{s,V} \to r_-U\kappa_- \tag{2.55}$$

$$\chi_s \to \frac{1}{r_-},\ \chi_{s,U} \to -\frac{\kappa_- V}{r_-^2},\ \chi_{s,V} \to -\frac{\kappa_- U}{r_-^2}. \tag{2.56}$$

We can now construct a solution to the Einstein equations using an iterative approach, taking the static Reissner-Nordström solution as the zeroth order solution ($\chi^{(0)} = \chi_s, \rho^{(0)} = \rho_s$) and substituting back into the Einstein equations to find the first order correction terms. Equations (2.50) can be integrated to solve for ρ:

$$\begin{aligned} \rho = \rho_s & \\ & - \int^V \frac{dV''}{\chi(U'',V'')}\int^{V''} dV'\chi(U',V')B''(V') \\ & - \int^U \frac{dU''}{\chi(U'',V'')}\int^{U''} dU'\chi(U',V')A''(U'). \end{aligned} \tag{2.57}$$

It is clear in our approximation scheme that (2.57) is the leading order contribution to the solution of the Einstein equations. The contribution from (2.48) will be of lower order.

Integration of (2.57) by parts gives the solution

$$\rho = \rho_s - (A+B) + \epsilon, \tag{2.58}$$

where ϵ is

$$\epsilon = \int^V \frac{dV''}{\chi_s(U'',V'')}\int^{V''} dV'\chi_{s,V'}(U',V')B'(V') + \int^U \frac{dU''}{\chi_s(U'',V'')}\int^{U''} dV'\chi_{s,U'}(U',V')A'(U') \tag{2.59}$$

$$\sim U\int BdV + V\int AdU \tag{2.60}$$

which is much smaller than $A+B$ in the remote past of CH. Using this approximation in the second equation of (2.48) and expanding to first order in A and B, allows the estimation

$$\chi = \chi_s + O(\frac{1}{r_s^2}(A+B)).$$

Substitution of this order of correction back into (2.57) yields a second order approximation

$$\rho = \rho_s \; - \; A(1 + O(\ln(-\kappa_- U))^{-p}) - B(1 + O\,(-\ln(-\kappa_- V))^{-q}).$$

Clearly, in the corner region where $(\ln(-\kappa_- U))^{-1} \sim (-\ln(-\kappa_- V))^{-1} \sim 0$, ρ is well approximated by the leading order solution (2.58).

To linear order in A and B, the mass function can be integrated from (2.51)

$$m(U, V) = \chi_s A' B'(1 + O(A + B)) + m'_{in} + m'_{out} - m_0$$

which in the limit $V \to 0$ is

$$m \sim \frac{\alpha\beta}{r_-\kappa_-^2}\frac{1}{UV}(\ln(-\kappa_- U))^{-p}\,(-\ln(-\kappa_- V))^{-q} \qquad (2.61)$$

showing the usual $1/V$ inflation found in earlier work [1].

The solutions for the original metric functions r and λ are

$$\begin{aligned} r &= r_s - \frac{(A+B)}{r_s} + \frac{2AB}{r_s^3} + O(A^2 + B^2) \\ \lambda &= \lambda_s + \frac{(A+B)}{r_s^2} + \frac{2AB}{r_s^4} + O(A^2 + B^2). \end{aligned} \qquad (2.62)$$

This approximation is not so good as the scattering surface is approached ($UV \to 1$, so that correction terms (2.59) are comparable to the first order terms in (2.58)). We already know that the solution near the scattering surface should be approximately described by the Reissner-Nordström solution. It is only after this region, deep into the blueshift region that an approximate solution is needed and this is where it is important that the solution be accurate. The solution that we have found is accurate where it matters, close to the Cauchy horizon.

2.8 The Scalar Field Solution

Using approximations similar to those just discussed for lightlike crossflow, we can develop an approximate analytic solution for the scalar field equations. As before, the physics tells us that the interior solution can be approximated well by the static Reissner-Nordström solution from the event horizon down to the scattering surface.

The Einstein equations for coupling to a massless scalar field are

$$\begin{aligned} r\phi_{,UV} &= -r_{,U}\phi_{,V} - r_{,V}\phi_{,U} \\ \rho_{,UV} &= -2\frac{1}{r\chi}\left(1-\frac{e^2}{r^2}\right) \\ (\ln(\chi))_{,UV} &= 8\pi\phi_{,U}\phi_{,V} - \frac{1}{r^3\chi}\left(1-\frac{3e^2}{r^2}\right) \\ (\chi\rho_{,U})_{,U} &= -8\pi\rho\chi\phi_{,U}^2 \\ (\chi\rho_{,V})_{,V} &= -8\pi\rho\chi\phi_{,V}^2 \\ m_{,UV} &= 16\pi^2\chi r^4\phi_{,U}^2\phi_{,V}^2 - 4\pi r f\phi_{,U}\phi_{,V}. \end{aligned} \tag{2.63}$$

Define functions $a(U)$ and $b(V)$ by setting their derivatives equal to

$$a'(U) \equiv \phi_{,U}|_b, \quad b'(V) \equiv \phi_{,V}|_b$$

where the subscript b refers to the value of the scalar field given by scattering at the underside of the potential barrier.

As before, the wave equations for ρ and χ have solutions which are finite and non-zero in the corner region as long as ϕ does not diverge. The initial conditions given by scattering (2.43,2.44) are that the scalar field is initially regular. Near the scattering surface the radius will be close to its Reissner-Nordström value, so using the scalar wave equation and the Reissner-Nordström radius (2.52), it can be seen that the UV mixed derivative of the scalar field, near the initial surface is

$$\begin{aligned} \phi_{,UV}|_b &= \frac{1}{2}\frac{f_s}{r_s\kappa_-}\left(\frac{\phi_{,V}}{U} + \frac{\phi_{,U}}{V}\right)|_b \\ &\sim (-\ln(-\kappa_- V))^{-q/2} + (\ln(-\kappa_- U))^{-p/2}. \end{aligned} \tag{2.64}$$

This derivative is small in the corner ($V \to 0, U \to -\infty$), so in the earliest regions the scalar field will not be changing rapidly from its initial value. This motivates us to make the ansatz that the leading order behaviour of the scalar field, near the Cauchy horizon should be

$$\begin{aligned} \phi_{,U}^{(0)} = a' &\equiv \sqrt{\frac{A''}{4\pi r_-^2}} \\ &\sim \frac{1}{\sqrt{4\pi r_o^2\kappa_-^2}}\frac{1}{(-U)}(\ln(-\kappa_- U))^{-p/2} \end{aligned}$$

$$\begin{aligned}\phi^{(0)}_{,V} = b' &\equiv \sqrt{\frac{B''}{4\pi r_-^2}} \\ &\sim \frac{1}{\sqrt{4\pi r_o^2 \kappa_-^2}} \frac{1}{(-V)} (-\ln(-\kappa_- V))^{-q/2}.\end{aligned} \tag{2.65}$$

With this ansatz, we can see that the scalar field will be small everywhere in the corner region, but that derivatives with respect to V will diverge near the Cauchy horizon.

As before we can calculate the first order correction terms by iterating the Einstein equations, again taking the zeroth order solutions for ρ and χ to be the same as the Reissner-Nordström solution. The solution for ρ is the same as the lightlike cross flow solution (2.58). Substitution of (2.58) and (2.65) into the scalar wave equation yields the first order equation

$$\phi_{,UV} = -\frac{1}{2\rho_s}((\rho_{s,U} - A')b' + (\rho_{s,V} - B')a')$$

which can be integrated asymptotically in the corner region, making use of (2.52)

$$\phi = a + b + \frac{1}{r_-^2}(Ab + Ba) + O(UV(a+b)). \tag{2.66}$$

Using the first order solutions for ϕ and ρ the wave equation for χ can be integrated

$$\ln\chi = \ln\chi_s + 8\pi ab + O(A+B). \tag{2.67}$$

To leading order the mass function is integrated to be

$$m(U,V) \sim \kappa_-^2/r_- A'B' \tag{2.68}$$

and the metric functions r and λ are

$$\begin{aligned} r &= r_s - \frac{(A+B)}{r_s} - \frac{AB}{r_s^3} + O(A^2+B^2) \\ \lambda &= \lambda_s - 8\pi ab + \frac{(A+B)}{r_s^2} + \frac{2AB}{r_s^4} + O(A^2+B^2). \end{aligned}$$

The existence of the Cauchy horizon in this solution can now be tested. Substitution of the solution for λ given by (2.69) into (2.39) gives the following asymptotic relation between the affine parameter, μ, and the coordinate, U,

$$\mu = \frac{U}{\kappa_-^2}(1 + O(a)) \ , U \to -\infty.$$

Condition (2.38) for the scalar field solution now reads

$$\lim_{\mu \to -\infty} \mu^2 T_{\mu\mu} = \lim_{U \to -\infty} \frac{(\ln(-\kappa_- U))^{-p+2}}{4\pi r_-^2 \kappa_-^6} \left(1 + O(\ln(-\kappa_- U))^{-p+1}\right) = 0.$$

Since condition (2.38) is satisfied, the Cauchy horizon exists in our approximate solution to the Einstein-scalar field equations. This is of course evident directly from the asymptotic form of the metric functions (2.69).

2.9 Evolution of the singularity

In the last 3 sections we have focussed on the structure of the black hole interior in the early corner region. What happens at later times as U increases along the Cauchy horizon? Consider again, equation (2.37). Since the outflux $T_{\mu\mu}$ is positive definite, the second derivative of r is negative. Since r is initially (at H) decreasing, it must continue to decrease to zero along the Cauchy horizon. We should note that $r = 0$ still represents a curvature singularity. But what is its character, is it spacelike, lightlike or timelike? This question is answered by examining the norm of $\partial_\alpha r$, given by f in equation (2.27). Near the Cauchy horizon, for small r

$$\partial_\alpha r \partial_\beta r g^{\alpha\beta} \sim -2m/r + e^2/r^2 \, .$$

When $r > e^2/m$ then r is spacelike. Since the mass is diverging at the Cauchy horizon, there is always a very small radius, $r = \epsilon$ which is a spacelike hypersurface. One would expect by continuity, that the hypersurface $r = 0$ should also be spacelike. The expectation is that the curvature singularity at the Cauchy horizon connects with a stronger spacelike $r = 0$ singularity as in figures 8 and 7.

In all of the considerations in this chapter we have always assumed that $r \neq 0$, so our solution can not describe the transition region, where the two singularities merge. It seems likely that the spacelike singularity would be described by the general oscillating BKL singularity [15]. In fact, there is a simple model of a spacelike singularity in the black hole interior, known as homogeneous mass inflation [36] which may be a good approximation to the singularity at $r = 0$. In this approximate solution, it is assumed that the solution depends only on r, which is a good approximation when r is very small. In the homogeneous mass inflation model, the solution oscillates violently as the spacelike singularity is approached.

The fact that we expect that the null singularity may merge with a spacelike singularity may explain the discrepancy between the different numerical results. The earliest integration by Gnedin and Gnedin [31] showed that there was no null portion of the singularity. But the Gnedins' code encounters its worst inaccuracies near $V = 0$, and within their margin of error, it is really impossible to decide whether their picture shows a completely spacelike singularity. It has also been shown recently [37] that the sort of code that was used is highly unstable, which casts a great deal of doubt on the conclusion drawn from this code. The other numerical integration by Brady and Smith [32] does show that a null singularity forms, similar to the one described in this chapter. It is interesting to see that the radius of $U =$ constant rays decreases as U increases and the trend suggested from their graphs is that the null singularity will indeed merge with a spacelike $r = 0$ singularity.

2.10 Conclusion

We have calculated the effect of the backreaction of scalar field perturbations propagating in the interior of a charged spherical black hole. We incorporated physical initial conditions, given by the scattering of the perturbations by the interior Reissner-Nordström potential barrier. The result of this analysis is that a null scalar curvature singularity forms at CH, as was found in the Poisson-Israel model [1].

The infinite distortion of the Penrose diagram near the point H of figure 7 can cause some confusion. The general picture of the processes involved can be summarised by the spacetime diagram 8. In this diagram two-spheres of constant r are represented by circles of radius r. Since r is timelike inside the black hole, we have labeled decreasing r as increasing time. In this diagram we show that the incoming radiation is partially scattered by the potential barrier into two streams. The scattered stream transversely crosses CH, causing it to contract to smaller r, eventually to $r = 0$. The transmitted stream travels parallel to CH, and is infinitely blueshifted, causing a curvature singularity to form at CH.

The Weyl curvature diverges in the limit $UV \to 0$ as,

$$\Psi_2 \sim \frac{1}{UV}(\ln(-U))^{-p/2}(-\ln(-V))^{-q/2}$$

at the segment CH, which is where the Cauchy horizon would be located if no perturbations were present. The divergence of Ψ_2 demonstrates that the tidal forces diverge at CH. However, the distortions are proportional to first integrals of the curvature $\sim \int dV \Psi_2 \sim (-\ln(-V))^{-q} + const.$ which are finite. In this sense the singularity is weak.

Chapter 3

Double-null dynamics

One of the fundamental goals of physics is to make predictions about the outcome of experiments. The means to this end is the initial value, or Cauchy formulation of the laws of physics, which provides a method to forecast the later state of a system once the appropriate initial conditions are known.

Our wish to predict the future is so great that we continue to use the Cauchy formulation, although the concept of time is not absolute: inertial observers measure clocks in different inertial frames to run slow. As a result, in order to make use of the initial value formulation correctly, it is important to keep the transformation properties of the laws of physics in mind. For example, consider Maxwell's theory of electromagnetism. One can obtain all the information about the electromagnetic field for all observers by solving Maxwell's equations for the field strength $F_{\alpha\beta}$. However, sometimes it is more useful to choose a special observer and project $F_{\alpha\beta}$ into electric and magnetic components and solve for the fields in the special reference frame. In order to describe the observations of a different inertial observer, it is necessary to use the Lorentz transformation laws to find the electric and magnetic fields in the new frame.

The situation is more complicated for the theory of general relativity where general transformations between the space and time coordinates are allowed. If the general covariance of the theory is to be respected, no observer should be considered special and no coordinate should be singled out as time. However, a generally covariant approach is complicated, for Einstein's field equations are a set of ten coupled second order non-linear partial differential equations for the spacetime metric. Except in special symmetric spacetimes, solutions to the field equations are very difficult to find. The use of the non-covariant initial value approach [38] to general relativity leads to conceptual simplifications which makes it easier to solve certain evolutionary problems.

The form and interpretation of the initial value problem for gravity introduced by Arnowitt, Deser and Misner (ADM) [39] is the current standard formulation used to study dynamical systems. The underlying fundamental structure that they imagined was a foliation of spacetime with a family of hypersurfaces of simultaneity, Σ_t. Each three dimensional spacelike Σ_t is labelled by a parameter t which is constant on the hypersurface. General relativity is now interpreted as the evolution of the intrinsic geometry on Σ_t to later hypersurfaces.

One drawback is that the hypersurfaces must be spacelike: it is impossible to describe the evolution of null hypersurfaces using the ADM formalism. This is a serious problem if one is interested in studying null hypersurfaces, such as the event and Cauchy horizons of a black hole. As the structure of the Cauchy horizon is the focus of this thesis, it would be advantageous to develop an imbedding formalism analogous to ADM's which can handle null hypersurfaces.

In this chapter we will present a new formalism [40] developed for a foliation of spacetime by two intersecting families of null hypersurfaces. The intersection of the families forms a two parameter collection of two dimensional spacelike surfaces, so this can be pictured as an evolution of the geometry on a surface in either of the lightlike directions normal to the surface. The null surface approach to field theory [41, 42, 43] (called variously "infinite momentum frame" and "light-front field theory") has led to important improvements in the understanding of field theories. Similarly, it is hoped that the double-null approach to gravity will clarify many problems. In this thesis, the double-null formalism presented in this chapter will be applied in chapters 4 and 5. In chapter 4 it will be used to describe the dynamics of quasi-local mass and to derive a general mass inflation law for black hole interiors. In chapter 5 the formalism will be used to solve the characteristic initial value problem in the black hole interior, which will show that a lightlike singularity forms at the Cauchy horizon of a nonspherical black hole.

The first formalism based on pairs of null directions was the generalized spin-coefficient formalism of Geroch, Held and Penrose (GHP) [44]. The GHP formalism is especially well suited for the study of algebraically special spacetimes. When the vectors defining the null directions are hypersurface orthogonal simpler formulations can be presented [45] - [49]. In this chapter we will present a double-null formulation of general relativity assuming hypersurface orthogonality. The content of the formulation is identical to [45] - [49] but the presentation of the field equations is much simpler conceptually. It is based on the classical description of surfaces imbedded in higher dimensional manifolds [50]. The formalism is essentially a generalization of a previous $2+1$ split of spacetime within a three dimensional null hypersurface [51] introduced to study cosmic censorship. The essential feature of our approach is that it maintains two-dimensional covariance while operating on objects with direct geometrical meaning.

This chapter is organised as follows. In section 1 the basic geometrical framework and notation are introduced. The first order imbedding relations (the Gauss-Weingarten equations) are derived in section 2. Commutation relations relating second derivatives of the metric functions are derived in section 3. In section 4 the Riemann and Ricci tensors are derived, through the Gauss-Codazzi relations. The contracted Bianchi identities, characteristic initial value problem and the Lagrangian for gravity are presented in sections 5, 6 and 7 respectively. More technical derivations are presented in the appendices.

3.1 Geometrical framework

We begin by imagining a foliation of spacetime by two intersecting families of null hypersurfaces, $\{\Sigma^0\}$ and $\{\Sigma^1\}$. Each hypersurface Σ^0 is defined as the locus of points on which the parameter u^0 has a constant value. Similarly the parameter u^1 labels the hypersurfaces Σ^1. The hypersurfaces Σ^0 and Σ^1 have null normal generators $\ell^{(0)}$ and $\ell^{(1)}$ respectively. (Here the bracketed numbers are labels and the spacetime indices are suppressed.) The intersection of two hypersurfaces from the two families occurs on a two dimensional spacelike surface, S, on which u^0 and u^1 are both constant. We introduce intrinsic coordinates on S, θ^2 and θ^3, so that the foliation of the four dimensional spacetime is described by the four imbedding relations

$$x^\alpha = x^\alpha(u^0, u^1, \theta^2, \theta^3), \tag{3.1}$$

where x^α are four dimensional spacetime coordinates.

Notation: Our conventions are: Greek indices $\alpha, \beta, \ldots$ run from 0 to 3; upper-case Latin indices $A, B, \ldots$ take values $(0, 1)$; and lower-case Latin indices $a, b, \ldots$ take values $(2,3)$. We adopt MTW curvature conventions [52] with signature $(-+++)$ for the spacetime metric $g_{\alpha\beta}$. When there is no risk of confusion we shall often omit the Greek indices on 4-vectors like $\ell^{(A)}_{\alpha}$ and $e^{\alpha}_{(a)}$: they are easily identifiable as 4-vectors by their parenthesized labels. Four-dimensional covariant differentiation is indicated either by ∇_α or a vertical stroke: $\nabla_\beta A_\alpha \equiv A_{\alpha|\beta}$. Four-dimensional scalar products are often indicated by a dot: thus, $\ell_{(A)} \cdot \ell_{(B)} \equiv g_{\alpha\beta}\, \ell^{\alpha}_{(A)}\, \ell^{\beta}_{(B)}$.

Intrinsic geometry of S

Consider, now, the intrinsic geometry of the surface S, by ignoring the extra imbedding dimensions. Given coordinates θ^a on S we can define basis one-forms $d\theta^a$ and a metric g_{ab} on S, so that the line element on S is

$$ds^2|_S = g_{ab} d\theta^a d\theta^b \ .$$

Quantities which transform as tensors under the "rotations" $\theta^a \to \theta^{a'} = \theta^{a'}(\theta^a)$, will be referred to as two-tensors. For example, the two-tensor $X_a{}^b$ transforms as

$$X_{a'}{}^{b'} = X_a{}^b \frac{\partial \theta^a}{\partial \theta^{a'}} \frac{\partial \theta^{b'}}{\partial \theta^b} \ .$$

The covariant derivative compatible with the metric g_{ab} is denoted by a semicolon. Christoffel symbols ${}^{(2)}\Gamma^a_{bc}$ and the Riemann tensor ${}^{(2)}R^a{}_{bcd} = \delta^a_{[c} g_{d]b}\, {}^{(2)}R$ are defined in the usual way.

Tangent vectors to S

Tangent vectors to S can be defined by reference to the imbedding (3.1)

$$e^{\alpha}{}_{(a)} = \frac{\partial x^\alpha}{\partial \theta^a} \ , \tag{3.2}$$

where the subscript a is treated here as a label, and α is a spacetime index. The tangent vectors act as projection operators: they project tensors in the four dimensional spacetime onto S. For example, the metric g_{ab} on S is obtained by the projection of $g_{\alpha\beta}$, the spacetime metric, ie.,

$$g_{ab} = g_{\alpha\beta} e^{\alpha}{}_{(a)} e^{\beta}{}_{(b)} \ . \tag{3.3}$$

Here we see that the two-tensor g_{ab} is viewed as a scalar function from the viewpoint of the four dimensional manifold.

The tangent vectors also act as pull-backs, mappings of two-tensors to the spacetime manifold. For example the two-tensor ${X_a}^b$ has the pull-back

$$X^\beta_\alpha = {X_a}^b {e_\alpha}^{(a)} {e^\beta}_{(b)} \,, \tag{3.4}$$

where ${e_\alpha}^{(a)} = g^{ab} g_{\alpha\beta} {e^\beta}_{(b)}$. From (3.4) we can see that two-tensors in general have a dual nature, they are tensors with respect to S but only tetrad-dependent scalars with respect to the full spacetime.

Normals to S

The normals to S, $\ell^{(A)}$ have uniquely defined directions, given by the gradients of the parameters u^A. However, since the normals are lightlike, they can be multiplied by an arbitrary scale factor

$$\ell^{(A)}_\alpha = e^\lambda \partial_\alpha u^A, \tag{3.5}$$

where λ is an arbitrary function. It is useful to allow for a normalisation of the two independent normals which is not unity. We have arbitrarily chosen the normalisation condition $\ell^{(0)} \cdot \ell^{(1)} := -e^\lambda$. Since the norm of a lightlike vector is zero, the inner product of the normals can be written

$$g^{\alpha\beta} \ell^{(A)}_\alpha \ell^{(B)}_\beta = e^\lambda \eta^{AB} \,, \tag{3.6}$$

where η^{AB} is the matrix

$$\eta^{AB} = \eta_{AB} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \,, \; \eta^{AB}\eta_{BC} = \delta^A_C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \,. \tag{3.7}$$

It is easily checked from the definitions of the tangent and normal vectors that their inner product is zero,

$$\ell^{(A)}_\alpha {e^\alpha}_{(a)} = 0 \,. \tag{3.8}$$

$(2+2)$-split of the metric

The vectors connecting two two-surfaces at different values of u^A are $\partial x^\alpha / \partial u^A$ which are not equal to $\ell^\alpha_{(A)}$. From (3.5) and (3.6) it can be seen that their difference is orthogonal to $\ell^\alpha_{(A)}$, or tangent to S. It is necessary to introduce shift vectors ${s_A}^a$, defined by

$$\frac{\partial x^\alpha}{\partial u^A} = \ell^\alpha_{(A)} + s^a_A \, e^\alpha_{(a)}, \tag{3.9}$$

which have been illustrated in figure 9. Similarly, equations (3.2) and (3.3) can be used to show that the difference between $e^{(a)}_\alpha$ and $\frac{\partial\theta^a}{\partial x^\alpha}$ is orthogonal to S. According to (3.9) the difference is:

$$e^{(a)}{}_\alpha = \frac{\partial\theta^a}{\partial x^\alpha} + e^{-\lambda} s_A{}^a \ell^{(A)}{}_\alpha \,, \tag{3.10}$$

This allows partial derivatives to be rewritten as

$$\partial_\alpha = e^{-\lambda}\ell^{(A)}_\alpha(\partial_A - s^a_A\partial_a) + e^{(a)}_\alpha\partial_a \,. \tag{3.11}$$

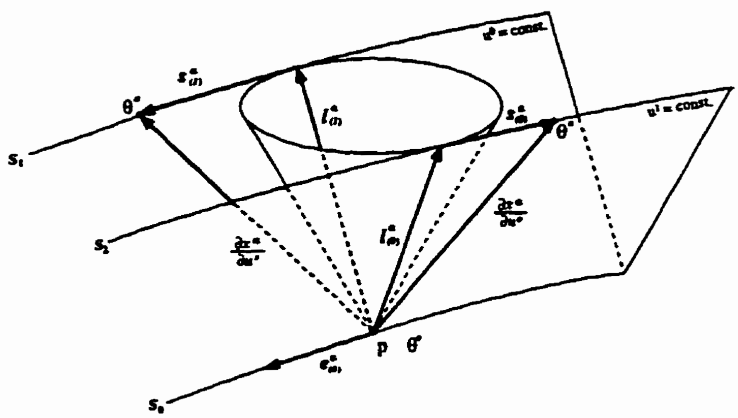

Figure 9: The light-cone in the double-null formalism. The intersection of two null hypersurfaces, u^0 = const. and u^1 = const. is shown on this diagram. The intersection (the line S_0) is a two dimensional spacelike surface, spanned by the two tangent vectors $e^\alpha_{(a)}$. The point p corresponds to a fixed value of the two coordinates θ^a on S_0. The future light-cone for p has lightlike tangent vectors $\ell^\alpha_{(0)}$ and $\ell^\alpha_{(1)}$. The vector $\ell^\alpha_{(0)}$ doesn't necessarily connect equal values of θ^a on the hypersurface u^1 = const. In general, equal values of θ^a are connected by the vector $\partial_0 x^\alpha$. The difference between the lightlike tangent and the connecting vector is the shift vector $s^\alpha_{(0)}$. The shift vector lies within the surface S. Similarly, a second shift vector $s^\alpha_{(1)}$ must be introduced.

An arbitrary displacement dx^α in spacetime is, according to (3.2) and (3.9), decomposable as

$$dx^\alpha = \ell^\alpha_{(A)}\,du^A + e^\alpha_{(a)}(d\theta^a + s^a_A\,du^A). \tag{3.12}$$

From (3.3), (3.6) and (3.8) we read off the completeness relation

$$g_{\alpha\beta} = e^{-\lambda}\eta_{AB}\ell^{(A)}_\alpha\ell^{(B)}_\beta + g_{ab}e^{(a)}_\alpha\, e^{(b)}_\beta \,. \tag{3.13}$$

Combining (3.12) and (3.13) shows that the spacetime metric is decomposable as

$$g_{\alpha\beta}dx^\alpha\, dx^\beta = e^\lambda \eta_{AB}\, du^A\, du^B + g_{ab}(d\theta^a + s^a_A\, du^A)(d\theta^b + s^b_B\, du^B). \tag{3.14}$$

First derivatives of the metric

We turn now to the definition of the extrinsic curvature and the twist, which are composed of first derivatives of the metric. First, it is useful to define a normal Lie derivative, D_A, which Lie propagates two-tensors at S along one of the normals $\ell_{(A)}$. Its action on the general two-tensor $X^{a\cdots}_{b\cdots}$ is defined by

$$D_A X^{a\cdots}_{b\cdots} := e^{(a)}_\alpha e^\beta_{(b)} \mathcal{L}_{\ell_{(A)}} X^{\alpha\cdots}_{\beta\cdots}\ , \tag{3.15}$$

where $X^{\alpha\cdots}_{\beta\cdots}$ is the pull-back defined by equation (3.4). It can be verified that this definition is equivalent to

$$D_A X^{a\cdots}_{b\cdots} = (\partial_A - {}^{(2)}\mathcal{L}_{s_A}) X^{a\cdots}_{b\cdots}\ , \tag{3.16}$$

where ${}^{(2)}\mathcal{L}_{s_A}$ is a two dimensional Lie derivative. In order to show the equivalence of the definitions (3.15) and (3.16), the following identities are useful. From the definitions of the normal (3.5) and tangent (3.2) vectors,

$$\mathcal{L}_{\ell_{(A)}} e_{(d)}{}^\alpha = e^\alpha_{(a)} \frac{\partial}{\partial\theta^d} s_A{}^a\ , \tag{3.17}$$

from which it follows that

$$e_{(d)}{}^\alpha \mathcal{L}_{\ell_{(A)}} e^{(a)}{}_\alpha = -e^{(a)}{}_\alpha \mathcal{L}_{\ell_{(A)}} e_{(d)}{}^\alpha = -\frac{\partial}{\partial\theta^d} s_A{}^a\ . \tag{3.18}$$

Substitution of (3.17) and (3.18) into (3.15) yields (3.16).

As examples of the use of this operator,

$$D_A\lambda = \partial_A\lambda - s^a_A \partial_a\lambda \tag{3.19}$$

$$D_A g_{ab} = \partial_A g_{ab} - 2 s_{A(a;b)}, \tag{3.20}$$

where a semi-colon denotes two-dimensional covariant differentiation.

The extrinsic curvatures K_{Aab} measure the change in the two-geometry as it is Lie propagated in either of the directions $\ell_{(A)}$ normal to S. Their values are

$$K_{Aab} := \frac{1}{2} D_A g_{ab}\ . \tag{3.21}$$

The dilation, K_A, measuring the change in area of a circle of light Lie transported along $\ell_{(A)}$, is the trace of the extrinsic curvature,

$$K_A := K_{Aab} g^{ab} = \partial_A \ln \sqrt{g} - s_{A\,;a}^{\ a} , \tag{3.22}$$

where g is the determinant of the metric on S. The shear, σ_{Aab}, measures the distortion of the circle and is defined as the traceless part of the extrinsic curvature

$$\sigma_{Aab} := \frac{1}{2}\sqrt{g}\,\partial_A \left(\frac{g_{ab}}{\sqrt{g}}\right) - s_{A(a;b)} + \frac{1}{2} g_{ab} s_{A\,;d}^{\ d} . \tag{3.23}$$

The twist ω^a is defined by

$$\epsilon_{AB}\omega^a e_{(a)} := \left[\ell_{(B)}, \ell_{(A)}\right] , \tag{3.24}$$

where ϵ is the completely antisymmetric matrix with component $\epsilon_{01} = \epsilon^{10} = 1$ and the square bracket denotes a Lie bracket. When the twist is zero then the curves tangent to $\ell_{(0)}$ and $\ell_{(1)}$ mesh together to form two-surfaces orthogonal to the surfaces S. An equivalent definition which follows directly from (3.24) and (3.9)

$$\omega^a = \epsilon^{AB}(\partial_B s_A^a - s_B^b s_{A\,;b}^{\ a}). \tag{3.25}$$

3.2 The Gauss-Weingarten relations

In the classical theory of surfaces imbedded in a three dimensional Riemannian manifold, the Gauss-Weingarten relations describe the variation of the tangent and normal vectors in the directions defined by these vectors. In this section we extend these equations to the imbedding of a surface in a four dimensional Lorentzian manifold. The complication in doing so is that there are now two normals which are both null. However, as we shall show in this section, the derivation of these relations is not particularly difficult.

Tangential variation of the basis vectors

We begin by computing the Gauss-Weingarten equations for the change in $\ell_{(A)}$ and $e_{(a)}$ as they are varied in a direction tangential to S. In other words, we wish to compute the covariant derivative in the direction of $e_{(b)}$. The results are that the tangential derivative of the tangent vectors is

$$e_{(b)} \cdot \nabla e_{(a)}^{\alpha} = {}^{(2)}\Gamma^c_{ab} e_{(c)}^{\alpha} - e^{-\lambda} K^A{}_{ab} \ell_{(A)}^{\alpha} , \tag{3.26}$$

while the tangential variation of the normal vectors is

$$e_{(b)} \cdot \nabla \ell^{\alpha}_{(A)} = K_{Aab} e^{(a)\alpha} + L_{ABb} \ell^{(B)\alpha} \tag{3.27}$$
$$L_{ABb} = \frac{1}{2}\eta_{AB}\partial_b \lambda + \frac{1}{2}\epsilon_{AB}\omega_b e^{-\lambda} \,.$$

In order to derive equations (3.26) and (3.27) the following identities are needed. From the definition of the tangent vectors (3.2) and the symmetry of the connection, it directly follows that

$$\begin{aligned} e_{(b)} \cdot \nabla e^{\alpha}_{(a)} &= e_{(a)} \cdot \nabla e^{\alpha}_{(b)} \,, \\ e^{(c)}_{\alpha} e_{(b)} \cdot \nabla e^{\alpha}_{(a)} &= {}^{(2)}\Gamma^{c}_{ab} \,. \end{aligned} \tag{3.28}$$

From the orthogonality of $\ell_{(A)}$ and $e_{(a)}$, it follows that

$$\ell_{(A)\alpha} e_{(b)} \cdot \nabla e^{\alpha}_{(a)} = -e^{\alpha}_{(a)} e_{(b)} \cdot \nabla \ell_{(A)\alpha} \,. \tag{3.29}$$

The definition of the Lie derivative allows the term on the right hand side of (3.29) to be written as

$$\begin{aligned} 2e^{\alpha}_{(a)} e_{(b)} \cdot \nabla \ell_{(A)\alpha} &= g_{d(a} e_{(b))}{}^{\alpha} \mathcal{L}_{\ell_{(A)}} e^{(d)}{}_{\alpha} - e_{\alpha((a)} \mathcal{L}_{\ell_{(A)}} e_{(b)}{}^{\alpha} + \ell^{\gamma}_{(A)} \nabla_{\gamma} g_{ab} \\ &= 2K_{Aab} \,. \end{aligned} \tag{3.30}$$

The second equality follows from the application of (3.21). The tangential variation of the tangent vectors (3.26), follows directly from equations (3.28), (3.29) and (3.30).

Since $\ell^{(B)}_{\alpha}$ is proportional to a gradient,

$$e^{\beta}_{(b)} \ell^{\gamma[A} \ell^{B]}{}_{[\gamma|\beta]} = e^{\beta}_{(b)} \ell^{\gamma[A} \ell^{B]}{}_{[\gamma} \partial_{\beta]} \lambda = 0.$$

From this result it follows that

$$\begin{aligned} e^{\beta}_{(b)} \ell^{\gamma}_{[A} \ell_{B]\gamma|\beta} &= e^{\beta}_{(b)} \ell^{\gamma}_{[A} \ell_{B]\beta|\gamma} \\ &= \frac{1}{2}\omega_b \epsilon_{BA} \,. \end{aligned} \tag{3.31}$$

The symmetric counterpart to (3.31) is

$$e^{\beta}_{(b)} \ell^{\gamma}_{(A} \ell_{B)\gamma|\beta} = \frac{1}{2}\partial_b \lambda \eta_{AB} \,. \tag{3.32}$$

The tangential variation of the normal vectors (3.27), follows from (3.30), (3.31) and (3.32).

Normal variation of the basis vectors

Similar results can be found for the variation of the tangent and normal vectors in a direction defined by $\ell_{(A)}$. The normal variation of the tangent vectors is

$$\ell_A \cdot \nabla e^\beta_a = (K_{Aa}{}^b + \partial_a s^b_A) e^\beta_b + L_{ABa} \ell^{B\beta} \,. \tag{3.33}$$

The normal variation of the normal vectors is

$$\begin{aligned} \ell_B \cdot \nabla \ell_{A\beta} &= N_{ABD} \ell^D_\beta - e^\lambda L_{BAa} e^a_\beta \\ N_{ABD} &= \eta_{D(A} D_{B)} \lambda - \frac{1}{2} \eta_{AB} D_D \lambda \,. \end{aligned} \tag{3.34}$$

In order to derive equation (3.33), note that from the definition of the tangent vectors and the symmetry of the connection,

$$\frac{\partial x^\alpha}{\partial u^A} \nabla_\alpha e^\beta_a = e^\alpha_a \nabla_\alpha \frac{\partial x^\beta}{\partial u^A} \,. \tag{3.35}$$

Substitution of (3.9) into (3.35) and making use the of the Gauss-Weingarten equation (3.27), results in equation (3.33).

The derivation of (3.34) requires the following formula which follows from (3.5):

$$\ell^A_{\beta|\alpha} = \ell^A_{\alpha|\beta} + \partial_\alpha \lambda \ell^A_\beta - \partial_\beta \lambda \ell^A_\alpha \,. \tag{3.36}$$

The symmetric component of (3.34) is computed using (3.36) and (3.11),

$$\ell^\alpha_{(B} \ell_{A)\beta|\alpha} = D_{(B} \lambda \ell_{A)\beta} - \frac{1}{2} \eta_{AB} (\ell^D_\beta D_D \lambda + e^\lambda e^a_\beta \partial_a \lambda) \,. \tag{3.37}$$

The antisymmetric component of (3.34) is just the Lie derivative given by equation (3.24). The normal variation of the normals follows from equations (3.36) and (3.24).

3.3 Commutation relations

In this section we derive the commutation relations between the different derivative operators. First we look at the commutator of the normal derivatives acting on a general two-tensor X^a_b, which by definition (3.15) is

$$D_{[A} D_{B]} X^a_b = e^a_\alpha e^\beta_b \mathcal{L}_{\ell_{[A}} \left(\Lambda^\alpha_\gamma \Lambda^\delta_\beta \mathcal{L}_{\ell_{B]}} (e^\gamma_i e^j_\delta X^i_j) \right) \,, \tag{3.38}$$

where the projector, Λ^α_γ is defined by

$$\Lambda^\alpha_\gamma = e^\alpha_c e^c_\gamma = \delta^\alpha_\gamma - e^{-\lambda}\ell^\alpha_A \ell^A_\gamma \ . \tag{3.39}$$

The Lie derivative of the projector follows from (3.39) and the definition of the twist (3.24),

$$\mathcal{L}_{\ell_B}\Lambda^\alpha_\gamma = -\mathcal{L}_{\ell_B} e^{-\lambda}\ell^\alpha_A \ell^A_\gamma = -e^{-\lambda}\ell^A_\gamma \epsilon_{AB}\omega^a e^\alpha_a \ .$$

From this result it follows that

$$D_{[A}D_{B]}X^a_b = e^a_\alpha e^\beta_b \mathcal{L}_{\ell_{[A}}\mathcal{L}_{\ell_{B]}}(e^\alpha_i e^j_\beta X^i_j) - e^{-\lambda}\ell^D_\alpha e^\beta_b \omega^a \epsilon_{D[A}\mathcal{L}_{\ell_{B]}}(e^\alpha_i e^j_\beta X^i_j) \ . \tag{3.40}$$

The second term vanishes since the Lie derivative of e^α_i (see equation (3.17)) has no normal component.

The first term in (3.40) can be simplified by noting that the commutator of two Lie derivatives is the Lie derivative of the commutator, ie.,

$$2\mathcal{L}_{\ell_{[A}}\mathcal{L}_{\ell_{B]}} = \mathcal{L}_{[\ell_A,\ell_B]} = \epsilon_{BA}\mathcal{L}_{\omega^a e_a} \ .$$

The action of this operator on the tangent vectors is

$$e^\alpha_i \mathcal{L}_{\omega^b e_b} e^a_\alpha = -e^a_\alpha \mathcal{L}_{\omega^b e_b} e^\alpha_i = \partial_i \omega^a \ .$$

The result is that

$$D_{[A}D_{B]}X^a_b = \frac{1}{2}\epsilon_{BA}\ {}^{(2)}\mathcal{L}_\omega X^a_b \ . \tag{3.41}$$

The following are examples of the action of this commutator:

$$D_{[B}D_{A]}\lambda = \frac{1}{2}\epsilon_{AB}\omega^a\partial_a\lambda \tag{3.42}$$

$$D_{[B}K_{A]ab} = \frac{1}{2}D_{[B}D_{A]}g_{ab} = \frac{1}{2}\,\epsilon_{AB}\,\omega_{(a;b)} \tag{3.43}$$

$$D_{[B}K_{A]} = \frac{1}{2}g^{ab}D_{[B}D_{A]}g_{ab} = \frac{1}{2}\,\epsilon_{AB}\,\omega^a{}_{;a} \ . \tag{3.44}$$

The commutation relation between the two dimensional covariant and Lie derivatives follows from their definitions and the two dimensional Ricci commutation relations:

$$\begin{aligned} {}^{(2)}\mathcal{L}_{s_A}X^a_{b;c} - ({}^{(2)}\mathcal{L}_{s_A}X^a_b)_{;c} &= X^d_b s_A{}^a{}_{;(cd)} - X^a_d s_A{}^d{}_{;(bc)} \\ &+\frac{1}{2}\,{}^{(2)}R\left(X^d_b s_A{}^a g_{cd} - X^a_d s_A{}^d g_{bc} + \frac{1}{2}X^a_c s_{Ab} - \frac{1}{2}X^d_b s_{Ad}\delta^a_c\right) \ . \end{aligned}$$

This can be simplified, by noting that the two dimensional Christoffel symbols are closely related to the extrinsic curvature. Defining $\bar{K}_{Aab} = \frac{1}{2}\partial_A g_{ab}$, the relation is

$$\begin{aligned} \partial_A {}^{(2)}\Gamma^c_{ab} &= 2\bar{K}_A{}^c{}_{a;b} - \bar{K}_{Aab}{}^{;c} \\ &= 2K_A{}^c{}_{a;b} - K_{Aab}{}^{;c} + s_A{}^c{}_{;ab} + \frac{1}{2}\,{}^{(2)}R(s_A{}^c g_{ab} - s_{A(a}\delta^c_{b)})\,. \end{aligned} \tag{3.45}$$

A little bit of algebra reveals that

$${}^{(2)}\mathcal{L}_{s_A} X^a_{b;c} - ({}^{(2)}\mathcal{L}_{s_A} X^a_b)_{;c} = X^d_b(\partial_A {}^{(2)}\Gamma^a_{dc} - \Gamma^a_{(A)dc}) - X^a_d(\partial_A {}^{(2)}\Gamma^d_{bc} - \Gamma^d_{(A)bc})\,, \tag{3.46}$$

where

$$\Gamma^a_{(A)bc} = 2K_{A(b}{}^a{}_{;c)} - K_{Abc}{}^{;a}. \tag{3.47}$$

To conclude, we note the rule for commuting D_A and the two-dimensional covariant derivative ∇_a, which follows directly from (3.46) and the definition (3.16) of D_A. The commutator $[D_A, \nabla_a]$, applied to any two-tensor, is formed by a pattern similar to its two-dimensional covariant derivative, but with ${}^{(2)}\Gamma^a_{bc}$ replaced by $\Gamma^a_{(A)bc}$. As examples:

$$[D_A, \nabla_a]\lambda = 0 \tag{3.48}$$

$$[D_A, \nabla_a]X^b = X^d\Gamma^b_{(A)da} \tag{3.49}$$

$$[D_A, \nabla_a]g_{bc} = -2\Gamma^d_{(A)a(b}g_{c)d} = -2K_{Abc;a}\,. \tag{3.50}$$

3.4 The Gauss-Codazzi relations

The Gauss-Codazzi relations are the integrability conditions of the system of first order (Gauss-Weingarten) differential equations (3.26), (3.27), (3.33) and (3.34). They express projections of the four-dimensional Riemann tensor in terms of K, L, N and their first derivatives. The most concise way of deriving these components in practice is through the Ricci commutation relations.

Let A^α, B^α, X^α and Y^α be arbitrary 4-vectors. The Ricci commutation relation is

$$\begin{aligned} R_{\alpha\beta\gamma\delta}X^\alpha A^\beta Y^\gamma B^\delta &= X^\alpha A^\beta Y^\gamma(\nabla_\alpha\nabla_\beta - \nabla_\beta\nabla_\alpha)B_\gamma \\ &= A^\alpha(X\cdot\nabla(Y\cdot\nabla B_\alpha) - Y\cdot\nabla(X\cdot\nabla B_\alpha)) \\ &\quad + (A^\alpha\nabla_\beta B_\alpha)\mathcal{L}_Y X^\beta\,, \end{aligned} \tag{3.51}$$

which relates covariant derivatives of the Gauss-Weingarten equations with the components of the Riemann tensor when the vectors in (3.51) are set equal to the tangent and normal vectors.

Similarly, the contracted Ricci commutation relation is

$$R_{\alpha\beta}A^{\alpha}B^{\beta} = \nabla_{\beta}(A^{\alpha}\nabla_{\alpha}B^{\beta}) - A^{\alpha}\nabla_{\alpha}(\nabla_{\beta}B^{\beta}) - (\nabla_{\beta}A^{\alpha})(\nabla_{\alpha}B^{\beta}). \tag{3.52}$$

Substitution of the normal and tangent vectors into this equation yields the components of the Ricci tensor. The calculation of the Ricci components is straight forward. The details of the calculation are presented in appendix A. Here we will only display the final results. Our notation for the tetrad components is typified by

$${}^{(4)}R_{ab} = R_{\alpha\beta}e^{\alpha}_{(a)}e^{\beta}_{(b)}, \qquad R_{aA} = R_{\alpha\beta}e^{\alpha}_{(a)}\ell^{\beta}_{(A)}.$$

The results are

$$\begin{aligned} {}^{(4)}R_{ab} &= \frac{1}{2}{}^{(2)}Rg_{ab} - e^{-\lambda}\left(D_A + K_A\right)K^{A}{}_{ab} \\ &\qquad +2e^{-\lambda}K_{A(a}{}^{d}K^{A}{}_{b)d} - \frac{1}{2}e^{-2\lambda}\,\omega_a\omega_b - \lambda_{;ab} - \frac{1}{2}\lambda_{,a}\lambda_{,b} \qquad (3.53) \\ R_{AB} &= -D_{(A}K_{B)} - K_{Aab}K_B{}^{ab} + K_{(A}D_{B)}\lambda \\ &\qquad -\frac{1}{2}\eta_{AB}\left[\left(D^E + K^E\right)D_E\lambda - e^{-\lambda}\omega^a\omega_a + (e^{\lambda})^{;a}{}_{a}\right] \qquad (3.54) \\ R_{Aa} &= K_{Aa}{}^{b}{}_{;b} - \partial_a K_A - \frac{1}{2}\partial_a D_A\lambda + \frac{1}{2}K_A\partial_a\lambda \\ &\qquad +\frac{1}{2}\epsilon_{AB}e^{-\lambda}\left[\left(D^B + K^B\right)\omega_a - \omega_a D^B\lambda\right]. \qquad (3.55) \end{aligned}$$

Equation (3.53), containing the term $D_A K^{A}{}_{ab}$ which is roughly of the form $\Box g_{ab}$, represents three equations which describe the evolution of the two-metric. The Raychaudhuri equations describing the focussing of light rays are encoded in the R_{00} and R_{11} components of equation (3.54). The equation $R_{00} = 8\pi(T_{00} - \frac{1}{2}g_{00}T_{\alpha\beta}g^{\alpha\beta})$ relates the change in the dilation rate, D_0K_0 on a $u^1 = constant$ hypersurface to $K_{0ab}K_0{}^{ab}$, the square of the dilation and shear and of the matter flowing transverse to the hypersurface. The component R_{01} is the trivial equation [53] (as we shall show when we discuss the characteristic initial value problem). The four equations represented by equation (3.55) are constraints on the values of the shift vectors on the null hypersurfaces.

The components of the Riemann tensor can be calculated using equation (3.51). A sample calculation of the component R_{AaBb} is included in appendix A, and the other components are calculated in a similar manner. The results are

$$
\begin{aligned}
{}^{(4)}R^{ab}{}_{cd} &= {}^{(2)}R\,\delta^a_{[c}\delta^b_{d]} - 2e^{-\lambda}K_{A[c}{}^{a}K^{Ab}{}_{d]} && (3.56)\\
R_{ABCD} &= \frac{1}{4}\epsilon_{AB}\,\epsilon_{CD}(2e^{\lambda}D^{E}D_{E}\lambda - 3\omega^{a}\,\omega_{a} + e^{2\lambda}\lambda^{,a}\lambda_{,a}) && (3.57)\\
R_{Aabc} &= 2K_{Aa[b;c]} - K_{Aa[b}\lambda_{,c]} - e^{-\lambda}\,\epsilon_{AB}K^{B}{}_{a[b}\omega_{c]} && (3.58)\\
R_{aABC} &= \frac{1}{2}\epsilon_{BC}\{D_{A}\omega_{a} + K_{Aab}\,\omega^{b} - e^{\lambda}\,\epsilon_{AE}(D^{E}\partial_{a}\lambda - K^{E}{}_{ab}\lambda^{,b}) - \omega_{a}D_{A}\lambda\} && (3.59)\\
R^{A}{}_{a}{}^{B}{}_{b} &= -D^{(A}K^{B)}{}_{ab} + K^{A}{}_{bd}K^{Bd}{}_{a} + D^{(A}\lambda\,K^{B)}{}_{ab} - \frac{1}{2}\eta^{AB}D_{E}\lambda\,K^{E}{}_{ab}\\
&\quad -\frac{1}{4}\eta^{AB}(e^{-\lambda}\omega_{a}\omega_{b} + e^{\lambda}\lambda_{,a}\lambda_{,b} + 2e^{\lambda}\lambda_{;ab})\\
&\quad -\frac{1}{2}\epsilon^{AB}(-\lambda_{,[a}\omega_{b]} + \omega_{[b;a]}). && (3.60)
\end{aligned}
$$

3.5 Comparison with the NP spin-coefficient formalism

In order to compare our notation with the more familiar notation of the NP [54] spin-coefficient formalism and the GHP [44, 55] formalism, we introduce the following complex null tetrad

$$
\begin{aligned}
&\ell^{\alpha} = e^{-\frac{1}{2}\lambda}\ell^{\alpha}_{(0)}\ ,\ n^{\alpha} = e^{-\frac{1}{2}\lambda}\ell^{\alpha}_{(1)}\ ,\ m^{\alpha} = e^{\alpha}_{(a)}m^{a}\\
&m\cdot\bar{m} = -\ell\cdot n = 1\ , && (3.61)
\end{aligned}
$$

and all other inner products are zero. The complex two-vector m_a is defined by

$$
\begin{aligned}
g_{ab} &= 2m_{(a}\bar{m}_{b)}\\
m^{b} &= m_{a}g^{ab}\\
m^{a}m_{a} &= m^{a}\bar{m}_{a} - 1 = 0\ . && (3.62)
\end{aligned}
$$

It is straightforward to calculate the NP and GHP spin coeffients from their definition given in reference [54, 44]. The spin coefficients will be denoted with bold Greek letters. Since the null vectors $\ell_{(A)}$ are tangent to geodesics, $\boldsymbol{\kappa} = \boldsymbol{\kappa}' = 0$. Furthermore, we assume in the double-null formalism that both $\ell_{(A)}$ are hypersurface orthogonal. As a result the rotation, or vorticity of the null vectors vanishes, i.e.,

$\rho - \bar{\rho} = \rho' - \bar{\rho}' = 0$. The remaining non-zero spin coefficients are

$$\sigma = m^a m^b e^{-\lambda/2}\sigma_{0ab} \quad ; \quad \sigma' = \bar{m}^a \bar{m}^b e^{-\lambda/2}\sigma_{1ab}$$
$$\rho = \frac{1}{2}e^{-\lambda/2}K_0 \quad ; \quad \rho' = \frac{1}{2}e^{-\lambda/2}K_1$$
$$\tau = \frac{1}{2}m^b(\lambda_{,b} + \omega_b e^{-\lambda}) \quad ; \quad \tau' = \frac{1}{2}\bar{m}^b(\lambda_{,b} - \omega_b e^{-\lambda})$$
$$\beta = \frac{1}{4}m^b\omega_b e^{-\lambda} \quad ; \quad \beta' = -\frac{1}{4}\bar{m}^b\omega_b e^{-\lambda}$$
$$\epsilon = -\frac{1}{4}e^{-\lambda/2}(D_0\lambda + \bar{m}^b D_0 m_b - m^b D_0 \bar{m}_b)$$
$$\epsilon' = -\frac{1}{4}e^{-\lambda/2}(D_1\lambda - \bar{m}^b D_1 m_b + m^b D_1 \bar{m}_b)\ .$$

The primed GHP coefficients are related to the Newman-Penrose coefficients [54] by

$$\kappa' = -\nu \quad \sigma' = -\lambda \quad \rho' = -\mu$$
$$\tau' = -\pi \quad \beta' = -\alpha \quad \epsilon' = -\gamma.$$

The Weyl scalars, as defined in references [44] and [54] are related to the components of the Riemann tensor by

$$\Psi_0 = -e^{-\lambda}m^a m^b R_{0a0b} \tag{3.63}$$
$$\Psi_1 = -e^{-\frac{3}{2}\lambda}m^a R_{010a} \tag{3.64}$$
$$\mathrm{Re}\Psi_2 = -\frac{1}{4}g^{ad}g^{bc}\,{}^{(4)}R_{abcd} - \frac{1}{4}g_{ab}\,{}^{(4)}R^{ab} + \frac{1}{12}\,{}^{(4)}R \tag{3.65}$$
$$\mathrm{Im}\Psi_2 = -iR_{[0}{}^{a}{}_{1]}{}^{b}\bar{m}_a m_b \tag{3.66}$$
$$\Psi_3 = e^{-\frac{3}{2}\lambda}\bar{m}^a R_{011a} \tag{3.67}$$
$$\Psi_4 = -e^{-\lambda}\bar{m}^a\bar{m}^b R_{1a1b} \tag{3.68}$$

When comparing the equations resulting from the spin-coefficient and double-null formalisms it should be remembered that the two formalisms assume opposite signatures. The presence of the derivatives of the shear axes in the ϵ and ϵ' coefficients cause many of the spin-coefficient field equations to be unneccessarily complicated. This complication occurs because the two-dimensionally covariant quantities (such as our shear tensor σ_{Aab}) are contracted with the axis vectors m_a in the NP formalism. Since we present the field equations in a two-dimensionally covariant form, they can be stated in the precise form of equations (3.53) - (3.55). The role of the equations in the double-null formalism presented in this chapter is fairly straightforward as will be shown in the next section, where we will examine the Bianchi identities, which will make it clear which equations are needed to find a solution to the Einstein field equations.

3.6 Bianchi identities

The Ricci components are linked by four differential identities, the contracted Bianchi identities

$$\nabla_\beta R^\beta{}_\alpha = \frac{1}{2}\partial_\alpha R, \tag{3.69}$$

where the four-dimensional curvature scalar $R = R^\alpha{}_\alpha$ is given by

$$R = e^{-\lambda} R^A{}_A + R^a{}_a, \tag{3.70}$$

according to (3.13).

As we show in Appendix B, projecting (3.69) onto $e_{(a)}$ leads to

$$(D_A + K_A)\, R^A{}_a = \frac{1}{2}\partial_a R^A{}_A + \frac{1}{2}e^\lambda \partial_a\, {}^{(4)}\!R^b{}_b - \left(e^\lambda\, {}^{(4)}\!R^b{}_a\right)_{;b}. \tag{3.71}$$

Projection of (3.69) onto $\ell_{(A)}$ similarly yields

$$\begin{aligned} D_B\left(R^B{}_A - \frac{1}{2}\delta^B_A R^D{}_D\right) - \frac{1}{2}e^\lambda D_A\, {}^{(4)}\!R^a{}_a \\ = e^\lambda\, {}^{(4)}\!R_{ab} K_A{}^{ab} - R^B{}_A K_B - \left(e^\lambda R^a{}_A\right)_{;a} + \epsilon_{AB}\omega^a R^B{}_a. \end{aligned} \tag{3.72}$$

Equations (3.71) and (3.72) express the four contracted Bianchi identities in terms of the tetrad components of the Ricci tensor.

We now look at the general structure of these equations.

For $A = 0$ in (3.72), $R_0{}^0$ does not contribute to the first (parenthesized) term, since

$$-R_{01} = R_0{}^0 = R_1{}^1 = \frac{1}{2}R_A{}^A. \tag{3.73}$$

This equation therefore takes the form

$$D_1 R_{00} + \frac{1}{2}e^\lambda D_0\, {}^{(4)}\!R_a{}^a = -K_0 R_{01} + \mathcal{L}({}^{(4)}\!R_{ab}, R_{00}, R_{0a}, \partial_a), \tag{3.74}$$

in which the schematic notation $\mathcal{L}$ implies that the expression is linear homogeneous in the indicated Ricci components and their two-dimensional spatial derivatives ∂_a.

The other ($A = 1$) component of (3.72) has the analogous structure

$$D_0 R_{11} + \frac{1}{2} e^{\lambda} D_1 \, {}^{(4)}R_a{}^a = -K_1 R_{01} + \mathcal{L}({}^{(4)}R_{ab}, R_{11}, R_{1a}, \partial_a). \tag{3.75}$$

The form of the remaining two Bianchi identities (3.71) is

$$D_0 R_{1a} + D_1 R_{0a} = \mathcal{L}({}^{(4)}R_{ab}, R_{01}, R_{Aa}, \partial_a). \tag{3.76}$$

The structure of (3.74)–(3.76) provides insight into how the field equations propagate initial data given on a lightlike hypersurface. Let us (arbitrarily) single out u^0 as "time," and suppose that the six "evolutionary" vacuum equations

$${}^{(4)}R_{ab} = 0, \qquad R_{00} = R_{0a} = 0 \tag{3.77}$$

are satisfied everywhere in the neighbourhood of a hypersurface $u^0 = 0$. (Bondi and Sachs [56, 53] refer to ${}^{(4)}R_{ab}$ as the "propagating" or "main" equations and to R_{00}, R_{0a} as "hypersurface equations.") Since R_{01} only appears algebraically in equation (3.74), the vanishing of the six evolutionary equations 3.77 in the region guarantee that R_{01} also vanishes. For this reason $R_{01} = 0$ is dubbed the "trivial equation" [56, 53].

Equations (3.75) and (3.76) imply that if R_{11} and R_{1a} vanish on the hypersurface $u^0 = 0$, then they automatically vanish on all other hypersurfaces in the region. The equations $R_{11} = R_{1a} = 0$ are known as the "subsidiary" or "supplementary" equations [56, 53].

The result is that the Bianchi identities guarantee that if the evolutionary equations are satisfied in a region and the subsidiary equations are satisfied on one hypersurface, then the vacuum Einstein equations are satisfied everywhere in the region. It should be noted however, that the numerical implementation of a double-null scheme is subject to instabilities [37]. When performing a numerical integration, it is necessary to check that the subsidiary equations are satisfied on all hypersurfaces during the evolution. In this thesis we are only concerned with analytic solutions of the field equations, so this problem is not relevant.

3.7 Characteristic initial value problem

In the characteristic formulation of general relativity, data are placed on two intersecting characteristics Σ^0 and Σ^1 and their spacelike intersection S_0, and evolved off the characteristics using the propagating Einstein equations $R_{ab} = 0$. (We have arbitrarily designated Σ^0 as the hypersurface at $u^0 = 0$ and Σ^1 as $u^1 = 0$.) In this section we discuss the formal solution of the characteristic initial value problem (CIVP).

Gauge fixing

As we have shown in section 2.6, the Bianchi identities act as constraints on the ten Einstein equations, leaving only the six equations (3.77) independent. The double-null metric (3.14) has eight free functions, so clearly there exists the freedom to make two global gauge conditions.

A natural coordinate condition is to demand that once the coordinates θ^a are defined on Σ^0 their values must remain the same if they are Lie transported off Σ^0. ie.,

$$0 = \mathcal{L}_{\ell_0}\theta^a = \ell_0^\alpha \partial_\alpha \theta^a = -s_0^a \,,. \tag{3.78}$$

This gauge choice leads to some nice simplifications. The normal Lie derivative operator becomes a simple partial derivative and the extrinsic curvature and twist take on the simple forms:

$$s_0^a = 0 \Rightarrow \quad D_0 = \partial_0 \tag{3.79}$$

$$K_{0ab} = \frac{1}{2}\partial_0 g_{ab} \tag{3.80}$$

$$K_0 = \partial_0 \ln\sqrt{g} \tag{3.81}$$

$$\sigma_{0ab} = \frac{1}{2}\sqrt{g}\,\partial_0\left(\frac{g_{ab}}{\sqrt{g}}\right) \tag{3.82}$$

$$\omega^a = -\partial_0 s_1^a \,. \tag{3.83}$$

In addition to the global gauge condition (3.78), we have the freedom to place coordinate conditions on the initial surface Σ^0. A related choice is to demand that the coordinates θ^a be kept constant when Lie transported along Σ^0. This is equivalent to stating that

$$\text{on } u^0 = 0\,, 0 = \mathcal{L}_{\ell_1}\theta^a = -s_1^a \,. \tag{3.84}$$

This leads to the following simplifications on Σ^0:

$$\text{on } u^0 = 0\,, s_1^a = 0 \Rightarrow \quad D_1 = \partial_1 \tag{3.85}$$

$$K_{1ab} = \frac{1}{2}\partial_1 g_{ab} \tag{3.86}$$

$$K_1 = \partial_1 \ln \sqrt{g} \tag{3.87}$$

$$\sigma_{1ab} = \frac{1}{2}\sqrt{g}\partial_1 \left(\frac{g_{ab}}{\sqrt{g}}\right) . \tag{3.88}$$

For the rest of this thesis we will assume that the gauge conditions (3.78) and (3.84) have been made and for simplicity will define $s^a := s^a_1$.

The hypersurface equations

With the gauge choice (3.78), the hypersurface equations reduce to

$$R_{00} = -\partial_0 K_0 - \frac{1}{2}K_0^2 + K_0\partial_0\lambda - \sigma_{0ab}\sigma_0{}^{ab} \tag{3.89}$$

$$R_{0a} = -\partial_0(e^{-\lambda}\sqrt{g}\omega_a) - \frac{1}{2}\partial_a K_0 + \sigma_{0a}{}^b{}_{;b} - \frac{1}{2}\partial_a\partial_0\lambda + \frac{1}{2}K_0\partial_a\lambda , \tag{3.90}$$

three equations for the six functions $K_0, \sigma_{0ab}, \lambda, \omega_a$. (Note that the shear (3.82) σ_{0ab} is traceless and has only two degrees of freedom.)

So far, the only restrictions made on the coordinates u^A is that they be lightlike. We still have the freedom to rescale the null coordinates by a function of themselves, $u^0 \to u^{0'}(u^0)$. Thus on Σ^1 we can make a coordinate choice which will simplify the integration of the hypersurface equations. For example, consider the parametrization of ℓ_0. Since from equation (3.34)

$$\ell_0 \cdot \nabla \ell_0^\alpha = \partial_0\lambda \ell_0^\alpha , \tag{3.91}$$

when $\partial_0\lambda = 0$, the parameter u^0 is affine. Thus we are free on Σ^1 to choose the "inaffinity" [57] of the parameter u^0. One special choice [57] is to set $\partial_0\lambda = \frac{1}{2}K_0$ which reduces (3.89) to a linear equation for K_0. However, linearity is not important. When u^0 is affine and given initial data σ_{0ab} on Σ^1 and the value of K_0 on S_0, equation (3.89) can be solved for K_0 everywhere on Σ^1. Alternately, if in a physical application there is a known form of the dilation K_0 on Σ^1, then (3.89) can be solved for λ (as long as $K_0 \neq 0$). Either way, after (3.89) has been solved, $K_0, \sigma_{0ab}, \lambda$ and all partial derivative ∂_a are known on Σ^1. From (3.81) and (3.82), $\sqrt{g}$ and g_{ab} are also known on Σ^1. Hence the last two hypersurface equations (3.90) are linear first order equations for ω_a. Once initial data for ω_a are set on S_0, then ω_a is known on all of Σ^1. The differential equations (3.83) can then be solved for the shift vector. The coordinate condition (3.84) sets $s^a = 0$ on S_0, so that s^a is also known everywhere on Σ^1. Hence, on Σ^1 all the metric functions, and their time and tangential derivatives ∂_0 and ∂_a are known.

The subsidiary equations

On the hypersurface Σ^0, where the condition (3.84) has been set, the supplementary equations reduce to

$$R_{11} = -\partial_1 K_1 - \frac{1}{2}K_1^2 + K_1\partial_1\lambda - \sigma_{1ab}\sigma_1{}^{ab} \tag{3.92}$$

$$R_{1a} = -\partial_1(e^{-\lambda}\sqrt{g}\omega_a) - \frac{1}{2}\partial_a K_1 + \sigma_{1a}{}^b{}_{;b} - \frac{1}{2}\partial_a\partial_1\lambda + \frac{1}{2}K_1\partial_a\lambda\,. \tag{3.93}$$

The remarks of the previous section on the hypersurface equations hold here. By choosing u^1 affine on Σ^0, and given initial data σ_{1ab}, the subsidiary equations can be integrated for K_1 and ω_a. Now all the metric functions and their derivatives ∂_1 and ∂_a are known on the hypersurface Σ^0. The problem remains to evolve the data to later times $u^0 > 0$, given the initial data on Σ^0 and the boundary data on Σ^1.

The propagating equations

The three propagating equations ${}^{(4)}R_{ab} = 0$ control the evolution of the shear and dilation off the initial characteristic Σ^0. After splitting the main equations into the trace and traceless parts, they can be written as

$$\begin{aligned} {}^{(4)}R^a_a &= 2e^{-\lambda}(D_1K_0 + K_0K_1) \\ &\qquad {}^{(2)}R - \frac{1}{2}\lambda_{,a}\lambda^{,a} - \lambda^{;a}{}_{;a} - \frac{1}{2}e^{-2\lambda}\omega^a\omega_a - e^{-\lambda}\omega^a{}_{:a} \end{aligned} \tag{3.94}$$

$$\begin{aligned} {}^{(4)}R^a_b - \frac{1}{2}\delta^a_b\, {}^{(4)}R^d_d &= e^{-\lambda}(2D_1\sigma_{0a}{}^b + K_0\sigma_{1a}{}^b + K_1\sigma_{0a}{}^b) + e^{-\lambda}\omega^a{}_{;b} \\ &\quad -\frac{1}{2}e^{-2\lambda}\omega^b\omega_a - \lambda^{;b}{}_{;a} - \frac{1}{2}\lambda_{;a}\lambda^{;b} \\ &\quad -\frac{1}{2}\delta^b_a\left(\frac{1}{2}e^{-2\lambda}\omega^a\omega_a - 2\lambda_{;a}{}^{;a} - \frac{3}{2}\lambda_{;a}\lambda^{;a}\right)\,. \end{aligned} \tag{3.95}$$

On Σ^0 the data

$$\mathcal{D} = \{g_{ab},\, \lambda,\, \omega_a,\, s^a_1,\, \partial_1,\, \partial_a\}$$

are known functions. In order to evolve $\mathcal{D}$ off Σ^0, all time derivatives ∂_0 of $\mathcal{D}$ must be known. Consider the trace (3.94) of the propagating equations. It is a linear first order equation for K_0 with all coefficients on Σ^0 known. The initial data for K_0 are given on Σ^1, so equation (3.94) can be integrated to solve for K_0. All derivative of $\sqrt{g}$ are now known.

Similarly, the traceless part of the propagating equations are linear first order equations for the shear $\sigma_{0a}{}^{b}$ with coefficients which are known on Σ^0 and initial data given on Σ^1. Thus equation (3.95) can be integrated to solve for the shear which through equation (3.82) determines all derivative of the two-metric g_{ab}.

This procedure can now be repeated on a later hypersurface, $\Sigma: \ u^0 > 0$. Given $\mathcal{D}$ on Σ, the propagating equations can be solved for K_0 and σ_{0ab}. The hypersurface equations can then be solved for $\partial_0\lambda$ and $\partial_0\omega_a$ at every point of Σ. The result is that $\mathcal{D}$ and all time derivatives of $\mathcal{D}$ are known everywhere. Thus, the CIVP is formally solved.

To reiterate, the CIVP is stated by specifying the following initial data :

$$
\begin{array}{ll}
\text{On } \Sigma^1 : & \sigma_{0ab}(u^0,\theta^a) \text{ , or } g_{ab}(u^0,\theta^a)/\sqrt{g} \\
\text{On } \Sigma^0 : & \sigma_{1ab}(u^1,\theta^a) \text{ , or } g_{ab}(u^1,\theta^a)/\sqrt{g} \\
\text{On } S_0 : & K_0(\theta^a), K_1(\theta^a), \omega_a(\theta^a), g_{ab}(\theta^a), \lambda(\theta^a) \ .
\end{array}
$$

3.8 Lagrangian

The Einstein field equations can be derived by varying the Einstein-Hilbert action with respect to the metric functions. It is fairly obvious that such a derivation with the $2+2$ formalism would be incomplete since the metric (3.14) contains only eight arbitrary functions: a variational principle can only derive eight of the ten Einstein equations. The reason for this problem is that we have already made two gauge fixing conditions by demanding that u^0 and u^1 be null. A fairly easy fix is to add two more functions to the metric, calculate the new Lagrangian, vary to find Einstein's equations, and then set the new functions to zero.

Consider the spacetime metric of equation (3.14), where the matrix η_{AB} is defined by

$$\eta_{AB}(x^\alpha) = \begin{pmatrix} \eta_{00}(x^\alpha) & -1 \\ -1 & \eta_{11}(x^\alpha) \end{pmatrix} , \tag{3.96}$$

replacing the definition of equation (3.7). Capital Latin indices are still raised by η^{AB} which is

$$\eta^{AB}(x^\alpha) = \frac{1}{\eta} \begin{pmatrix} \eta^{11}(x^\alpha) & 1 \\ 1 & \eta^{00}(x^\alpha) \end{pmatrix} , \tag{3.97}$$

where η is the determinant of η_{AB}.

The Einstein-Hilbert Lagrangian density is now $\mathcal{L} = \sqrt{{}^{(4)}g}R^{\alpha}_{\alpha} = e^{\lambda}\sqrt{g}\sqrt{-\eta}(R^{A}_{A}e^{-\lambda}+ R^{a}_{a})$, where $\sqrt{g}$ is the square root of the the determinant of g_{ab}. In order to derive the full Lagrangian it is necessary to rederive the Ricci scalar as in Appendix A, with η_{AB} given by equation (3.96). This amounts to simply replacing L_{ABa} in equation (A.1) with

$$L_{ABa} = \frac{1}{2}e^{-\lambda}(\partial_a(\eta_{AB}e^{\lambda}) + \omega_a\epsilon_{AB})\ , \tag{3.98}$$

and N_{ABD} in equation (A.3) with

$$N_{ABD} = e^{-\lambda}(D_{(A}(e^{\lambda}\eta_{B)D}) - \frac{1}{2}D_D(e^{\lambda}\eta_{AB})\ . \tag{3.99}$$

The Ricci scalar can be derived by a calculation similar to that presented in appendix A. Total derivatives can be isolated from the expression, by noting that

$$\begin{aligned}(e^{-\lambda}X^A\ell^{\alpha}_{A})_{|\alpha} &= e^{-\lambda}(D_AX^A + X^A(K_A + D_A\ln\sqrt{-\eta})\\ (y^a e^{\alpha}_{a})_{|\alpha} &= y^a_{;a} + y^a(\lambda_{,a} + \partial_a\ln\sqrt{-\eta})\ .\end{aligned}$$

After some algebra, the Lagrangian density can be written as

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 + \text{total derivatives}\ , \tag{3.100}$$

where the first term is the value of the Lagrangian if η_{AB} is assumed constant,

$$\begin{aligned}\mathcal{L}_1 &= g^{\frac{1}{2}}\sqrt{-\eta}\left[e^{\lambda}({}^{(2)}R + \frac{1}{2}\lambda_{,a}\lambda^{,a}) + \eta^{AB}(K_AK_B - K_A{}^{ab}K_{Bab} + K_BD_A\lambda)\right]\\ &\quad + \frac{g^{\frac{1}{2}}}{2\sqrt{-\eta}}e^{-\lambda}\omega^a\omega_a.\end{aligned} \tag{3.101}$$

Variation of $\mathcal{L}_1$ with respect to the functions λ, g_{ab} and s^a_A produces the vacuum Einstein equations $G_{01} = G_{ab} = G_{Aa} = 0$. The second term,

$$\mathcal{L}_2 = g^{\frac{1}{2}}\sqrt{-\eta}K_BD_A\eta^{AB}\ , \tag{3.102}$$

must be varied along with (3.101) in order to produce the equations $G_{00} = G_{11} = 0$. The last term is

$$\begin{aligned}\mathcal{L}_3 &= g^{\frac{1}{2}}\sqrt{-\eta}\left[2K^AD_A\ln\sqrt{-\eta} + \frac{1}{2}\eta\epsilon_{CF}\epsilon_{E(B}D_{A)}\eta^{EF}D^C\eta^{AB} + \frac{1}{4}\eta\epsilon_{AE}\epsilon_{BF}D^C\eta^{AB}D_C\eta^{EF}\right]\\ &\quad g^{\frac{1}{2}}\sqrt{-\eta}e^{\lambda}\left[3\lambda^{,a}\partial_a\ln\sqrt{-\eta} + \frac{1}{4}\eta\epsilon_{AE}\epsilon_{BF}\partial_a\eta^{AB}\partial^a\eta^{EF}\right]\ ,\end{aligned} \tag{3.103}$$

which doesn't contribute to the dynamics since, after varying, η_{AB} is set to a constant. Consider the variation of $\mathcal{L}_3$ with respect to η^{00} and η^{11}. Since for any derivative operator, $\partial(\sqrt{-\eta}) = -\frac{1}{2}\eta_{AB}\partial\eta^{AB}$, the variation of the first term is

$$\begin{aligned}\int d^4x \frac{\delta}{\delta\eta^{AB}} g^{\frac{1}{2}}\sqrt{-\eta}2K^A D_A \ln\sqrt{-\eta} \\ = \int d^4x (2g^{\frac{1}{2}}\sqrt{-\eta}K_A\partial_B \ln\sqrt{-\eta} + \eta_{AB}\partial_D(g^{\frac{1}{2}}K^D)) = 0\ ,\end{aligned} \tag{3.104}$$

when η_{AB} is set to the constant matrix of equation (3.7). Similarly the variation of the rest of the terms of $\mathcal{L}_3$ do not contribute to the classical field equations in the double-null gauge.

The variation of $\mathcal{L}_2$ is

$$\int d^4x \frac{\delta}{\delta\eta^{AB}}\mathcal{L}_2 = -\int d^4x g^{\frac{1}{2}}(D_{(A}K_{B)} + K_A K_B)\ , \tag{3.105}$$

while the same variation of $\mathcal{L}_1$ yields

$$\int d^4x \frac{\delta}{\delta\eta^{AB}}\mathcal{L}_1 = \int d^4x g^{\frac{1}{2}}(K_A K_B - K_{Aab}K_B{}^{ab} + K_{(A}D_{B)}\lambda)\ . \tag{3.106}$$

The vacuum field equations result:

$$\begin{aligned} G_{00} = R_{00} = \frac{\delta}{\delta\eta^{00}}(\mathcal{L}_1 + \mathcal{L}_2) = 0\ , \\ G_{11} = R_{11} = \frac{\delta}{\delta\eta^{11}}(\mathcal{L}_1 + \mathcal{L}_2) = 0\ , \end{aligned}$$

which agrees with equation (3.54).

Setting η_{AB} to the constant matrix and varying $\mathcal{L}_1$ yields the remaining equations. Variation with respect to λ yields

$$\begin{aligned} \frac{\delta}{\delta\lambda}\mathcal{L}_1 &= \sqrt{g}e^{\lambda}({}^{(2)}R - \frac{1}{2}\lambda_{,a}\lambda^{,a} - \lambda^{;a}{}_{;a} - \frac{1}{2}e^{-2\lambda}\omega^a\omega_a) \\ &\quad -\sqrt{g}(D_A K^A + K_A K^A) \\ &= e^{\lambda}G_{01} = \frac{1}{2}R^a_a\ . \end{aligned}$$

Variation of $\mathcal{L}_1$ with respect to s^a_A produces the field equations $G^a_A = R^a_A = 0$. In order to vary $\mathcal{L}_1$ correctly, it should be remembered that the extrinsic curvature, normal Lie derivative and the twist all depend on s^a_A implicitly. Their variations are (where Φ_A and $\Psi_{A(ab)}$ are arbitrary functions)

$$
\begin{aligned}
\frac{\delta}{\delta s^a_A}(\sqrt{g}K_B\Phi^B) &= -\frac{\delta}{\delta s^a_A}(\sqrt{g}s^b_{B;b}\Phi^B) = \sqrt{g}\partial_a\Phi^A \\
\frac{\delta}{\delta s^a_A}(\sqrt{g}K_B{}^{bc}\Psi^B{}_{bc}) &= -\frac{\delta}{\delta s^a_A}(\sqrt{g}s_B{}^{(b;c)}\Psi^B{}_{bc}) = \sqrt{g}\Psi^A{}_a{}^d{}_{;d} \\
\frac{\delta}{\delta s^a_A}(\sqrt{g}\Phi^B D_B\lambda) &= -\frac{\delta}{\delta s^a_A}(\sqrt{g}\Phi^B s^b_B\partial_b\lambda) = -\sqrt{g}\Phi^A\partial_a\lambda \\
\frac{\delta}{\delta s^a_A}(\sqrt{g}e^{-\lambda}\omega^b\omega_b) &= -2\epsilon^{AB}\sqrt{g}e^{-\lambda}(D_B\omega_a + (K_B - D_B\lambda)\omega_a)\ .
\end{aligned}
$$

Similarly, the variation of $\mathcal{L}_1$ with respect to g^{ab} yields the Einstein equations $G_{ab} = R_{ab} - \frac{1}{2}g_{ab}(R^d_d + e^{-\lambda}R^D_D)$. The variation is straightforward when the two-dimensional identity

$$\frac{\delta}{\delta g^{ab}}(\sqrt{g}\phi\,{}^{(2)}R) = \sqrt{g}(g_{cd}\phi^{;a}{}_{;a} - \phi_{;cd}) \tag{3.107}$$

is used, where ϕ is any scalar function.

The Lagrangian and Hamiltonian dynamics of 2+2 splittings of gravity have been examined in more detail by various authors [57, 58, 59].

3.9 Conclusion

In this chapter we have presented a formalism for describing the geometry of spacetime in terms of a foliation by two families of lightlike hypersurfaces. The Einstein field equations (3.53) - (3.55) are presented as three concise equations relating the geometry of a two-surface to its imbedded spacetime in a two-dimensionally covariant manner. The definitions and equations presented in this chapter will be used extensively in the next two chapters to discuss the quasi-local gravitational mass (chapter 4) and the nature of a black hole's Cauchy horizon (chapter 5).

Chapter 4

Dynamics of the mass function

The most fundamental concept of general relativity is the equivalence principle [52]: no local experiment can distinguish between a gravitational field and uniform acceleration in flat space. As a result, it is not possible to define local observables for the gravitational field, such as an energy density or a stress tensor. For if it were possible to measure the energy density of the gravitational field at one point, a non-zero result would reveal that a gravitational field is present, violating the equivalence principle.

Suppose that we tried to introduce a definition of an energy density for gravitation. On dimensional grounds alone [60], the energy density must be quadratic in the first derivatives of the metric. In general relativity there are no local coordinate independent quantities involving first derivatives of the metric. For example, the square of certain Christoffel symbols may seem like a good candidate for a measure of energy, a priori. But one of the most fundamental theorems of differential geometry states that locally a coordinate system (Riemann normal coordinates [61]) can always be found in which the Christoffel symbols vanish. As a result, the "Christoffel measure" of energy can always be transformed to zero.

Contrast this with the situation in electromagnetism. In this case the electromagnetic field strength is a local observable, and there is a well defined stress tensor which is quadratic in the field strength. As a result, it is possible, at every point in space to determine the energy of the electromagnetic field. This allows us to discuss the local energy carried away from a time dependent source by electromagnetic radiation.

The problem for the gravitational field is more difficult. A dynamical gravitational source creates a time dependent gravitational field. In analogy with the electromagnetic problem, we expect that gravitational radiation will be produced which will carry information about the change of the source. However, there is no unique way to locally split the gravitational field into a radiation field which propagates on a background gravitational field, except in special circumstances, such as in spherical symmetry [62] and high frequency gravitational radiational waves [29]. Hence any local stress tensor defined through such a split would not be unique. The key difference between electromagnetism and gravity is that the electromagnetic field is locally observable, while the gravitational field is not.

On the other hand, analysis of the observations of the binary pulsar PSR 1913+16 [63] gives credence to the view that gravitational waves carry energy away from a time dependent gravitational source. The subtlety is that gravitational energy is a global concept. The energy carried by gravitational radiation can be measured in the wave zone, but is not well defined in the near zone. Far away from an isolated source, the total gravitational energy (or equivalently the mass) of the source can be defined. The Bondi [56] and ADM [39] gravitational masses measure the total energy of the gravitational field at null and spacelike infinity respectively in an asymptotically flat spacetime. If an otherwise static source is dynamic for a period of time, the Bondi mass can be measured before and after the activity, and the change in mass is equal to the energy radiated away by the gravitational waves.

A compromise between the local and global definitions is a quasi-local definition of mass which is defined as an average over a two dimensional spacelike surface. The premise behind a quasi-local definition is that two local observers can measure the geodesic deviation between themselves and together determine the spacetime geometry in their neighbourhood. The problem with defining a gravitational mass in this neighbourhood, is that there are an infinite number of ways to do so. Criteria for a reasonable definition have been listed [64], but it is probably impossible for any definition to satisfy all the criteria. Two definitions of quasi-local mass, by Hawking [65] and Hayward [66] are the most useful (in our opinion) and their properties will be explored in this chapter. Neither definition of mass can be unequivocally interpreted as a local energy. It is more conservative to interpret them as a measure of the focussing and shearing power respectively of the gravitational field.

Of interest is the application of these definitions of mass to the interior of the black hole. One of the more astonishing results of classical general relativity is the mass inflation [1] effect: when realistic gravitational perturbations are taken into account in the charged spherical black hole interior, the local mass function diverges at the hole's Cauchy horizon. We are interested in a generalization of the mass inflation effect to non-spherical black holes. In this chapter, we derive a general formalism for equations describing the dynamics of the quasi-local mass functions of Hawking and Hayward which can be used to deduce the mass inflation phenomenon. The equations are quite general and can be used in other applications, such as asymptotically flat geometries.

The quasi-local definitions of energy which we will examine are constructed out of quadratic combinations of the extrinsic curvature of a two dimensional spacelike surface. In chapter 3 we have presented in detail a formalism especially designed to describe the dynamics of such surfaces. In this chapter we will make extensive use of the notation and results of chapter 3.

This chapter is organised as follows. In section 1 the concept of the total mass of the spacetime is reviewed and the ADM and Bondi masses are defined. Quasi-local definitions of mass are discussed in section 2. In sections 3 and 4, formulae describing the variation of the quasi-local masses defined by Hawking and Hayward are derived. In section 5 a wave equation for Hawking's mass is derived and is used to demonstrate the mass inflation effect inside non-spherical black holes.

4.1 The ADM and Bondi masses

Consider an isolated star which is initially static but undergoes a period of activity from retarded times $u^1 = u^1_i$ to u^1_f during which it emits gravitational radiation. After time u^1_f it returns to a static state. In the static regions the star's metric is approximated by the Schwarzschild metric (2.1) at distances far from the star. The motion of a satellite in an orbit far from the star will be described by Kepler's third law, and the mass calculated from the orbit is asymptotically the same as the Schwarzschild mass. An observer at spacelike infinity (i^0) measures the state of the star in the remote past when it was static. A measurement of the mass at i^0 will determine the original total mass of the star. This mass is called the ADM mass, M_{ADM}, and is a measurement of the total mass of the spacetime [39]. An invariant definition is [67]

$$M_{ADM} = \lim_{r\to\infty} \frac{r}{4\pi} \int_{S_r} d^2\theta \sqrt{g} \mathrm{Re}\Psi_2 \,, \tag{4.1}$$

where S_r is a two-sphere of radius r and g is the determinant of the two-dimensional metric on S_r. The real part of Ψ_2 is known as the Coulomb component of the Weyl tensor, due to the analogy with electromagnetism. The definition of the Coulomb component is (in vacuum)

$$\mathrm{Re}\Psi_2 = \frac{1}{4}({}^{(2)}R + e^{-\lambda}K_0K_1 - 2e^{-\lambda}\sigma_{0ab}\sigma_1{}^{ab})) , \tag{4.2}$$

where, in the notation of chapter 3, ${}^{(2)}R$ is the intrinsic curvature of the two-surface, the function λ is the metric component defined in (3.14), and K_A and σ_A are the extrinsic curvatures (3.22 - 3.23) of the two-surface. It has been proved [68, 69] that the ADM mass must always be positive, as must be the case if M_{ADM} really represents the total mass of the spacetime.

Observers at future null infinity (at any point on $\mathcal{J}^+$) can measure the mass left in the star at any finite retarded time u^1. This mass is named the Bondi mass, $M_B(u^1)$ [56]. The Bondi mass is equal to the ADM mass minus the energy carried away by gravitational radiation [70]. In order to define the Bondi mass, it is assumed that the metric can be expanded in powers of $1/r$. To leading order, the metric is flat and described by the metric

$$\begin{aligned} ds^2|_{flat} &= -2du^0du^1 + r^2d\Omega^2 \\ &= -(du^1)^2 - 2drdu^1 + r^2d\Omega^2 , \end{aligned}$$

where the advanced time u^0 is related to the coordinate r by

$$2du^0 = du^1 + 2dr . \tag{4.3}$$

It should be noted that the non-standard definitions for the advanced and retarded times have been taken,

$$u^1 = t - r \, , 2u^0 = t + r.$$

At order $1/r$ the metric contains a dynamic, non-spherically symmetric term. The assumption is that the asymptotic behaviour of the metric functions (using the notation of chapter 2) are

$$\begin{aligned} g_{ab} &= r^2\Omega_{ab} + \frac{C_{ab}(u^1,\theta^a)}{r} + O(1/r^2) \\ s^a_A &= O(1/r^2) \\ -2e^{-\lambda}\partial_0 r\partial_1 r &= 1 - \frac{2m_B(u^1,\theta^a)}{r} + O(1/r^2) , \end{aligned} \tag{4.4}$$

where Ω_{ab} is the metric on the unit sphere, C_{ab} is a traceless two-tensor and the Bondi mass aspect m_B is defined to agree with the Schwarzschild mass (2.27) in spherical symmetry. The tensor C_{ab} is known as the news function. It encapsulates the two degrees of freedom of the gravitational field, since it determines all of the physically meaningful quantities.

From the definitions (4.3) and (4.4), the asymptotic behaviour of the curvatures can be derived,

$$\begin{aligned} {}^{(2)}R &= \frac{2}{r^2} + O(1/r^3) \\ K_0 &= \partial_0 \ln r^2 = \frac{2}{r} + O(1/r^2) \\ K_1 &= \partial_1 \ln r^2 = -\frac{1}{r} + O(1/r^2) \\ \sigma_{0ab} &= -\frac{C_{ab}(u^1, \theta^a)}{r^2} + O(1/r^3) \\ \sigma_{1ab} &= \frac{\partial_1 C_{ab}(u^1, \theta^a)}{r} + O(1/r^2) \\ \omega^a &= O(1/r^3) \, . \end{aligned} \tag{4.5}$$

Substituting these expansions into the Coulomb component of the Weyl tensor, we see that [56, 71]

$$\mathrm{Re}\Psi_2 = \frac{1}{r^3}(m_B + \frac{1}{2} C^{ab} \partial_1 C_{ab}) + O(1/r^4) \, . \tag{4.6}$$

To highest order, the Coulomb component of the curvature is determined by the mass aspect of the star at time u^1 and the rate of change of the two-metric. A coordinate independent definition of the Bondi mass is the average of the Bondi mass aspect over a sphere at infinity:

$$M_B(u^1) = \lim_{r\to\infty} \frac{r}{4\pi} \int_{S_r} d^2\theta \sqrt{g} (\mathrm{Re}\Psi_2 + \frac{1}{2} e^{-\lambda} \sigma_{0ab} \sigma_1{}^{ab}) \, . \tag{4.7}$$

Asymptotically, the largest component of the Weyl tensor is [56, 71]

$$\Psi_4 = e^{-\lambda} \bar{m}^a \bar{m}^b \frac{\partial_1 \partial_1 C_{ab}}{r} + O(1/r^2) \, , \tag{4.8}$$

which motivates the identification of Ψ_4 as the radiative part of the gravitational field.

The Bondi-Sachs mass loss formula [56, 73] can be derived from the Raychaudhuri (3.92) equation $R_{11} = 8\pi T_{11}$ where T_{11} is the stress tensor of the material flowing out of the star. Substituting the asymptotic expansions (4.4) and (4.5) into Raychaudhuri's equation, we find that

$$\partial_1 M_B(u^1) = -\frac{1}{2} \lim_{r\to\infty} \frac{1}{4\pi r^2} \int_{S_r} d^2\theta \sqrt{g} \left(\partial_1 C_{ab} \partial_1 C_{cd} g^{ac} g^{bd} + 8\pi T_{11} \right) \, . \tag{4.9}$$

In vacuum, the rate of change of the traceless part of the two-metric uniquely determines the loss of mass from the star. This is interpreted as the rate that gravitational radiation carries energy away from the source. The right hand side of (4.9) is negative semi-definite (if the stress tensor obeys the dominant energy condition), so that the mass of an isolated star can not increase. It has been proved [72, 74, 69] that the Bondi mass must always be positive, in other words, the star can not radiate away more mass than it originally started with.

4.2 Quasi-local definitions of mass

As we mentioned in the introduction, there can be no general local definition of the mass (or equivalently, energy) of the gravitational field. The closest type of definition of a gravitational mass is a "quasi-local" mass. A quasi-local definition is not local, but is usually defined with respect to a spacelike closed two-surface which provides a notion of quasi-locality. There is no obvious canonical prescription for a quasi-local definition: in fact there are an infinite number of ways in which a definition can be made. A review of the various definitions would be pointless, since most are not particularly useful for our purposes. Instead, as a guide, we list a number of properties that a reasonable quasi-local mass should possess, and discuss the definitions which come closest to fulfilling the desired properties. The following list of properties is a modification of Eardley's list [64]. Here we assume that the definition is made with respect to a two dimensional spacelike surface S which has area A. A "good" definition of a quasi-local mass should:

i) reduce to zero if A reduces to zero.

ii) reduce to zero in Minkowski spacetime, regardless of the shape of S.

iii) reduce to the Schwarzschild mass (2.27) in a spherically symmetric spacetime.

iv) reduce to the ADM mass (4.1) at spatial infinity in an asymptotically flat spacetime.

v) reduce to the Bondi mass (4.7) at null infinity in an asymptotically flat spacetime.

vi) be equal to the irreducible mass of a horizon when S is an apparent horizon, where $m_{irr} = (A/16\pi)^{\frac{1}{2}}$.

vii) increase if A increases and S is outside of a black hole.

viii) reproduce the Bondi mass loss formula when varying with respect to time.

No present definition satisfies all of these restrictions and it is probably impossible to invent a new definition which satisfies all of these points.

Relation between local and quasi-local mass

We wish to investigate the properties of three quasi-local definitions of gravitational mass which involve an integration over a surface. Quasi-local quantities of this type can be awkward to manipulate, so for convenience, we will associate with each quasi-local mass $M(u^A)$ a local mass aspect $m(u^A, \theta^a)$. The quasi-local definitions which we will discuss make use of a quasi-local luminosity distance $l(u^A)$ (or area radius), defined by

$$4\pi l^2 = \int_S d^2\theta \sqrt{g}\,. \tag{4.10}$$

The ratio of the quasi-local mass, $M(u^A)$ to the luminosity distance is set equal to the average over S of a local function of the extrinsic and intrinsic curvatures of S which has dimension $1/L^2$. (There are an infinite number of local functions which meet these requirements.) For convenience, we will define this local function to be m/r^3, so that

$$\frac{M}{l} = \frac{1}{4\pi}\int d^2\theta \sqrt{g}\frac{m}{r^3}. \tag{4.11}$$

It is useful to introduce a local function $r(x^\alpha)$ defined by reference to the characteristic initial value problem (CIVP). The CIVP is typically formulated by stating the initial conditions for the metric functions on two intersecting characteristics and on S_0 the spacelike intersection of the initial characteristics (see section 3.7). From (4.10), it follows that the area of S_0 is $4\pi l_0^2$. The relation between r and l_0 is

$$\sqrt{g(x^\alpha)} := \frac{\sqrt{g_0}}{l_0^2} r^2(x^\alpha)\,. \tag{4.12}$$

When all of the surfaces S are spheres of radius l, then $r = l$.

Substituting (4.12) into (4.11), we find that the relation between the quasi-local mass M and the local mass aspect m is

$$M = \frac{l}{4\pi l_0^2}\int_S d^2\theta \sqrt{g_0}\frac{m}{r}\,. \tag{4.13}$$

Recalling the definition (3.22) of the dilations in the double null formalism,

$$D_A l^2 = \partial_A l^2 = \frac{1}{4\pi} \int_S d^2\theta \sqrt{g} K_A \,, \tag{4.14}$$

we see that K_A measures the change in the area of S in a direction normal to the surface. The dilation is related to r^2 by

$$K_A = D_A \ln r^2 - \Delta s_A \,, \; \Delta s_A = \frac{1}{\sqrt{g_0}} \partial_a (\sqrt{g_0} s^a_A) \,, \tag{4.15}$$

In most applications of interest Δs_A will be negligible.

The Schwarzschild mass

The most naive definition of a quasi-local mass is a generalisation of the Schwarzschild mass to non-spherical spacetimes. Recalling that in spherical symmetry the dilation is $K_A = \partial_A \ln r^2$, we define the Schwarzschild mass aspect, $m_S(x^\alpha)$ to be

$$\frac{2m_S}{r} = \frac{1}{2} e^{-\lambda} K_0 K_1 + 1 \,. \tag{4.16}$$

This definition was originally introduced by Misner and Sharp [75]. Details of the properties of m_S in spherically symmetric spacetimes have been described in reference [76]. The quasi-local mass, $M_S(u^A)$ associated with (4.16) is found by replacing M by M_S and m by m_S in equation (4.13),

$$M_S = \frac{l}{8\pi l_0^2} \int_S d^2\theta \sqrt{g_0} \left(\frac{1}{2} e^{-\lambda} K_0 K_1 + 1 \right) \,. \tag{4.17}$$

Clearly, the definition of M_S reduces to the definition (2.27) of the mass in spherical symmetry.

The Hawking mass

A slight modification of the Schwarzschild mass is Hawking's quasi-local mass [65]. If our basic demand on a quasi-local mass is that it should reduce to Bondi's mass (4.6) in the limit $r \to \infty$, then we are led to Hawking's definition of the mass aspect, $m_H(x^\alpha)$,

$$2\frac{m_H}{r} = \frac{1}{2} r^2 \left(e^{-\lambda} K_0 K_1 + {}^{(2)}R \right) \,. \tag{4.18}$$

The quasi-local version of Hawking's mass, $M_H(u^A)$

$$M_H = \frac{l}{8\pi l_0^2}\int_S d^2\theta\sqrt{g_0}\frac{1}{2}r^2\left(e^{-\lambda}K_0K_1 + {}^{(2)}R\right) . \qquad (4.19)$$

is exactly the same as M_S when the topology of the surfaces S are spherical. This is a result of the Gauss-Bonnet theorem, which states that

$$\int_S d^2\theta\sqrt{g}\,{}^{(2)}R = 8\pi(1-g) , \qquad (4.20)$$

where g is the genus of S. When S has spherical topology, $g = 0$. Hawking's quasi-local mass can be thought of as a generalization of the Schwarzschild mass to arbitrary topology.

Because of the Gauss-Bonnet theorem, when S has spherical topology, the two definitions have the same properties. Both the Schwarzschild and Hawking masses satisfy most of the properties (i) - (viii) but fail property (ii). It has been proved [60] that for small spheres, the Hawking mass reduces to zero when the surface area of S shrinks to zero. The Hawking mass reduces to the Schwarzschild, Bondi and irreducible masses in the appropriate limits, by definition. It reduces to the ADM mass if it is demanded [70] that the shear falls off quickly enough at spacelike infinity. Variation of Hawking's mass [79] shows that it increases as r increases and reproduces the Bondi-Sachs mass loss formula. As we will show in section 4.3 the Bondi-Sachs mass loss formula is applicable inside the black hole as well, where it becomes a mass gain formula.

The Hayward mass

The Hawking and Schwarzschild definitions of the quasi-local mass are zero in Minkowski spacetime only if S is spherical [66, 60]. Suppose that we set up a foliation of flat space with surfaces with non-zero shear. Consider the definition (4.2) of Ψ_2. In Minkowski space $\Psi_2 = 0$, which implies that

$$M_H = M_S = \frac{l}{8\pi l_0^2}\int_S d^2\theta\sqrt{g_0}\frac{1}{2}r^2e^{-\lambda}\sigma_{0ab}\sigma_1{}^{ab} \text{ , in flat space.} \qquad (4.21)$$

As a result it is possible to set up a foliation of flat spacetime by highly distorted surfaces which the Hawking definition would register as a fictitious mass.

Hayward has suggested [66] that the real part of Ψ_2 has energy-like properties and it may be worthwhile to consider a new mass aspect, m', defined by

$$2\frac{m'}{r} = \frac{1}{2}r^2({}^{(2)}R + e^{-\lambda}K_0K_1 - 2e^{-\lambda}\sigma_{0ab}\sigma_1{}^{ab}) = 2\frac{m_H}{r} - r^2e^{-\lambda}\sigma_{0ab}\sigma_1{}^{ab} \qquad (4.22)$$

which is always zero in flat space independent of the foliation S.

By definition, this mass reduces to the ADM mass at spacelike infinity. When the appropriate limit [66] is taken, the Bondi mass is also found. When the surfaces S are spheres, the shear term disappears, so that the Schwarzschild and irreducible masses are recovered. In the limit of small surface area, Hayward's mass goes to zero from the negative direction [77], which is not a very good feature. In section 4.4 we will compute the variation of m' in order to discuss properties (vii) and (viii).

A further generalisation [66] is to add a term to m' proportional to $e^{-2\lambda}\omega_a\omega^a$, where ω^a is the twist defined by equation (3.25). A new analysis [78] of the symplectic structure of the double-null formalism suggests that the dynamical degrees of freedom of the gravitational field can be encoded in the twist instead of the shear, as is more usual. If this line of thought is developed further, it may be interesting to examine the dynamics of a quasi-local mass containing a twist term. However, we are most interested in problems where the twist is small, such as asymptotically flat spacetimes and the black hole interior, and we will not consider this sort of generalisation.

4.3 Variation of Hawking's mass

Hawking's quasi-local mass has the property that it reduces to Bondi's mass at $\mathcal{I}^+$ in an asymptotically flat geometry. In the same limit the change in Hawking's mass reduces to the Bondi-Sachs energy loss formula [79]. This suggests that far from a radiating source, the change in Hawking's quasi-local mass can be interpreted as the energy carried by the gravitational radiation, to a good approximation. The result connecting the variation of M_H with the Bondi-Sachs mass loss formula was derived [79] using special assumptions which limits the result to untrapped regions. A formula of this sort is useful for the derivation of the mass inflation phenomenon in the black hole interior. In this section we will derive an equation for the normal Lie derivative of m_H.

The aim of this section is to produce an expression for D_Am_H. First, it is useful to derive an expression for the normal derivatives of the dilations, D_AK_B. The equation can be derived from algebraic manipulations of the components of the Einstein field equations normal to S. Forming the combination $G_{AB} - g_{AB}G^D_D$ where G_{AB} is given by equations (3.53) and (3.54) and substituting the definition of the mass aspect (4.18), we find that

$$\begin{aligned} D_{(A}K_{B)} &= -8\pi(T_{AB} - g_{AB}T + \tau_{AB}) + 2g_{AB}\frac{m_H}{r^3} \\ &\quad +K_{(A}D_{B)}\lambda - \frac{1}{2}\eta_{AB}K^E D_E\lambda - \frac{1}{2}K_A K_B + g_{AB}\mu \qquad (4.23) \\ \tau_{AB} &= \frac{1}{8\pi}\left(\sigma_{Aa}{}^{b}\sigma_{Bb}{}^{a} - \frac{1}{2}\eta_{AB}\sigma_{Da}{}^{b}\sigma^{D}{}_{a}{}^{b}\right) \qquad (4.24) \\ \mu &= -(\frac{1}{4}e^{-2\lambda}\omega^a\omega_a + \frac{1}{2}\lambda_{;a}^{;a} + \frac{1}{4}\lambda^a\lambda_a)\,. \qquad (4.25) \end{aligned}$$

The term represented by μ is zero in spherical symmetry and negligible in an asymptotically flat region of spacetime. In chapter 5 we will show that the term μ is of order unity and that $e^\lambda \to 0$ at the Cauchy horizon of a black hole. The matrix τ_{AB} plays the role of an effective stress tensor for gravitational radiation. When the gravitational radiation is highly blueshifted, then the average of the tensor $\tau_{AB}\ell^{(A)}\ell^{(B)}$ over many wavelengths reduces to the Isaacson [29] effective stress tensor.

Now, note the following identity for the normal derivative of the quantity $K_BK^B = -2K_0K_1$:

$$D_A(K_BK^B) = 2K^B(D_{(A}K_{B)} + D_{[A}K_{B]})\,. \qquad (4.26)$$

The antisymmetric term is related to the twist by equation (3.44), while the symmetric term is given by contracting expression (4.23) with K^B,

$$2K^BD_{(A}K_{B)} = -16\pi K^B(T_{AB}-g_{AB}T+\tau_{AB})+K_BK^B(-\frac{3}{2}K_A+D_A\lambda)+e^\lambda K_A(\,^{(2)}R+2\mu)\,. \qquad (4.27)$$

The normal derivative of Hawking's mass aspect can be calculated by operating directly on the definition (4.17). The result after making use of equations (4.26) and (4.27) is

$$\begin{aligned} D_A m_H &= 2\pi r^3 e^{-\lambda}K^B(T_{AB} - g_{AB}T + \tau_{AB}) + \mu_A + \nu_A \qquad (4.28) \\ \mu_A &= \frac{r^3}{8}\left(-e^{-\lambda}K^B\epsilon_{BA}\omega^a{}_{;a} - 2\mu K_A\right) + \frac{3}{2}\Delta s_A m_H \qquad (4.29) \\ \nu_A &= \frac{rl_0^2}{4\sqrt{g_0}}\left(\partial_A(\sqrt{g}\,^{(2)}R) - \sqrt{g}(s_A^a\,^{(2)}R)_{;a}\right)\,. \qquad (4.30) \end{aligned}$$

Equation (4.28) should be compared with the equation (2.32) derived in spherical symmetry. The terms grouped together as μ_A vanish in spherical symmetry and will be negligible in the applications of interest to us. The terms grouped together as ν_A vanish when integrated over S,

$$\int_S d^\theta \sqrt{g_0} \frac{1}{r} \nu_A = 0 .$$

The first term of ν_A vanishes as a result of the Gauss-Bonnet theorem and the second term is an integral of a divergence, which vanishes over a closed surface.

Consider an asymptotically flat vacuum spacetime. Substituting in the asymptotic expansions (4.4) and (4.5) into (4.28), it can be seen that $\mu_A = O(1/r)$ and

$$\int_S d^\theta \sqrt{g_0} \frac{\partial_1 m_H}{r} = -\frac{1}{2} \frac{1}{4\pi r^2} \int_S d^\theta \sqrt{g} \left(\partial_1 C_{ab} \partial_1 C_{cd} g^{ac} g^{bd} \right) , \tag{4.31}$$

which is the Bondi-Sachs mass loss formula (4.9).

The normal derivative of Hawking's quasi-local mass can be computed, by substituting this result into the derivative of equation (4.19):

$$D_A M_H = \frac{l}{4\pi l_0^2} \int_S d^2\theta \sqrt{g_0} \left(\frac{1}{r} D_A m_H + \frac{1}{2} \frac{m_H}{r} (D_A \ln l^2 - K_A - 3\Delta s_A) . \right) \tag{4.32}$$

Outside of a black hole, a special foliation of spacetime by surfaces of mean constant curvature can be made [79]. Surfaces of mean constant curvature have the property that K_A is a constant on the surface S, so that $D_A \ln l^2 = K_A$. As a result, the variation of M_H yields the Bondi-Sachs mass loss formula [79]. Since outside of a black hole $D_0 M_H > 0$ and $D_1 M_H < 0$, the mass increases as r increases [79].

Now, consider the variation of Hawking's mass in a region inside a black hole. close to its Cauchy horizon. The stationary Kerr metric,

$$\begin{aligned} ds^2 &= \Sigma \left(\frac{dr^2}{\Delta} + d\theta^2 \right) + \left((r^2 + a^2) + \frac{2mra^2 \sin^2\theta}{\Sigma} \right) \sin^2\theta d\phi^2 \\ &\quad - \frac{4mar}{\Sigma} \sin^2\theta d\phi dt - \left(1 - \frac{2mr}{\Sigma} \right) dt^2 \\ &\quad \Sigma = r^2 + a^2 \cos^2\theta \\ &\quad \Delta = r^2 - 2mr + a^2 \end{aligned} \tag{4.33}$$

can be simplified on the axis of symmetry, where $\theta = 0$. On the axis, the metric is

$$ds^2|_{\theta=0} = \frac{r^2 + a^2}{\Delta} dr^2 - \frac{\Delta}{r^2 + a^2} dt^2 . \tag{4.34}$$

Null Eddington-Kerr coordinates, similar to the Reissner-Nordström coordinates (2.3) can be defined

$$dv = dt + \frac{r^2 + a^2}{\Delta} dr , du = -dt + \frac{r^2 + a^2}{\Delta} dr , \tag{4.35}$$

and Kruskal coordinates,

$$\kappa_- U = -e^{-\kappa_- u}\ ,\ \kappa_- V = -e^{-\kappa_- v}\ , \tag{4.36}$$

defined such that the metric near CH is finite,

$$\begin{aligned} ds^2|_{\theta=0} &\sim -2e^{-\kappa_-(u+v)}dudv && (4.37)\\ &\sim -2dUdV\ . && (4.38)\end{aligned}$$

Compare the metric near CH with the general form of the double-null metric (3.14). In Eddington-Kerr coordinates, CH corresponds to the limit $v \to \infty$ and the metric function $e^{\lambda} = e^{-\kappa_-(u+v)} \sim 0$. Suppose that a solution to the field equations in the interior can be written as a perturbation series around the stationary black hole solution. We would expect that close to the Cauchy horizon the metric function λ is such that $e^{-\lambda} \to \infty$. Taking this limit in equation (4.28) we find that in vacuum,

$$\begin{aligned} \lim_{v\to\infty} \partial_v m_H &= -2\pi r^3 e^{-\lambda} K_u \tau_{vv} + O(1) && (4.39)\\ \lim_{v\to\infty} \partial_u m_H &= -2\pi r^3 e^{-\lambda} K_v \tau_{uu} + O(1)\ , && (4.40)\end{aligned}$$

where u is the internal retarded time (2.3). The right hand sides of equations (4.39) and (4.40) are positive since the components (4.24) of the effective stress tensor τ_{AB} are positive, and the dilations K_A are negative. We are assuming that the event horizon conjecture [51] (which has only been proven in special circumstances) is generally true, so that the region inside an event horizon will always be trapped and the dilations must be negative.

The positivity of the variations of m_H reveals that in the presence of perturbations, the mass must increase as the Cauchy horizon is approached. As well, as u increases along the Cauchy horizon, the mass must increase. The increase in mass is independent of the form of the perturbations, as long as they are weak enough that the perturbation approach is justified, which is the case in the collapse of a star to a black hole. Thus inside the black hole, the Hawking variational formula is a mass gain formula.

In the stationary solution λ diverges linearly in the external advanced time, so that $e^{-\lambda} \sim e^{\kappa_- v}$. The shear of the ingoing null generators, σ_{vab} represents the perturbations of the interior caused by the incoming gravitational wave tail. The tail has the form of an inverse power law, $\sigma_{vab} \sim v^{-q/2}$, $(q > 0)$. These perturbations are scattered in the interior by the curvature in the hole. The backscattered radiation causes shearing of the outgoing null generators (discussed in section 4.3) which has an inverse power law form $\sigma_{uab} \sim u^{-p/2}$. Substituting these perturbative values for the shears and for λ into (4.39) we find that as a first approximation, Hawking's mass will diverge exponentially.

This illustrates the duality between observations made inside and outside a black hole formed by the collapse of a star. Gravitational radiation emitted by the star as it collapses is partially scattered by the external potential barrier. The gravitational radiation which is transmitted to $\mathscr{I}^+$ carries mass away from the star, so that observers outside the forming black hole see the mass of the hole decreasing with an inverse power law in time. Observers entering the black hole and falling freely towards the Cauchy horizon see the Hawking mass increase without bound as the backscattered gravitational radiation is infinitely blueshifted.

4.4 Variation of Hayward's mass

Since Hawking's mass is not necessarily zero in flat spacetime, it is possible to create fictitious gravitational radiation by choosing a foliation of spacetime with non-spherical surfaces. In this section we will derive a variational formula for Hayward's mass which is always zero in flat spacetime.

First, we define the difference between Hayward's mass aspect, m', Hawking's mass aspect, m_H,

$$\Delta m = m' - m_H = \frac{1}{4} r^3 e^{-\lambda} \sigma_{Ba}{}^b \sigma^B{}_b{}^a \ . \tag{4.41}$$

Then the normal derivative of Δm is

$$D_A(\Delta m) = \Delta m (\frac{3}{2} D_A \ln r^2 - D_A \lambda) + \frac{1}{2} r^3 e^{-\lambda} \sigma^B{}_b{}^a D_A \sigma_{Ba}{}^b \ . \tag{4.42}$$

This expression can be simplified by noting that the symmetric normal derivative of the shear is related to the Riemann tensor by equation (3.60). In particular,

$$\begin{aligned} \sigma^B{}_b{}^a D_{(A}\sigma_{B)a}{}^b &= -\sigma^B{}_b{}^a R_{AaB}{}^b - 4\pi K^B \tau_{AB} + \frac{\Delta m}{r^3} e^\lambda (-3K_A + 2D_A\lambda) \\ & \quad -\frac{1}{4} \sigma_{Aab} e^\lambda (e^{-2\lambda} \omega^a \omega^b + \lambda^{,a} \lambda^{,b} + 2\lambda^{;ab}) \ . \end{aligned} \tag{4.43}$$

The antisymmetric normal derivative of the shear is given by the commutation relations (3.43) and (3.44). Some algebra reveals the result

$$D_A(\Delta m) = r^3 e^{-\lambda}(-\frac{1}{2}\sigma^{Bab}R_{AaBb} - 2\pi K^B \tau_{AB}) + \mu'_A \tag{4.44}$$

$$\begin{aligned} \mu'_A &= \frac{1}{4} r^3 e^{-\lambda}\epsilon_{BA}\sigma^{Bab}\omega_{a;b} + \frac{3}{2}\Delta s_A \Delta m \\ &\quad -\frac{1}{8} r^3 \sigma^{Aab}(e^{-2\lambda}\omega_a\omega_b + \lambda_{,a}\lambda_{,b} + 2\lambda_{;ab}) . \end{aligned} \tag{4.45}$$

The final result is that the normal derivative of m' is

$$D_A m' = r^3 e^{-\lambda}\left(-\frac{1}{2}\sigma^{Bab}R_{AaBb} + 2\pi K^B(T_{AB} - \eta_{AB}T)\right) + \mu_A + \nu_A + \mu'_A \tag{4.46}$$

In an asymptotically flat region, the expansions (4.4) and (4.5) can be substituted into (4.46) to find

$$\begin{aligned} \partial_u M'(u) &= \frac{1}{8\pi}\int d^2\theta \sin\theta \; C^a{}_b \partial_u \partial_u C^b{}_a \\ &= \frac{1}{8\pi}\int d^2\theta \sin\theta \left(\frac{1}{2}\partial_u\partial_u |C|^2 - \partial_u C^a{}_b \partial_u C^b{}_a\right) . \end{aligned} \tag{4.47}$$

The sign of this expression depends on the second time derivative of the magnitude of the gravitational perturbations. Consider the case of the collapse of a star to a black hole. Generically, a gravitational wave tail forms which has the Price power law fall off at late time,

$$|C| \sim u^{-p/2} \, , \; u \to \infty \tag{4.48}$$

where p is a positive integer, typically $p = 12$ for gravitational radiation. Substituting the power law into the formula for the rate of change of Hayward's mass, we find

$$\partial_u M' \sim \left(\frac{1}{2}p(p+1) - \frac{p^2}{4}\right) u^{-(p+2)} > 0 \, , \; p > 0. \tag{4.49}$$

Hence, Hayward's mass increases (slowly) at late times after the complete gravitational collapse of a star. This property is rather counter-intuitive, and leads us to conclude that M' is not a very good measure of the gravitational mass of the space-time. Clearly, this formula does not reproduce the Bondi-Sachs mass loss formula. It is not surprising that variation of Hayward's mass can't reproduce the usual law for the change in energy due to gravitational radiation. This is because m' is a component of the Riemann tensor. If we take a derivative of the Riemann tensor, the Bianchi identities will relate the derivative to other components of the Riemann tensor. But the Bondi-Sachs formula relates the derivative of a mass to the effective stress tensor of gravitational radiation. The effective stress tensor is not built out of any components of the Riemann tensor, so it is impossible for the variation of Hayward's mass to depend on the effective stress tensor. For this reason, m' probably is not a very good representation of the gravitational energy of the system.

While the interpretation of m' as a mass is debatable, it does have an invariant geometrical meaning: it is the Coulomb component of the Weyl tensor. The solution of (4.46) can reveal some information about the geometry of the spacetime. Consider the equation in the limit of the Cauchy horizon, as discussed in the previous section. In vacuum, the solution of (4.46), with $u^A = v$ is

$$\lim_{v\to\infty} \mathrm{Re}\Psi_2 = \lim_{v\to\infty} m'/r^3 = \frac{1}{2r^3}\int^{\infty} dv r^3 e^{-\lambda}\sigma_u{}^{ab}R_{vavb}\,. \tag{4.50}$$

In a Petrov type D spacetime (such as a stationary black hole), the component R_{vavb} vanishes. The solution of (4.50) for a type D spacetime is that Ψ_2 is a constant near the Cauchy horizon of the spacetime. Now consider a spacetime which is a perturbation of a stationary black hole, so that R_{vavb} is small, but non-zero. If $e^{-\lambda}$ diverges faster than the rate that R_{vavb} goes to zero, then the magnitude of the right hand side of (4.50) will diverge. Since we expect λ to diverge linearly and the perturbation in the Riemann tensor to go to zero as an inverse power law, this argument leads us to suspect that the magnitude of the Ψ_2 component of the curvature will diverge exponentially at the Cauchy horizon. The results of the next chapter will prove that this is the case.

4.5 Wave equation for the mass

In spherical symmetry, the derivation of a wave equation (2.33) for the mass function led to the conclusion that the introduction of perturbations to the interior will cause the internal mass of the black hole to inflate exponentially as the Cauchy horizon is approached, signalling a curvature singularity. In this section we will derive a similar equation for Hawking's mass aspect which holds in a general spacetime.

To begin, we rearrange the contracted Bianchi identity (3.72) to the form of a conservation law for the stress tensor,

$$D_B T_A{}^B + K_B T_A{}^B = -(e^{\lambda}T_A{}^a)_{;a} + \epsilon_{AB}\omega_a T^{Ba} + e^{\lambda}\sigma_{Aab}T^{ab} + K_A e^{\lambda}P + \frac{1}{2}D_A(e^{\lambda})\,T$$

$$2P := T_{ab}g^{ab}\ ,\ T := T_{AB}\eta^{AB}e^{-\lambda}\ . \tag{4.51}$$

A similar equation for the effective stress tensor can be derived. Taking the divergence of $\tau_A{}^B$ as defined in (4.44) yields

$$8\pi D_B\tau_A^B = \sigma_{Aa}{}^b D_B\sigma_b{}^{Ba} + \sigma_{Aa}{}^b(D_B\sigma_{Aa}{}^b - D_A\sigma_{Ba}{}^b)\ . \tag{4.52}$$

The first term on the right hand side of (4.52) is related to the traceless part of the propagation equation (3.95), while the second term is given by the commutator (3.43). The final result is that

$$D_B\tau_A{}^B + K_B\tau_A{}^B = -\frac{1}{16\pi}K_A\sigma_{Dab}\sigma^{Dab} - \sigma_{Aab}T^{ab}e^{\lambda} + \frac{e^{\lambda}}{8\pi}\alpha_A \tag{4.53}$$

where the terms grouped together as α_A,

$$\alpha_A = \epsilon_{AB}\sigma^{Bab}\omega_{a;b} - \sigma_{Aab}(\frac{1}{2}e^{-2\lambda}\omega^b\omega^a + \lambda^{;ba} + \frac{1}{2}\lambda^{;a}\lambda^{;b})\ . \tag{4.54}$$

are small inside the black hole or in an asymptotically flat region.

It is now a matter of some algebra to find the two dimensional wave operator's action on m_H. Operating on (4.18) with the operator D^A, and making use of equations (4.23), (4.51) and (4.53), we find that

$$\begin{aligned} D^AD_Am &= -(4\pi)^2r^3e^{-\lambda}(T_{AB}+\tau_{AB})(T^{AB}+\tau^{AB}) \\ &\quad +(m - \frac{r^3}{4}\,{}^{(2)}R)\sigma_{Bab}\sigma^{Bab} \\ &\quad +\alpha^{(0)} + \alpha^{(1)} + \alpha^{(2)} \end{aligned} \tag{4.55}$$

The terms grouped together in $\alpha^{(0)}$ are terms which are zero in spherical symmetry only if the stress tensor has the property that $T = P = 0$,

$$\alpha^{(0)} = 2\pi r^3 K_AK^A(2P - T) - 2\pi r^3K^AD_A(T) - 4\pi mTe^{\lambda}\ . \tag{4.56}$$

The terms grouped together as $\alpha^{(1)}$ are zero in vacuum,

$$\begin{aligned} \alpha^{(1)} &= 3\pi r^3e^{-\lambda}K^B\Delta s^A(T_{AB} - g_{AB}T) - 2\pi r^3\mu T \\ &\quad +2\pi r^3e^{-\lambda}K^A(-(e^{\lambda}T_A{}^a)_{;a} + \epsilon_{AB}\omega_aT^{Ba})\ . \end{aligned} \tag{4.57}$$

The terms grouped together as $\alpha^{(2)}$ are small near the Cauchy horizon of a black hole.

$$\alpha^{(2)} = \frac{1}{4}r^3K^A\alpha_A + D^A\mu_A + D^A\nu_A + 3\pi r^3e^{-\lambda}K^B\Delta s^A\tau_{AB}\ . \tag{4.58}$$

The wave equation (4.55) for m_H can be compared with the corresponding equation (2.33) for the mass in spherical symmetry. Clearly, the two equations agree in the spherical limit. The wave equation in spherical symmetry can be inverted, as in equation (2.36), since it is only two dimensional. Technically, equation (4.55) can not be inverted to solve for m_H in a general spacetime. However, consider again the perturbation approach discussed in section 4.3. In this approach we assumed that the metric functions of the perturbed solution are close to the stationary solution. On the axis of the Kerr solution, the shift vectors are zero, so to highest order, the normal Lie derivatives reduce to regular partial derivatives, ie., $D_A \rightarrow \partial_A$. As stated earlier, in the stationary solution $e^{-\lambda} \rightarrow \infty$ at the Cauchy horizon. In this approximation the solution becomes effectively two dimensional and the mass function can be approximated, in vacuum, by the formal solution of the integral equation

$$m_H = \int du \int dv \left(\frac{1}{8} r^3 e^{-\lambda} \sigma_{vab} \sigma_v{}^{ab} \sigma_{ucd} \sigma_u{}^{cd} + m_H \sigma_{vab} \sigma_u{}^{ab}\right) + O(1) , \quad (4.59)$$

for $r \neq 0$. Suppose that m_H is approximated by the first term of (4.59). If the behaviour of the shear and λ are as discussed in section 4.3, then as a first approximation, equation (4.59) yields that

$$\lim_{v \rightarrow \infty} m_H = e^{\kappa_-(v+u)} v^{-q} u^{-p} . \quad (4.60)$$

It is easily verified that $m_H \sigma_{vab} \sigma_u{}^{ab}$ is much smaller than the first term of (4.59).

This handwaving discussion suggests that the effect of perturbations propagating in the interior is to cause m_H to diverge exponentially near the Cauchy horizon. In order to make this solution for the mass function more rigorous, it is necessary to solve the characteristic initial value problem for the interior of a general black hole. We will do this in the next chapter. The result is that we find the mass function does diverge approximately as described by equation (4.60).

4.6 Conclusion

In this chapter we have discussed the problems associated with defining a local energy density for the gravitational field. Although it is impossible to define a local gravitational energy density, the quasi-local definitions made by Hawking and Hayward have many of the characteristics which we would expect such an energy density to have.

We have derived equations for the normal Lie derivatives of both quasi-local masses and a wave equation for Hawking's mass. We have shown that general arguments based on the behaviour of perturbations in a black hole spacetime suggest that the magnitudes of both mass definitions will diverge at the Cauchy horizon of a perturbed black hole. Since Hayward's mass is a curvature invariant, its divergence signals the presence of a curvature singularity at the Cauchy horizon. In the next chapter we will solve the characteristic initial value problem inside the black hole and show that in general, a curvature singularity does form, as suggested by the arguments presented in this chapter.

Chapter 5

The generic black hole singularity

The principal goal of this thesis is to describe the interior of an isolated black hole formed from the collapse of a rotating star. The exterior geometry of the black hole is given by the stationary Kerr-Newman family of solutions. How deep into the black hole does the Kerr-Newman solution approximate the interior of a perturbed black hole? As discussed in chapter 2, the analytic extension of Kerr across the event horizon has an unphysical timelike singularity which lies behind a Cauchy horizon. As in the static Reissner-Nordström solution, the Kerr Cauchy horizon is a surface of infinite blueshift, where the energy of perturbations measured by free falling observers diverges [26, 80]. The aim of this chapter is to present a backreaction calculation in which the effect of the blueshifted perturbations is taken into account.

The causal structure of the Kerr black hole is similar to the Reissner-Nordström black hole. In chapter 2 we found that general spherical perturbations of Reissner-Nordström result in a null scalar curvature singularity forming at the location of the Reissner-Nordström Cauchy horizon. This singularity acts as a brick wall rendering the extension of the spacetime beyond the singularity meaningless.

The assumption of spherical symmetry in chapter 2 plays a minor role compared to the causal structure of the black hole. For this reason, it might be expected that a similar singularity will be found at the Cauchy horizon of Kerr when perturbations are present. In the analysis of this chapter the results of chapter 2 will be used as a guide.

The key question is whether the spherical backreaction models are stable to perturbations. It has been observed that the black hole interior is isomorphic to the interaction region of a colliding plane wave spacetime. Since the Cauchy horizon in a plane wave spacetime is unstable to perturbations and is generically replaced by a spacelike singularity (see figure 1) [81], it has been suggested [30] that a similar phenomenon may also occur inside the black hole. Although this is a nice argument, no analytical backreaction calculation has shown any evidence of an all-encompassing spacelike singularity. In order to discuss this argument, we introduce a plane wave metric and consider plane symmetric perturbations of Reissner-Nordström . The plane wave analysis also serves as a simple model, which is a remarkably good description of the general black hole solution which will be presented later in this chapter.

Several calculations have suggested that the null singularity found in spherical symmetry may be a generic feature of black holes formed by gravitational collapse. Bonanno[82] matched two Kerr solutions along a thin null shell in the interior and showed that the mass diverges along a null hypersurface. His analysis assumed that the hole's angular momentum is small and thus can't be considered a general solution. Ori [83] has shown that gradients of the metric perturbations of the Kerr solution diverge at the Cauchy horizon, suggesting that a null singularity forms. Brady and Chambers [84] have solved the Einstein equations on the Cauchy horizon and an intersecting null hypersurface. Their solution shows that a singularity forms on the Cauchy horizon, but they did not evolve the equations off the initial characteristics. Recent results of Ori and Flanagan [85] show that the Einstein equations admit a generic family of null spacetime singularities. The arguments presented in section 4.5 of this thesis suggest that the backreaction of perturbations in the Kerr black hole will cause Hawking's quasi-local mass to diverge. In a general spacetime the Weyl curvature is not uniquely determined by the mass, as it is in spherical symmetry, but this suggests that a curvature singularity may form.

Our method is to model the innermost region of the black hole by solving the Einstein equations for a completely general metric near the Cauchy horizon. This complicated task can be simplified by noting that the structure of the black hole interior is ideally suited for a double-null decomposition of the spacetime metric. Applying the $2+2$ formalism of chapter 3 to the interior problem results in a simplification of the Einstein equations and allows us to find a solution near the Cauchy horizon. We shall show that the initial conditions given by the collapse of a star lead to a null singular solution with the requisite number of arbitrary functions to be considered general. All components of the resulting Weyl curvature tensor diverge at CH. As the singularity is approached, the Kretschmann invariant is dominated by the Ψ_0, Ψ_2 and Ψ_4 components of the Weyl tensor.

We find that close to the Cauchy horizon, the metric is approximated by a simple plane symmetric spacetime with a null shock-like singularity. The resulting picture is of a collision of ingoing and scattered gravitational radiation interacting with the geometry to create a lightlike singularity. The singularity is mild in the same sense as was found in spherical symmetry: the metric can be written in coordinates which leave all components finite and non-zero. Thus, tidal distortions of observers remain finite at the singularity.

The organisation of this chapter is as follows. In section 1 we discuss the general collapse problem which provides the initial conditions for the black hole interior. In section 2 we discuss the plane wave approximation on which the general solution presented in section 3 is based on.

5.1 Collapse with angular momentum

The general features of the collapse of a star with angular momentum are similar to the non-rotating collapse discussed in section 2.2. The gravitational field of a rotating star may be very complicated but after it has settled into a stationary state, the black hole's exterior gravitational field is completely described by its total mass, electric charge and angular momentum.

Price's analysis [11] of the radiation of the star's irregularities was done on a spherical background which is only valid for collapse with zero angular momentum. However, the presence of the power law tails is due to the power law behaviour of the curvature potential at large distances away from the black hole. The large distance behaviour of the Kerr potential is similar, so it is expected that power law tails should develop [19].

A recent numerical integration has shown that in the linear approximation power law tails do develop in the exterior of the Kerr solution [20]. In this study, it was found that for slow rotation, tails develop almost exactly as they do in spherical symmetry. The analysis for quickly rotating black holes is complicated by the mixing of different l-modes. The study showed that the dominant term in the $l = 2$ mode tail has angular dependence at intermediate times, but this dependence dies out at late times. The late time behaviour is a power law, $t^{-\mu}$, with $\mu \sim 2.9$. If the background were spherically symmetric, $\mu = 7$. However, the value of μ is unimportant for our analysis. The important point is that the power law form is generic.

The analysis of scattering on the spherical black hole's exterior [21, 22] showed that the power law wave tails enter the black hole. While this has not been explicitly shown for the power law tails which develop outside of Kerr, the spherical analysis is general enough to suggest that the power law tails enter the Kerr black hole.

An important effect is the propagation of the wave tail in the black hole interior. The Kerr potential is more complicated than Reissner-Nordström, but for low energy modes the potential appears qualitatively similar. The scattering occurs for these modes in a thin band at a radius much larger than the Cauchy horizon. Since the scattering will occur long before the modes are infinitely blueshifted, the results of scattering on a stationary background[86, 87, 80] will serve as a reasonable approximation. We expect then that near the Cauchy horizon, for late times, the metric perturbations fall off in an inverse power law. Ori's analysis [83] of the metric perturbations confirms this picture.

5.2 The plane wave approximation

The spherical model of the black hole interior has several key features which should be typical of the generic situation. Gravitational collapse will generally produce a weak tail of gravitational radiation which is backscattered into the hole. These weak infalling perturbations will appear to be infinitely blueshifted to freely falling observers at the Cauchy horizon of the hole. The Cauchy horizon of Kerr-Newman is characterised by a surface gravity κ_- which is independent of the angular coordinates θ and ϕ. This suggests that the exponential blueshift function will be independent of angular location on the Cauchy horizon. The influx will interact with the the curvature as it propagates inwards, producing a scattered outflux of the form modeled in the previous section. It seems reasonable that a null curvature singularity will also form in the general case when the backreaction of the wave tail is taken into account.

A simple model of the black hole interior can be developed by approximating the incoming and scattered gravitational radiation as plane gravitational waves. The region near the black hole's Cauchy horizon is modelled as a colliding plane wave spacetime. As we shall see in the next section, this is more than a toy model: the dominant terms of the general solution are identical to the plane wave spacetime.

On length scales, l, which are much smaller than the Cauchy horizon radius r_-, the Cauchy horizon appears locally to be flat. To be specific, take the two-sphere at the Cauchy horizon $r_-^2 d\Omega^2$, and transform to coordinates x, y defined by

$$x + iy = r_- e^{i\phi} \sin(\theta) \tag{5.1}$$

so that $r_-^2 d\Omega^2 \sim dx^2 + dy^2$.

Thus, close to the Cauchy horizon, the static black hole interior can be approximated by the metric

$$ds^2 = f_s(r) du dv + \frac{r^2}{r_-^2}(dx^2 + dy^2) , \tag{5.2}$$

where $r = r(u, v)$. Cauchy data placed outside of the hole can only be evolved as far as the Cauchy horizon at $r = r_-$ where $f_s(r_-) = 0$.

Now consider perturbations of this metric, corresponding to a collision of plane parallel polarized gravitational waves propagating in the interior [88]

$$ds^2 = -2e^{\lambda} du^0 du^1 + \frac{r^2}{r_-^2}(e^{2\beta} dx^2 + e^{-2\beta} dy^2) , \tag{5.3}$$

where Λ, r, β are only functions of u^A. Our aim is to test the stability of the Cauchy horizon at $f_s(r_-) = 0$ under gravitational perturbations. If it is stable, it will be possible to find a solution to the vacuum field equations which to leading order is of the form $e^{\lambda} \sim f_s(r_-)$ and $r \sim r_-$. The metric (5.3) has only one degree of freedom. In order to model the non-linear aspects of the gravitational radiation, it is necessary to introduce a second degree of freedom, the function $\gamma(u^A)$. The metric can be written in a standard form [81]

$$ds^2 = -2e^{\lambda} du^0 du^1 + \frac{r^2}{r_-^2}\left(\cosh\gamma(e^{2\beta} dx^2 + e^{-2\beta} dy^2) - 2\sinh\gamma dx dy\right) , \tag{5.4}$$

which represents plane-symmetric gravitational waves.

The details of the choice of two-metric are unimportant. We will write the plane wave metric in a less coordinate specific way,

$$ds^2 = -2e^{\lambda} du^0 du^1 + \frac{r^2}{r_-^2} h_{ab} d\theta^a d\theta^b , \tag{5.5}$$

where h_{ab} has two degrees of freedom, since we define $\det h_{ab} = 1$. Comparing with the standard form (3.14) of the $2+2$ metric, we see that

$$g_{ab} = \frac{r^2}{r_-^2} h_{ab} \qquad \sqrt{g} = \frac{r^2}{r_-^2}$$
$$s_A^a = 0 \quad \omega^a = 0 \quad D_A = \partial_A . \tag{5.6}$$

The two dimensional Ricci scalar vanishes, as a result of the plane symmetry. The extrinsic curvatures are

$$K_A = \partial_A \ln \frac{r^2}{r_-^2} \tag{5.7}$$

$$\sigma_{Aab} = \frac{1}{2}\frac{r^2}{r_-^2}\partial_A h_{ab} \,. \tag{5.8}$$

The vacuum Einstein equations can now be easily written using the double-null equations derived in chapter 3.

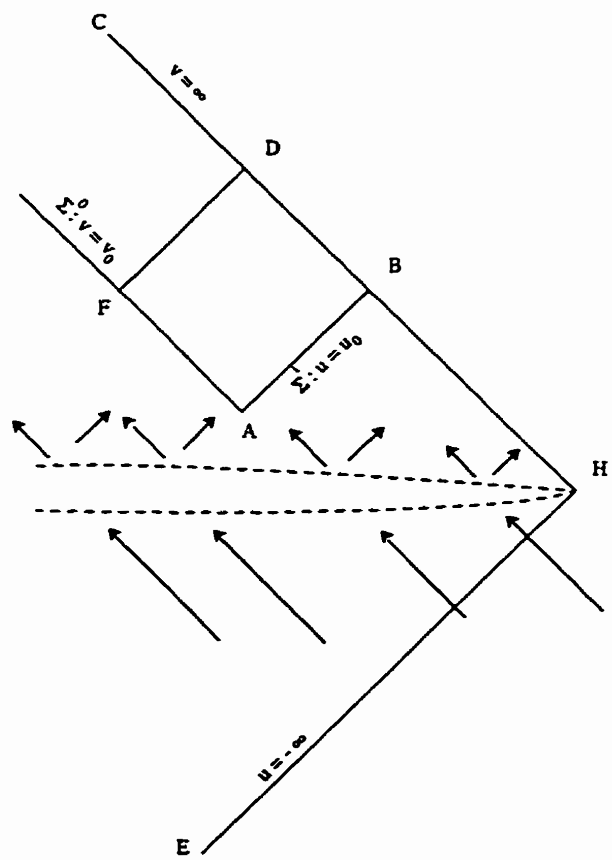

Figure 10: The characteristic initial data problem for the black hole interior. Initial data (four functions of three variables) are placed on the initial characteristics AF and AB and evolved using the vacuum Einstein equations to the final characteristics DF and DB. The initial data correspond to gravitational radiation which has been scattered by the black hole's internal gravitational field.

Initial conditions for the gravitational perturbations are given by specifying the form of the conformal metric h_{ab} on the initial characteristics Σ^0 and Σ^1 (see figure 10). Equivalently, σ_{1ab} can be specified on Σ^0 and σ_{0ab} on Σ^1. The goal is then to evolve the shear off the initial characteristics using the CIVP integration procedure explained in section 3.7.

Before solving the CIVP, we should consider the gauge freedom in the choice of null coordinates u^A. Consider the vacuum propagation equation (3.94)

$$0 = R_a{}^a = 2\frac{e^{-\lambda}}{r^2}\partial_0\partial_1 r^2 \,. \qquad (5.9)$$

The solution for r^2 depends on two arbitrary functions of one variable,

$$r^2 = f(u^0) + g(u^1) + \text{constant} \,.$$

The arbitrariness in f and g is a manifestation of our freedom in choosing coordinates u^A. Our aim is to study the stability of the spherical solution to plane-symmetric perturbations. For this reason we choose r^2 to have the same form (2.58) as was found for the spherically symmetric solution,

$$r^2 = r_-^2 - 2(A(u) + B(v)) \,. \qquad (5.10)$$

In chapter 2 we solved the equations using Kruskal coordinates, in which all components of the metric are finite and non-zero near the Cauchy horizon. In this chapter we choose to use the Eddington-Finkelstein coordinates $u^1 = u$ and $u^0 = v$ (2.3) for the reason that in these coordinates, the static metric has $e^\lambda = f_s(r)$. At the Cauchy horizon $e^\lambda \to 0$. In the general solution we will exploit this fact in order to expand the general equations in powers of e^λ, which will be small.

In this coordinate system, the functions A and B are

$$A(u) = \frac{\kappa_-^{-2}}{(p-1)}(-\kappa_- u)^{-p+1}\left(1 - \frac{p-1}{-\kappa_- u} + \ldots\right) \qquad (5.11)$$

$$B(v) = -\frac{\kappa_-^{-2}}{(q-1)}(\kappa_- v)^{-q+1}\left(1 - \frac{q-1}{\kappa_- v} + \ldots\right) . \qquad (5.12)$$

As initial conditions, we assume that the shear is determined by the results of scattering on a stationary background. We define shape functions $a(u)$ and $b(v)$ which describe the initial behaviour of the perturbations,

$$(b'(v))^2 := \frac{\kappa_-}{4\pi r_-^2}B'(v) \sim (\kappa_- v)^{-q}$$

$$(a'(u))^2 := \frac{\kappa_-}{4\pi r_-^2}A'(u) \sim (-\kappa_- u)^{-p} \,. \qquad (5.13)$$

The shear of the ingoing null rays initially has the value

$$\sigma_{vab}(u_0, v) = Z_{ab} b'(v) \tag{5.14}$$

where Z_{ab} is a constant traceless two-tensor. Similarly, the initial value of the shear of the outgoing null rays is

$$\sigma_{uab}(u, v_0) = Y_{ab} a'(u) \tag{5.15}$$

where Y_{ab} is constant and traceless.

The hypersurface equation (3.89) on Σ^1 can be rearranged to form a first order ODE for λ, as long as $K_v \neq 0$, (which is always true inside a black hole),

$$\partial_v \lambda = \partial_v \ln K_v + \frac{1}{2} K_v + \frac{|\sigma_v|^2}{K_v} , \tag{5.16}$$

where the positive definite norm is defined by

$$|\sigma_v|^2 := \sigma_{va}{}^b \sigma_{vb}{}^a . \tag{5.17}$$

Substituting the initial conditions (5.14) and (5.15) on Σ^1 into the hypersurface equation, we find that

$$\lambda(u_0, v) = \lambda_0 + \frac{1}{2} \ln r^2/r_0^2 + \ln |B'(v)/B'(v_0)| - \kappa_-(v - v_0) , \tag{5.18}$$

where the subscript 0 refers to the value of a function on the initial surface S_0. We have used the freedom to rescale v by a positive constant to set the coefficient of v in the last term of (5.18) to κ_-. The rescaling is equivalent to setting $|Z|^2 = 8\pi$. This solution shows that on the initial surface $\lim_{v \to \infty} e^\lambda \to 0$.

Similarly, the subsidiary equation $R_{11} = 0$ can be solved on Σ^0, yielding

$$\lambda(u, v_0) = \lambda_0 + \frac{1}{2} \ln r^2/r_0^2 + \ln |A'(u)/A'(u_0)| - \kappa_-(u - u_0) . \tag{5.19}$$

It remains to evolve the shear to later hypersurfaces $u > u_0$. If the power law behaviour remains, integration of (5.16) will result in a general solution for λ with behaviour similar to the initial value (5.18). The gravitational degrees of freedom are propagated by the traceless equations (3.95),

$$\begin{aligned} 0 &= R_a{}^b - \frac{1}{2}\delta_a{}^b R_d^d = e^{-\lambda}\left(2\partial_u \sigma_{va}{}^b + K_v \sigma_{ua}{}^b + K_u \sigma_{va}{}^b\right) && (5.20) \\ &= e^{-\lambda}\left(h^{bc}\partial_u\partial_v h_{ac} + \partial_u h^{bc}\partial_v h_{ac} + \frac{1}{2r^2}(\partial_v r^2 h^{bc}\partial_u h_{ac} + \partial_u r^2 h^{bc}\partial_v h_{ac})\right) && (5.21) \end{aligned}$$

This is a non-linear coupled system of equations for the two degrees of freedom denoted by h_{ab}. In general, no closed form solution of (5.21) is known, however, global existence and uniqueness has been proven for these equations [81] in the case of colliding plane gravitational waves.

When h_{ab} is diagonal, as for the simple parallel polarized gravitational wave spacetime (5.3), the equation for h_{ab} has an exact solution. In this case, $\sigma_{Ax}{}^{x} = -\sigma_{Ay}{}^{y} = \partial_A\beta$ and equation (5.20) is linear and reduces to

$$0 = 2r^2\partial_u\partial_v\beta + \partial_u r^2\partial_v\beta + \partial_v r^2\partial_u\beta \;, \tag{5.22}$$

which can be solved in terms of Hankel functions of zero order,

$$\beta = \int d\omega \left(c(\omega)e^{i\omega\chi}H_0^{(1)}(\omega r^2) + d(\omega)e^{-i\omega\chi}H_0^{(2)}(\omega r^2)\right) \;, \tag{5.23}$$

$$\chi = A(u) - B(v). \tag{5.24}$$

The solution (5.23) can also be written as a function of the initial data [88] which demonstrates that given power law initial data, β continues to have a power law form when evolved off the initial characteristics.

The evolution of the shear off the initial hypersurfaces controls the character of the metric. For the square of the shear $|\sigma_v|$ makes a contribution to the hypersurface equation (5.16) which must be solved on the later hypersurfaces $u > u_0$. If the shear continues to have a power law behaviour on the later surfaces then a solution very similar to the mass inflation singularity will result. If the shear should develop singular behaviour, then the mass inflation picture will not be stable. We can prove the following theorem concerning the behaviour of the shears.

Theorem: 1 *If the initial data for the shears are such that*

i) *σ_v is an inverse power law in v on the hypersurface $u = u_0$,*

ii) *σ_u is an inverse power law in $|u|$ on the hypersurface $v = v_0$,*

and the solution for r^2 is given by (5.10), then the leading order solution of the propagation equations (5.21) yields a power law behaviour on later hypersurfaces $u > u_0$ and $v > v_0$.

Proof: In chapter 4 we introduced an effective stress tensor for gravitation τ_{AB} (4.24) which has components given by

$$\tau_{AB} = \frac{1}{8\pi}\text{diag}(|\sigma_u|^2, |\sigma_v|^2) \;. \tag{5.25}$$

Using the traceless part of the propagation equations $R_{ab} = 0$, we derived a "conservation" law (4.53) for τ_{AB} which has the exact form for the plane wave metric

$$\partial_u(r^2|\sigma_v|^2) = -r^2 K_v \sigma_{uab}\sigma_v{}^{ab} \tag{5.26}$$

$$\partial_v(r^2|\sigma_u|^2) = -r^2 K_u \sigma_{uab}\sigma_v^{ab} . \tag{5.27}$$

If the right hand sides of (5.26) and (5.27) were zero, then it would be possible to show that the evolution of the shear preserves the power law decay of the wave tail. Although the right hand sides of these equations are not zero, it is not difficult to find an upper bound for $|\sigma_{uab}\sigma_v{}^{ab}|$.

First, define the function ξ

$$\xi = \sigma_{uab}\sigma_v{}^{ab}. \tag{5.28}$$

Integration of (5.26), making use of the initial condition (5.14) and the solution for r^2 (5.10), yields the equation

$$|\sigma_v(u,v)|^2 = \frac{(|Z|r(u_0,v)b'(v))^2}{\kappa_- r^2(u,v)}\left(\kappa_- + \int_{u_0}^{u}\xi du\right) . \tag{5.29}$$

Similarly, integration of (5.27) yields

$$|\sigma_u(u,v)|^2 = \frac{(|Y|r(u,v_0)a'(u))^2}{\kappa_- r^2(u,v)}\left(\kappa_- + \int_{v_0}^{v}\xi dv\right) . \tag{5.30}$$

Consider the Schwartz inequality

$$|\xi|^2 \leq |\sigma_u|^2|\sigma_v|^2 . \tag{5.31}$$

When the two-metric g_{ab} is diagonal (as in the plane wave metric (5.3) with one degree of freedom) the equality holds. Substitution of the solutions (5.29) and (5.30) into (5.31) yields the inequality

$$\xi^2 \leq \mu^2(u,v)\left(\kappa_- + \int_{u_0}^{u}\xi du\right)\left(\kappa_- + \int_{v_0}^{v}\xi dv\right) , \tag{5.32}$$

where the positive function μ^2 is defined to be

$$\mu^2(u,v) = \left(\frac{r(u_0,v)r(u,v_0)|Y||Z|a'(u)b'(v)}{\kappa_- r^2(u,v)}\right)^2 . \tag{5.33}$$

Suppose that at the point $(u,v) = (u',v')$ in ABDF the function ξ has its maximum value, $\xi_{max} = \xi(u',v')$. Then the following inequalities are satisfied

$$\int_{u_0}^{u'}\xi du \leq (u'-u_0)\xi_{max} , \int_{v_0}^{v'}\xi du \leq (v'-v_0)\xi_{max} . \tag{5.34}$$

The following inequality for ξ_{max} can be derived by substituting (5.34), into (5.32):

$$\alpha\,\xi_{max}^2 - \beta\,\xi_{max} - \gamma \leq 0 \tag{5.35}$$

where the coefficients α, β, γ have been defined by

$$\begin{aligned}
\alpha &= 1 - \mu^2(u',v')(u'-u_0)(v'-v_0) \\
\beta &= \kappa_- \mu^2(u',v')\left((u'-u_0)+(v'-v_0)\right) > 0 \\
\gamma &= \kappa_-^2 \mu^2(u',v')\,.
\end{aligned}$$

From the inequality (5.35), it can be seen that there are two cases:

$\alpha \leq 0$ If α is negative, then (5.35) can't be used to place a bound on ξ_{max}.

$\alpha > 0$ In this case the upper bound on ξ_{max} is

$$\xi_{max} \leq \frac{\beta}{2\alpha} + \frac{\sqrt{\beta^2 + 4\alpha\gamma}}{2\alpha}\,. \tag{5.36}$$

In order to determine the sign of α, it is necessary to consider the magnitude of the term $\mu^2(u'-u_0)(v'-v_0)$ which appears in the coefficient α. The functions $|Y|$ and $|Z|$ are bounded and approximately of order unity. The diamond ABDF of figure 10 is the region where $|u| \leq v$ and the magnitudes of both coordinates are large, ie. $\kappa_-|u| \sim \kappa_- v >> 1$. As a result, the functions $a'(u) \sim b'(v) << 1$ and the radius r (5.10) and is always close to the value of the Kerr Cauchy horizon radius, $r \sim r_-$. The characteristic length of the segment AF is approximately r_-, so that at most. $(u'-u_0) \leq r_-$. Hence

$$(a'(u))^2(u-u_0) \sim (-\kappa_- u)^{-p}\frac{1}{r_-} << 1\,. \tag{5.37}$$

Although the interval $(v'-v_0)$ can be infinite, this does not have a disastrous effect, since

$$(b'(v))^2(v-v_0) \sim \frac{1}{r_-^2\kappa_-}(\kappa_- v)^{-q+1}(1-v_0/v) << 1\,. \tag{5.38}$$

Making use of these two inequalities, we can write

$$\mu^2(u-u_0)(v-v_0) << \frac{1}{r_-^3\kappa_-^3} \sim 1\,. \tag{5.39}$$

Substituting (5.39) into (5.36), it is clear that $\alpha \sim 1$ and ξ_{max} is given by (5.36).

Since μ is very small everywhere in ABDF, we can expand (5.36) to lowest order in μ to approximate

$$\xi_{max} \sim \gamma^{1/2} \sim \kappa_- r_-^2 \mu << r_-^2 \ . \tag{5.40}$$

This allows an estimation of the upper bound for ξ at any point in ABDF by neglecting the integrals in equations (5.29) and (5.30). The function ξ then satisfies

$$|\xi(u,v)|^2 \leq |\sigma_u(u,v_0)|^2 |\sigma_v(u_0,v)|^2 = (|Y||Z|a'(u)b'(v))^2 \tag{5.41}$$

which is the required bound on ξ. Hence, by elementary calculus, equations (5.29) and (5.30) have the solutions

$$|\sigma_v|^2 \leq (|Z|b'(v))^2 + O(ab^3) \tag{5.42}$$

$$|\sigma_u|^2 \leq (|Y|a'(u))^2 + O(a^3 b) \ . \tag{5.43}$$

The result is that when the initial data for the shear is of the Price inverse power law form, the evolution preserves the power law fall off at later characteristic slices. QED.

The prediction of the theorem has been verified numerically [90] with initial data which include perturbations of a power law. Substitution of the solution (5.42) into the differential equation for λ on slices $u > u_0$ and matching with the boundary data on Σ^0, we find that the general solution for λ is

$$\lambda = \lambda_0 - \kappa_-(v+u) + \ln(A'(u)B'(v)) - \ln(A'(u_0)B'(v_0)) + \frac{1}{2}\ln r^2/r_0^2 + O(ab) \tag{5.44}$$

The general solution for the conformal metric can be found by integrating (5.8).

$$h_{ab}(u,v) = h_{0ab} + 2Z_{ab}(b(v) - b(v_0)) + 2Y_{ab}(a(u) - a(u_0)) \ , \tag{5.45}$$

while the determinant of the two-metric is

$$\sqrt{g} = \frac{r^2}{r_-^2} = 1 - \frac{2}{r_-^2}(A(u) + B(v)) \ . \tag{5.46}$$

This solution is singular in the limit $v \to \infty$, as can be seen by substituting the solution (5.44) - (5.46) into the components of the Weyl tensor, (3.63) - (3.68). The components Ψ_1 and Ψ_3 are identically zero. The non-zero components,

$$\begin{aligned} \Psi_0 &\sim \kappa_- e^{\kappa_-(u+v)} b'(v) \\ \Psi_2 &\sim -4r_-^3 e^{\kappa_-(u+v)} a'(u) b'(v) \\ \Psi_4 &\sim \kappa_- e^{\kappa_-(u+v)} a'(u) \ . \end{aligned} \tag{5.47}$$

The asymptotic behaviour of the square of the Weyl tensor is

$$C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta} \sim e^{-2\lambda}b'(v)a'(u) \sim e^{2\kappa_-(v+u)}(\kappa_- v)^{-q/2}(-\kappa_- u)^{-p/2} . \tag{5.48}$$

The curvature diverges exponentially in the limit $v \to \infty$.

The gravitational wave tail, $b(v)$, which enters the black hole is infinitely blueshifted at the Cauchy horizon. When the tail interacts with the scattered radiation, $a(u)$, transversely crossing the Cauchy horizon, a gravitational shock wave forms, creating a curvature singularity. The specific form of the function $a(u)$ is unimportant. The importance of the scattered radiation is that it serves as a catalyst. If $a(u) = 0$, then Ψ_0 is the only non-zero component of the Weyl tensor and the square of the Weyl tensor is zero. The absence of scattered radiation produces a coordinate dependent singularity. As long as $a(u) \neq 0$ the Kretschmann invariant diverges at the Cauchy horizon. The black hole's curvature will always scatter the incoming radiation, so in general the function $a(u)$ will never be zero for all u.

It is known [88, 81] that the general singularity formed in the collision of plane waves is spacelike. It may seem counter-intuitive that our solution which describes a collision of plane waves has a lightlike singularity. Yurtsever's theorems [88] describe the behaviour of the metric (5.3) close to $r = 0$. By studying the solution for β (5.23), he has proved that under generic perturbations of β a spacelike singularity forms at $r = 0$ and that no other singularities can precede it. But this is only true if in the unperturbed spacetime the initial characteristic $u = u_0$ extends as far as $r = 0$. Contrast this with the case of a black hole. Inside of a black hole, initial data can only be evolved as far as the Cauchy horizon at $r = r_- \neq 0$. In other words, the initial characteristic necessarily ends at $r = r_-$. The point is that a Cauchy horizon is a relative concept: it depends on the choice of initial Cauchy surface [89]. A Cauchy surface in the Reissner-Nordström exterior is not equivalent to a Cauchy surface for the interior (including $r = 0$) of Reissner-Nordström.

5.3 The general solution

Our approach is to model the region close to the Cauchy horizon using the metric

$$ds^2 = -2e^{\lambda}dudv + 2s_a dud\theta^a + g_{ab}d\theta^a d\theta^b + s_a s^a du^2 \tag{5.49}$$

where the six metric functions are functions of all four variables. The double-null formalism of the previous section is a natural choice to decompose the Einstein equations to a tractable form, since the Cauchy horizon is a null hypersurface. Using this formalism we will show that in the limit of the Cauchy horizon the metric reduces to the form (5.5) of the interacting plane waves which we examined in section 5.2. This is not surprising since we are essentially studying the interaction of gravitational radiation in the black hole interior.

If the Kerr metric is written in null coordinates u and v (defined in equation (4.35), then the g_{uv} component of the metric takes the form

$$g_{uv} \sim -e^{-\kappa_-(u+v)}\ , v \to \infty \tag{5.50}$$

near the Cauchy horizon $r = r_- = m_0 - \sqrt{m_0^2 - a^2}$, where $\kappa_- = (r_-^2 + a^2)^{-1}\sqrt{m_0^2 - a^2}$ is the surface gravity of the inner horizon. Thus we see that in this coordinate system, close to the inner horizon the metric function $e^{-\lambda} \sim e^{\kappa_-(u+v)} \to \infty$. This suggests that all factors of $e^{-\lambda}$ be pulled out and the Ricci tensor be expanded as the series

$$R_{\alpha\beta} = R^{(0)}_{\alpha\beta} e^{-\lambda} + R^{(1)}_{\alpha\beta} + \ldots \tag{5.51}$$

A solution of the vacuum field equations asymptotic to the Cauchy horizon is found by solving the equation $R^{(0)}_{\alpha\beta} = 0$.

In this limit, the vacuum field equations (3.53) - (3.55) reduce to a form similar to the field equations for a plane wave spacetime. The similarity will be exploited in the solution of the characteristic initial value problem (CIVP) which will be presented in this section.

The new complication, as compared to the plane wave spacetime of the previous section, is that we do not assume that the spacetime has any symmetry. All six metric functions depend on the four spacetime coordinates. In addition to the four metric functions λ and g_{ab} of the plane wave spacetime, the general spacetime has a shift vector s^a. As a result, the four equations $R_{Aa} = 0$ are not trivially satisfied.

Our notation and general method for finding a solution will be identical to the analysis presented for the plane wave spacetime.

The hypersurface equations

In the discussion of the plane wave spacetime, we made a number of remarks about the expected behaviour of the initial data. We assume that on the initial characteristics, the gravitational perturbations can be expanded in an inverse power series in the advanced time v,

$$\begin{aligned}\sigma_{vab}(u_0, v, \theta^a) &= (\kappa_- v)^{-q/2} \Sigma_{n=0}^{\infty} \sigma_{vab}^{(n)}(\theta^a) v^{-n} \\ &\sim Z_{ab}(\theta^a) b'(v) \,, \end{aligned} \tag{5.52}$$

where the function $b(v)$ (5.13) describes the shape of the tail, to highest order and the traceless two-tensor Z_{ab} is now a function of the angular coordinates θ^a.

The two-metric is split as before into $g_{ab}(x^\alpha) = \sqrt{g(x^\alpha)} h_{ab}(x^\alpha)$, and a scalar function $r(x^\alpha)$ introduced,

$$\sqrt{g(x^\alpha)} = \sqrt{g_0(\theta^a)} \frac{r^2(x^\alpha)}{r_0^2(\theta^a)} \,, \tag{5.53}$$

where the subscript '0' denotes the value of a function on the initial two-surface S_0. The shear and dilation of the ingoing null rays are related to the two-metric by

$$K_v = \partial_v \ln \frac{r^2}{r_0^2} \tag{5.54}$$

$$\sigma_{vab} = \frac{1}{2} \sqrt{g} \partial_v h_{ab} \,. \tag{5.55}$$

Our Ansatz is that the v dependence of r^2 on the initial surface $\Sigma^{(1)}$ in these coordinates should be the same as in the plane wave model (5.10)

$$r^2(u_0, v, \theta^a) = r_0^2(\theta^a) + \frac{K_{v0}(\theta^a)}{r_0^2(\theta^a) B'(v_0)} (B(v) - B(v_0)) \,, \tag{5.56}$$

where $B(v)$ is given by (5.12) and the function K_{v0} is the value of the dilation at S_0. Since the black hole interior is a trapped region, the dilation is negative for all values of the angular coordinates.

The value of the function λ on $\Sigma^{(1)}$ is given by the solution of the hypersurface equation $R_{vv} = 0$, which is the same equation (5.16) for λ in the plane wave spacetime. The solution, given the initial data for the shear and dilation, is

$$\lambda(u_0, v, \theta^a) = -\kappa_v(\theta^a)(v - v_0) + \ln |B'(v)/B'(v_0)| + \frac{1}{2} \ln \frac{r}{r_0} + \lambda_0(\theta^a) \tag{5.57}$$

$$\kappa_v(\theta^a) = \frac{\kappa_- |Z|^2 B'(v_0)}{|K_{v0}|} \,. \tag{5.58}$$

Note that the solution (5.58) for the function λ in the general case is very similar to the solution (5.18) for λ in the plane wave spacetime. The main difference is the appearance of an arbitrary function of angular coordinates, $\kappa_v(\theta^a)$. Since $B'(v)$ is a positive function, the function κ_v is positive definite. As a result, the function e^{λ} diverges in the limit $v \to \infty$ on the initial surface $\Sigma^{(1)}$. If κ_v is not constant, then the strength of the divergence of $e^{-\lambda}$ at CH will depend on the angle at which the horizon is approached. It follows from (5.57) that the rate of change of λ with respect to angular position diverges, since

$$\lim_{v\to\infty} \partial_a \lambda = - \lim_{v\to\infty} \partial_a \kappa_v \, (v - v_0). \tag{5.59}$$

This situation is very different from that of the spherical black hole, where the divergence doesn't depend on angle. However, we will soon prove that the evolution equations constrain κ_v to be a constant.

We turn now to the constraint equations $R_{va} = 0$, which are identically zero in the plane wave spacetime. These can be written as a first order ODE for ω_a:

$$\partial_v(r^2 e^{-\lambda}\omega_a) = r^2 \left(2\sigma_{va}{}^{b}{}_{;b} - \partial_a\partial_v\lambda - \partial_a\partial_v \ln r^2 + \partial_a\lambda\partial_v \ln r^2\right) . \tag{5.60}$$

Once the two-metric g_{ab} is specified on S_0 the two-dimensionally covariant derivative (denoted by ;) is defined. Substituting in the asymptotic behaviour of the initial data, this equation can be integrated to yield,

$$\omega^a = \frac{e^{\lambda}}{r^2}\left(\omega_0^a(\theta^a)e^{-\lambda_0} + \partial_a\kappa_v \, (v - v_0)\right)(1 + O(b)) \tag{5.61}$$

where $\omega_0^a(\theta^a)$ is an integration function. Since the twist is related to the shift vector by (3.83), we have the result

$$s^a = -\frac{e^{\lambda}}{\kappa_v r^2}\left(\omega_0^a(\theta^a)e^{-\lambda_0} + \partial_a\kappa_v \, (v - v_0)\right) + \frac{\omega_0^a}{\kappa_v} , \tag{5.62}$$

where the form of the last term was chosen to set $s^a = 0$ on $v = v_0$. The results of equations (5.61) and (5.62) show that the shift and twist vectors are exponentially suppressed on $\Sigma^{(1)}$.

The subsidiary equations

The procedure for solving the subsidiary equations $R_{uu} = R_{ua} = 0$ on Σ^0 is similar to that for the hypersurface equations. We have made use of the coordinate freedom (discussed in section 3.6) to set the shift vector to zero on Σ^0. As a result, the shear and dilation of the outgoing null rays on Σ^0 have the same simple form as equation (5.54) and (5.55) with v replaced by u. Off Σ^0, the shear and dilation are more complicated, since the shift vector is not zero.

The initial data for the shear are

$$\begin{aligned} \sigma_{uab}(u, v_0, \theta^a) &= (-\kappa_- u)^{-p/2} \Sigma_{n=0}^{\infty} \sigma_{uab}^{(n)}(\theta^a)(-\kappa_- u)^{-n} \\ &\sim Y_{ab}(\theta^a) a'(u) , \end{aligned} \tag{5.63}$$

where Y_{ab} is traceless and a is given by (5.13). The Ansatz for the form of the function r on Σ^0 is that it takes the same functional form as in the plane wave spacetime.

$$r^2(u, v_0, \theta^a) = r_0^2(\theta^a) + \frac{K_{u0}(\theta^a)}{r_0^2(\theta^a) A'(u_0)}(A(u) - A(u_0)) , \tag{5.64}$$

where $A(u)$ is given by (5.11) and the (negative definite) function K_{u0} is the value of the dilation at S_0.

The equation $R_{uu} = 0$ on Σ^0 is exactly the same as the equation for the plane wave spacetime. The solution for λ on Σ^0 is

$$\lambda(u, v_0, \theta^a) = -\kappa_u(\theta^a)(u - u_0) + \ln|A'(u)/A'(u_0)| + \ln\frac{1}{2}\frac{r}{r_0} + \lambda_0(\theta^a) \tag{5.65}$$

$$\kappa_u(\theta^a) = \frac{\kappa_- |Y|^2 A'(u_0)}{|K_{u0}|} , \tag{5.66}$$

where κ_u is positive definite.

The value of the twist on Σ^0 is given by the solution of the equation $R_{ua} = 0$. The formal solution is

$$\omega_a(u, v_0, \theta^a) = e^{\lambda} r^2 \int du r^2 \left(2\sigma_{ua}{}^{b}{}_{;b} - \partial_a \partial_u \lambda - \partial_a \partial_u \ln r^2 + \partial_a \lambda \partial_u \ln r^2\right) . \tag{5.67}$$

To summarise, we have shown that once the functions $g_{ab}, \partial_v s^a, \partial_v r^2, \partial_u r^2$ and λ are specified on S_0 and $\sigma_{uab}, \sigma_{vab}$ on the initial characteristics, the constraint equations on the characteristics are satisfied.

Solution of the propagation equations

The propagation equations ${}^{(4)}R_{ab} = 0$ (3.95) may appear rather complicated. However, we have just shown that on the characteristics, $\omega^a \sim s^a \sim e^{\lambda}$, which allows the expansion of the field equations (5.51) discussed earlier. Our procedure is to assume that a solution for λ of the form given by the solutions (5.66) and (5.57) on the initial hypersurfaces holds everywhere near the Cauchy horizon so that $\lambda \to -\infty$ and $e^{\lambda} \to 0$. We shall show that this assumption is self-consistent.

We first apply this limit to the two dimensional trace of the evolution equations:

$$\begin{aligned} {}^{(4)}R^a_a &:= g^{ab\,(4)}R_{ab} = e^{-\lambda} g^{ab\,(4)}R^{(0)}_{ab} + g^{ab\,(4)}R^{(1)}_{ab} \\ &= -e^{-\lambda}\left(D_A K^A + K_A K^A\right) + {}^{(2)}R - \frac{1}{2}e^{-2\lambda}\omega^a\omega_a - \lambda^{;a}{}_{;a} - \frac{1}{2}\lambda_a\lambda^a \,. \end{aligned} \quad (5.68)$$

The term $e^{-2\lambda}\omega^a\omega_a \sim O(1)$. The dilation is $K_u = \partial_u \ln r^2 + O(e^{\lambda})$, and the Lie derivative operator is $D_u = \partial_u + O(e^{\lambda})$. Hence the Ricci tensor is split into the terms

$$g^{ab\,(4)}R^{(0)}_{ab} = \frac{2}{r^2}\partial_u\partial_v r^2 \quad (5.69)$$

$$\begin{aligned} g^{ab\,(4)}R^{(1)}_{ab} &= -\frac{1}{2}\lambda_a\lambda^a - \lambda^{;a}{}_{;a} + {}^{(2)}R - \frac{1}{2}e^{-2\lambda}\omega^a\omega_a \\ &\quad -e^{-\lambda}\left(\partial_v(s^a{}_{;a}) + s^a\partial_a\partial_v \ln r^2 - 2s^a{}_{;a}\partial_v \ln r^2\right). \end{aligned} \quad (5.70)$$

The solution of the zeroth order vacuum equation $g^{ab\,(4)}R^{(0)}_{ab} = 0$ is

$$r^2(u,v,\theta^a) = r_0^2(\theta^a) + \frac{K_{v0}(\theta^a)}{r_0^2(\theta^a)B'(v_0)}(B(v) - B(v_0)) + \frac{K_{u0}(\theta^a)}{r_0^2(\theta^a)A'(u_0)}(A(u) - A(u_0)) \,, \quad (5.71)$$

which satisfies the initial Ansatz (5.56) and (5.64).

For self-consistency of the field equations, it is necessary for the lower order terms of the Ricci tensor to vanish, i.e., $g^{ab\,(4)}R^{(1)}_{ab} = 0$. All terms in (5.70) are of order unity, except for the first two, since as shown in equation (5.59), the function $\lambda_{,a}$ diverges at the Cauchy horizon. Taking the limit of the Cauchy horizon in the lower order field equation, we find that

$$0 = \lim_{v\to\infty} g^{ab\,(4)}R^{(1)}_{ab} = -\frac{1}{2}\lim_{v\to\infty}(v - v_0)^2 \partial_a\kappa_v \partial_b\kappa_v g^{ab}. \quad (5.72)$$

This equation has a solution only if $\partial_a\kappa_v = 0$. From its definition (5.58), κ_v is positive, so the freedom to rescale the coordinate v by a positive constant allows us to set

$$\kappa_v = \kappa_- \,. \quad (5.73)$$

This is the important result that the field equations force the metric functions to diverge uniformly. The singularity is equally strong at any value of θ^a on the Cauchy horizon. This result makes sense intuitively, since the infinite blueshift in the Kerr black hole is controlled by the surface gravity of the Cauchy horizon which is a constant. A similar restriction can be placed on the function κ_u by considering the behaviour of equation (5.68) in the limit that the initial surface Σ^1 is moved backwards to earlier times, $u_0 \rightarrow -\infty$. The result is that κ_u must be constant, and can be set equal to κ_-. This places a restriction on the initial values of the dilations K_{v0} and K_{u0}, since they are related to κ_v and κ_u through equations (5.58) and (5.66). As a result the general solution for r^2 will be

$$r^2(x^\alpha) = r_0^2(\theta^a) + |Z(\theta^a)|^2(B(v_0) - B(v)) + |Y(\theta^a)|^2(A(u) - A(u_0)) \ . \qquad (5.74)$$

The final propagation equation to be considered is the traceless equation (3.95) which controls the shear. In the limit $e^\lambda \rightarrow 0$ (3.95) reduces, in lowest order, to the propagation equation (5.21) for the shear in the plane wave spacetime. The conservation laws (4.53) for τ_{AB} reduce in this limit to

$$\partial_u(r^2|\sigma_v|^2) = -r^2 K_v \sigma_{uab}\sigma_v^{ab} + O(e^\lambda) \qquad (5.75)$$
$$\partial_v(r^2|\sigma_u|^2) = -r^2 K_u \sigma_{uab}\sigma_v^{ab} + O(e^\lambda) \ . \qquad (5.76)$$

The theorem presented in the previous section applies to this case, if we make the trivial change that all functions depend on θ^a and that the tensors $Y_{ab}(\theta^a)$ and $Z_{ab}(\theta^a)$ are bounded. The result is that the right hand sides of (5.75) and (5.76) are always small in the diamond ABDF of figure 10 and these equations can be integrated to show that the shear maintains its initial power law behaviour, as in equations (5.42) and (5.43) for the shear in the plane wave spacetime. Integration of the hypersurface equations yields the solution for λ found in the plane wave spacetime.

To summarise, the solution of the vacuum field equations, asymptotic to the Cauchy horizon ($v \rightarrow \infty$), is given by the metric functions:

$$\begin{aligned}
\lambda &= \lambda_0 - \kappa_-(v+u) + \ln r/r_0 + \ln(A'(u)B'(v)) - \ln(A'(u_0)B'(v_0)) + O(ab) \\
\sqrt{g(x^\alpha)} &= \sqrt{g_0(\theta^a)}\left(1 + |Z(\theta^a)|^2(B(v) - B(v_0)) + \frac{K_{u0}(\theta^a)}{A'(u_0)}(A(u) - A(u_0))\right) \\
g_{ab} &= g_{ab0}(\theta^a) + 2Z_{ab}(\theta^a)(b(v) - b(v_0)) + 2Y_{ab}(\theta^a)(a(u) - a(u_0)) \\
s^a &= -\frac{e^{\lambda-\lambda_0}}{\kappa_- r^2}\omega_0^a(\theta^a) + \frac{\omega_0^a}{\kappa_-} \ ,
\end{aligned}$$

where $Y_{ab}(\theta^a)$, $Z_{ab}(\theta^a)$ are restricted by $Y_{ab}g^{ab} = Z_{ab}g^{ab} = 0$. First order quantities derived from the metric functions are

$$\sigma_{vab} = Z_{ab}b'(v)(1 + O(a^2b^2))$$

$$\begin{aligned} \sigma_{uab} &= Y_{ab}a'(u)(1+O(a^2b^2)) \\ \omega^a &= \frac{e^{\lambda-\lambda_0}}{r^2}\omega_0^a(\theta^a)(1+O(ab)) \,. \end{aligned}$$

The functions A and B are given by (5.11) and (5.12) and the functions a and b are defined by (5.13).

The components of the Weyl tensor can be found by substituting this solution into the formal expressions (3.63) - (3.68). To highest order, the components are

$$\begin{aligned} \Psi_0 &\sim \kappa_- m^a m^b e^{\kappa_-(v+u)} Z_{ab} b'(v) \\ \Psi_1 &\sim -\frac{1}{2} m^a e^{\frac{1}{2}\kappa_-(v+u)} Z_{ab} \lambda^{,b} b'(v) \\ \Psi_2 &\sim -\frac{1}{2} e^{\kappa_-(v+u)} Y_{ab} Z^{ab} a'(u) b'(v) \\ \Psi_3 &\sim -\frac{1}{2} \bar{m}^a e^{\frac{1}{2}\kappa_-(v+u)} Y_{ab} \lambda^{,b} a'(v) \\ \Psi_4 &\sim \kappa_- \bar{m}^a \bar{m}^b e^{\kappa_-(v+u)} Y_{ab} a'(v) \end{aligned}$$

The asymptotic behaviour of the square of the Weyl tensor is given approximately by the product $\Psi_0\Psi_4$,

$$C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta} \sim e^{-2\lambda} b'(v) a'(u) \sim e^{2\kappa_-(v+u)} (\kappa_- v)^{-q/2} (-\kappa_- u)^{-p/2} \,. \tag{5.77}$$

The curvature diverges exponentially in the limit $v \to \infty$.

The solution will still be valid when matter is present if the stress tensor has the following asymptotic fall-off near the Cauchy horizon:

$$T_{\alpha\beta} = e^{-\lambda}(v^{-n}\partial_\alpha v \partial_\beta v + |u|^{-n}\partial_\alpha u \partial_\beta u) + T^{(1)}_{\alpha\beta} \,, \tag{5.78}$$

where $T^{(1)}_{\alpha\beta}$ is of order unity.

5.4 Conclusion

The general description of the black hole singularity that we have presented here is remarkably similar to a lightlike shock wave singularity formed by the interaction of highly blueshifted gravitational waves. It is reasonable to assume that in the collapse of a general rotating object to form a black hole, scattering of the perturbations on the background geometry should cause gravitational wave tails of the Price form to enter the black hole. On physical grounds one would expect the influx of gravitational waves to be infinitely blueshifted at the black hole's Cauchy horizon and to cause a curvature singularity to form. In section 5.3 we have presented a solution of the Einstein equations for a metric with six functions of four variables. Under the assumption that the initial perturbations fall off with the Price law we have shown that it is possible to find a solution of the field equations asymptotic to the Cauchy horizon. A simple model of the region of the general solution near the Cauchy horizon is the parallel polarized plane wave metric. At the Cauchy horizon, we find that the curvature invariants diverge. The curvature singularity is lightlike and weak with a simple structure reminiscent of the spherically symmetric mass inflation model. Although gravitational mass is not well defined in non-spherical spacetimes, the concept of quasi-local mass can be useful. We have shown (in chapter 4) that the mass aspect function and Hawking's quasi-local mass diverge at the singularity. For this reason it is reasonable to refer to this solution as mass inflation.

The simple form of the mass inflation singularity should be contrasted with the strong spacelike and oscillatory singularity of BKL. It is expected, though, that as gravitational radiation transversely crosses the Cauchy horizon, its generators will be eventually focussed (through the Raychaudhuri equation) to zero radius. At this point the singularity would become spacelike and possibly of the BKL form. The solution presented in this chapter should be thought of as a null precursor to a stronger spacelike singularity.

Chapter 6

Quantum effects in the black hole interior

It has often been suggested that the inclusion of the backreaction of quantum fields on classically singular spacetimes could tend to weaken singularities. This is due to the loop-hole in the singularity theorems [2] which prove that the complete gravitational collapse of a star generically produces a singularity. The theorems require that matter obeys the dominant energy condition. Classical matter always obeys this condition, but it is possible for the stress tensor of quantum fields to violate the condition without violating the conservation law $\nabla_\mu T^\mu_\nu = 0$. This opens up the possibility that when quantum effects are considered, the collapse of a star may not necessarily produce a singularity.

Quantum effects near classical singularities may be important for another reason. Recall that the notion of a continuous manifold requires that the measurement of position be made arbitrarily precise. However, through the Heisenberg uncertainty principle, particles with Planck scale momentum will be produced if length scales of the order of the Planck length (10^{-33} cm) are probed. The backreaction onto the geometry of these particles would cause such a large fluctuation in the curvature that the classical picture of a continuous manifold will probably break down [91]. One of the signatures of a singularity is that the Riemann curvature diverges. If the characteristic length scale of the spacetime is defined as $\ell \sim |R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}|^{-1/4}$, then close to a singularity ℓ will approach the Planck length. Hence, we expect that a purely classical description of a singularity cannot be adequate and a quantum theory of gravity must be invoked.

The implication is that the classical mass inflation scenario describing a rotating black hole's singularity is not complete. In the preceding chapters we ignored the contribution of quantum effects to the black hole singularity. At moderate curvatures the classical approach is justified, for causality protects us from the region of strong curvature. Since the singularity is lightlike, curvature increases as time increases so that the region of Planckian curvature described by the unknown quantum theory of gravity lies to the future and cannot influence the region of moderate curvature. However, in the region of moderate curvature where classical relativity is still a good description, effects such as the production of elementary particles and vacuum polarization may contribute significantly to the curvature of spacetime.

It is held by many that quantum gravity should have a self-regulatory effect [92], i.e., that quantum effects should help to weaken singularities. However an examination of the literature does not produce any definite proof of any self-regulatory property. The effect of quantized fields on other spacetime singularities produced by gravitational collapse has been considered by various authors. Ford and Parker [93] studied the production of particles by naked singularities, although they didn't calculate the backreaction of the created particles on the geometry. They considered shell crossing singularities and showed that quantum effects do not tend to remove the singularity. They also examined the $|e| > m$ Reissner-Nordström solution and showed that the singularity is not damped, but that quantum effects may cause an event horizon to form. Frolov and Vilkovisky [94] studied the collapse of a spherically symmetric null shell governed by a quadratic effective Lagrangian. They found evidence suggesting that the collapse to $r = 0$ can produce a regular solution. Anderson, Brady and Camporesi [95] calculated the effects of vacuum polarization in the homogeneous mass inflation model (HMI), briefly discussed in section 2.9. The HMI model is a simplified model of the spacelike $r = 0$ singularity which joins to the mass inflation singularity. They found that the effect of vacuum polarization is to intensify the strength of the singularity.

In this chapter we will calculate the expectation value for the stress tensor of quantized fields propagating on a simple mass inflation background. We will work in the semi classical approximation where the gravitational field is treated classically while all other fields are quantum in nature [96]. A few other calculations of this sort have been done by others. Balbinot and Brady [97] found in the (1+1) dimensional analogue of the Reissner-Nordström solution that quantum effects tend to make the singularity stronger. A (2+1) dimensional calculation by Steif [98] came to a similar result. In (3+1) dimensions, Balbinot and Poisson [99] ignored non-local effects by using a quadratic Lagrangian model and found that quantum effects either strengthened or weakened the singularity, depending on the sign of the quadratic terms in the Lagrangian. We find [100] that the quantum stress tensor exhibits a divergence which is exponentially stronger than the rate of divergence of the classical stress tensor. However, we are unable to ascertain, in this formalism, the sign of the divergence. If the quantum influx were to diverge to positive infinity, the quantum effects would tend to increase the strength of the singularity, while a negative divergence would tend to weaken the singularity. As we will show, the origin of the ambiguity is in the non-local contribution to the quantum stress tensor, which dominates over the local terms. Non-local effects are typically due to the dominance of low energy quanta which probe long distance scales. The classical mass inflation effect is mainly due to the scattering of low energy fields, so it is interesting that low energy effects are also important in the quantum domain.

The organisation of this chapter is as follows. Particle creation by black holes will be discussed in section 1. In section 2 Horowitz's formula [17] for the quantum stress tensor in linearized gravity will be introduced. In section 3 we will use this formula to calculate the expectation value of the stress tensor of non-gravitational quantum fields in the Ori model of mass inflation. Concluding remarks will be made in section 4.

6.1 Particle creation in a black hole spacetime

The modern view of the vacuum, gained through the study of quantum field theory [101], is that it is not really a vacuum: virtual particle-antiparticle pairs are constantly being created and annihilated. The Heisenberg uncertainty principle allows particles of rest energy mc^2 to live briefly for a time of order $\hbar/mc^2$. These virtual particles can have measurable consequences when an external field is present. Heuristically, the effect of an external field is to lend energy to the virtual pair, allowing them to exist for a longer period of time. For example, in the Casimir effect, an electric field applied to the vacuum region between two conductors polarizes the vacuum. The vacuum polarization is responsible for a measurable force on the conductors which can't be explained classically.

It seems reasonable that if the gravitational field is treated as an external force, similar particle creation effects will occur [102]. However, the particle concept is not well defined in a general curved spacetime, so the external field concept must be applied with care. The problem is that the definition of a positive energy particle state depends on the choice of observer. For an inertial observer in Minkowski space, the definition is clear: positive energy states are eigenstates of the Killing vector $\partial/\partial t$. where t is the Minkowski time coordinate. The definition is Poincaré invariant, so that all inertial observers agree on their observations of particles. The introduction of a non-inertial observer serves as an example of the effects which can occur in a curved spacetime. The non-inertial observer's time, τ is in general a complicated function of the inertial observer's time, so that eigenstates of $\partial/\partial t$ are not generally eigenstates $\partial/\partial\tau$. The result is that accelerated observers will detect particles in the inertial observer's vacuum state [103]. The situation in a general curved spacetime is similarly ambiguous, for if there are different nonequivalent observers, their definitions of a particle will not agree.

The situation which is most straightforward to analyze is a spacetime which is initially stationary, undergoes a period of evolution, and afterwards settles down to a final stationary state. We will refer to the initial stationary period as the "in" state and the final stationary period as the "out" state. The eigenfunctions of the Killing vector for the "in" state are the one parameter family $u_\omega^{in}(x^\mu)$. A similar family $u_\omega^{out}(x^\mu)$ can be defined for the "out" region. Both families form a complete basis, so that any quantum scalar field can be "second quantized" by an expansion into the normal modes of either basis,

$$\Phi(x^\mu) = \Sigma_\omega(a_\omega u_\omega^{in} + a_\omega^\dagger u_{\omega^*}^{in}) = \Sigma_\omega(b_\omega u_\omega^{out} + b_\omega^\dagger u_{\omega^*}^{out}) \ . \tag{6.1}$$

Where the operators $a_\omega^\dagger$ and a_ω respectively create and annihilate quanta of energy ω in the "in" state. Similarly $b_\omega^\dagger$ and b_ω are creation and annihilation operators for the "out" state. The definition of the "in" vacuum (or no-particle) state is the set of conditions

$$a_\omega|0_{in}>= 0 \text{ , for all } \omega \ . \tag{6.2}$$

Similarly the "out" vacuum is defined by

$$b_\omega|0_{out}>= 0 \text{ , for all } \omega \ .$$

The operators in the two representations are related by the linear Bogoliubov transformation

$$b_\omega = \Sigma_{\omega'}(\alpha^*_{\omega\omega'} a_{\omega'} - \beta^*_{\omega\omega'} a^\dagger_{\omega'}) \tag{6.3}$$

where $\alpha^*_{\omega\omega'}$ and $\beta^*_{\omega\omega'}$ are the Bogoliubov coefficients. If any of the $\beta^*_{\omega\omega'}$ coefficients are non-zero, then the transformation is nontrivial and there will be mixing between states. It can easily be shown, using equations (6.2) and (6.3) that the expectation value of the number of "out" particles in the "in" vacuum state is

$$< 0_{in}|n^{out}_\omega|0_{in} >=< 0_{in}|b^\dagger_\omega b_\omega|0_{in} >= \Sigma_{\omega'}|\beta_{\omega\omega'}|^2 \; . \tag{6.4}$$

The result is that if the Bogoliubov coefficients are non-zero, and the system begins in a vacuum state, observers in the "out" region will observe particles.

Consider a star which collapses to a stable radius larger than the Schwarzschild radius. The Killing vectors for the regions before and after the collapse are defined with respect to the Schwarzschild time and are identical. It follows that the eigenfunctions in the "in" and "out" regions are identical and the Bogoliubov transformation is trivial. No particle creation results in this case [104], except for transitory particle production during the non-stationary period.

The case which is of interest to us is the collapse of a star to form a black hole. Naively, we might define the "in" state at $\mathscr{I}^-$ and the "out" state at $\mathscr{I}^+$. However, the collapse to a black hole has changed the topology of the spacetime non-trivially, and $\mathscr{I}^+$ no longer constitutes a Cauchy surface. A complete Cauchy surface consists of the union of $\mathscr{I}^+$ with the future event horizon. Hawking [105] showed that the Bogoliubov transformation is non-trivial when a black hole is formed. Although observers measure the quantum state to be vacuous before the collapse begins, observers near the event horizon, will measure the "in" vacuum state to be full of particles. Some of these particles will fall into the horizon and become causally disconnected from their anti-particle partners. The partners can now be considered real particles that can travel to the future static region and be measured by future static observers. Hawking's calculations show that in the far future, the observers will measure thermal radiation corresponding to a temperature of $\kappa_+/2\pi$ coming from the black hole.

The Hawking radiation has consequences inside the black hole as well. Conservation of energy implies that a flux of positive energy radiation outside the hole is accompanied by an influx of negative energy radiation inside the hole [106]. The quantum influx is infinitely blueshifted at the inner horizon, so that (in Kruskal coordinates (2.6) for the inner horizon) the stress tensor for the Hawking influx is

$$T_{\alpha\beta}^{Hawking} = -\left(\frac{\kappa_+}{2\pi}\right)^4 \left(\frac{1}{\kappa_- V}\right)^2 \partial_\alpha V \partial_\beta V \,.$$

As we have discussed in section 1.2, the classical influx of gravitational radiation created by the collapse has the Price tail form (2.35), so that the Hawking radiation becomes comparable in size to the Price tail at a time V given by

$$\ln|\kappa_- V| = -\left(\frac{\kappa_+}{2\pi}\right)^{1/3}$$

for $p = 12$. Substituting this value of V into the expression for the Weyl curvature (2.29), it can be seen that for black holes with exterior mass larger than approximately 100 kg, the Weyl curvature has grown to Planckian levels at which point the semi-classical approximation is not valid. Hence we conclude that the effect of the Hawking radiation on the interior geometry is only important for black holes which have mass less than 100 kg [107]. When the mass is larger than this critical value, the classical picture presented in the preceding chapters is valid. We will restrict our attention to the study of quantum effects inside astrophysical scale black holes, where the effect of the influx of Hawking radiation will be negligible.

Another important cause of quantum particle creation is the electric field present in the Reissner-Nordström black hole. Markov and Frolov [108] have shown that particle creation by the electric field will very rapidly neutralize the Reissner-Nordström black hole. This suggests that a realistic model of the Reissner-Nordström black hole interior should look similar to the Schwarzschild interior [109]. However. our use of the Reissner-Nordström black hole has always been as a toy model of the more complicated Kerr black hole. As we have shown in chapter 5, the non-spherical black hole interior is very similar to the charged black hole interior. In this chapter we will consider quantum effects on the background of a charged black hole with the understanding that our results should be indicative of the more general situation.

6.2 Vacuum polarization inside the black hole

The general goal of semi-classical gravity is to calculate a finite expectation value for the stress tensor, $< T_{\alpha\beta} >$ of a quantum field on a fixed manifold. This expectation value can then be used as a source term for the semi-classical Einstein equations

$$G_{\mu\nu} = 8\pi(T_{\mu\nu}^{class.} + \hbar < T_{\mu\nu} >)$$

which can be solved for the classical spacetime metric. This quantum backreaction problem is very difficult to solve in general. The calculation of the expectation value of the quantum stress tensor is hampered by the fact that there is no formal expression for a finite quantum stress tensor. The standard approach to this problem (see for example reference [96]) is to substitute the quantized field (such as the expression (6.1)) into the expression for the stress tensor for the equivalent classical field. The classical stress tensor is quadratic in derivatives of the field. For example, the stress tensor for a massless scalar field is

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi \ .$$

Substitution of the infinite mode sum (6.1) into this expression will yield an infinite result from the infinite zero point contribution of terms of the form $a_\omega a_\omega^\dagger$. Whatever method is used to calculate the quantum stress energy, the result is always divergent. The divergent expression can be separated into two terms, one finite and the other infinite by one of many regularization prescriptions. A finite, renormalized, stress tensor is then obtained by retaining just the finite portion of the regularized result. Of course, there is some ambiguity in subtracting one infinite expression from another. A priori, it seems that there is no reason for different regularization schemes to yield the same finite result.

Wald's axiomatic approach has shown that the results of various regularization schemes agree within a reasonable degree of freedom. Wald's axioms [110, 111] are that a reasonable definition of a renormalized quantum stress tensor in the "in-in" state should have the following properties.

i) The difference between the expectation value for different states should be well defined.

ii) The expectation value for the stress tensor is zero in Minkowski space.

iii) The expectation value is conserved.

iv) The expectation value at a point p depends only on the geometry within the past light cone of p.

Wald has shown that these axioms define a quantum stress tensor up to an arbitrary, but conserved local tensor depending on the curvature of spacetime.

Quantum backreaction

Our aim is to estimate the effect in the black hole interior of the backreaction due to vacuum polarization. In order to do so, we will compute the renormalized one-loop expectation value of the quantum stress tensor for the "in-in" (or Unruh) vacuum state, $< 0_{in}|T_{\mu\nu}|0_{in} >$, which will abbreviate to $< T_{\mu\nu} >$. The "in-in" vacuum state, in the black hole context means that the vacuum was in an unexcited state before the collapse of the star occurred. The result of the specification of this vacuum state is the influx of Hawking radiation, which (as discussed in the previous section) is negligible. The calculation of $< T_{\mu\nu} >$ in the general non-spherical black hole background presented in chapter 5 is a very difficult problem. We have shown that the simple spherically symmetric Ori model captures the essence of the more complicated black hole interior. It seems reasonable that a study of the quantum backreaction problem in spherical symmetry should also serve as a good model of the general situation.

Recall that in spherical symmetry, the mass function completely specifies the Weyl curvature. Following Simon's prescription [112] for finding perturbative solutions to the semi-classical Einstein equations we assume that they yield a solution for the metric functions of the form

$$m^{semi-class.} = m^{class.} + \hbar m^{quant} \tag{6.5}$$

$$r^{semi-class.} = r^{class.} + \hbar r^{quant} \tag{6.6}$$

The wave equation for the mass function (2.33) depends on the two-dimensional square of the stress tensor,

$$T_{AB}T^{AB} = T_{UU}^{class.}T_{VV}^{class.} + \hbar(T_{UU}^{class.} < T_{VV} > + T_{VV}^{class.} < T_{UU} >) + O(\hbar^2), \tag{6.7}$$

where $< T_{\alpha\beta} >$ corresponds to the quantum corrections to the stress tensor. Thus to order $\hbar$, the semi-classical field equation for the quantum correction to the mass is

$$\begin{aligned} \partial_U\partial_V m^{quant} &= \frac{1}{2}(4\pi)^2 r^4 \left(T_{UU}^{class.} < T_{VV} > + T_{VV}^{class.} < T_{UU} >\right) \\ &+ \frac{1}{2}(4\pi)^2 r^4 4 T_{UU}^{class.} T_{VV}^{class.} r^{quant.}/r, \end{aligned} \tag{6.8}$$

where quantities without a superscript are understood to be classical. The quantum corrections to the other metric functions are roughly of the order

$$r^{quant.} \sim -\int\int < T_{VV} > dVdV, \tag{6.9}$$

which follows from Raychaudhuri's equation (2.37). If we find that the solution for the quantum correction $m^{quant.}$ is positive, then we will conclude that the backreaction increases the strength of the mass inflation singularity. A negative result for $m^{quant.}$ will result in a damping of the singularity.

Horowitz's formula for linearized gravity

In the case of linearized gravity, Wald's axioms have been successfully applied by Horowitz [17] to deduce a relatively simple formula for the renormalized stress tensor. Horowitz's formula is valid for spacetimes which can be decomposed as a perturbation of Minkowski space,

$$g_{\alpha\beta} = \eta_{\alpha\beta} + \gamma_{\alpha\beta}$$

where $\eta_{\alpha\beta}$ represents the Minkowski metric. All quantities second order in γ are smaller than quantities first order in γ. The Poincaré invariance of the Minkowski metric simplifies the problem. A further assumption, which Horowitz makes is that $< T_{\alpha\beta} >$ can not depend on derivatives of γ which are of sixth order or higher. This restriction fixes the arbitrariness in the stress tensor down to two tensors $\dot{A}_{\alpha\beta}$ and $\dot{B}_{\alpha\beta}$ which are the linearized variations ,

$$\dot{A}_{\alpha\beta} = \frac{1}{\sqrt{-g}}\frac{\delta}{\delta g^{\alpha\beta}}\int \sqrt{-g}C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta}d^4x \tag{6.10}$$

$$\dot{B}_{\alpha\beta} = \frac{1}{\sqrt{-g}}\frac{\delta}{\delta g^{\alpha\beta}}\int \sqrt{-g}R^2 d^4x\,, \tag{6.11}$$

where a dot denotes a linearized quantity. These two tensors are the only conserved linearized tensors with fewer than six derivatives of the metric [17]. Horowitz finds that the most general stress tensor which satisfies Wald's axioms is

$$< T_{\alpha\beta}(x) >= \hbar\left(\int H_\lambda(x-x')[a\dot{A}_{\alpha\beta}(x') + b\dot{B}_{\alpha\beta}(x')]d^4x' + \beta\dot{B}_{\alpha\beta}(x)\right)\,, \tag{6.12}$$

where $H_\lambda(x-x')$ is a distribution with support on the past light cone of x, and will be defined in the next section. The constants a and b are positive and are known for all massless quantum fields [17]. The constant β is arbitrary, as is the parameter λ on which the distribution depends. This formula has been derived by a number of authors using a variety of alternate methods [113, 114].

An approximate black hole metric

It is probably not obvious that a formula based on linearized gravity can have any application to the black hole interior. The presence of the mass inflation singularity means that the metric is decidedly not flat! However, the singularity has a mild integrable form. Recall that in section 1.3 we showed that the Ori metric can be written in coordinates in which the metric has no singular components. Furthermore, the diverging mass function has the special form

$$m(V) = m_0 \frac{1}{-\kappa_- V} |\ln(\kappa_- V)|^{-p}, \ V \to 0_- , \tag{6.13}$$

where V is the Kruskal advanced time. The presence of the logarithmic damping factor has the result that

$$\frac{d^{(i)}m(V)}{dV^{(i)}} >> \frac{d^{(j)}m(V)}{dV^{(j)}} \frac{d^{(k)}m(V)}{dV^{(k)}} \text{, for } i \geq j + k + 1 \, . \tag{6.14}$$

In this restricted sense, terms non-linear in m can be neglected compared to terms linear in m.

Recall that in the simple Ori model of mass inflation, the metric after the shell is

$$\begin{aligned} ds^2 &= dV(f(r,V)dV - 2dr) + r^2 d\Omega^2, \\ f(r,V) &= 1 - 2m(V)/r + e^2/r^2, \end{aligned} \tag{6.15}$$

which can be written as a flat metric plus a perturbation term linear in the mass function. All curvature quantities in the Vaidya spacetime are linear in m, so that it seems that the Horowitz formula can be applied to this spacetime. There is, however, a very serious problem with this approach. The light cone structure of the Ori metric (6.15) is very different from the light cone structure of Minkowski space. For example, the coordinate r is spacelike in Minkowski space and timelike in the Ori metric. But Horowitz's formula involves an integral over the past light cone of the observation point. If the light cone of the perturbed metric is radically different from the light cone for Minkowski space, the result of the formula (6.12) may yield incorrect results.

For this reason we will not attempt to calculate quantum effects in the Ori background metric. Instead, we will show that there is a spacetime which approximates the Ori metric and has the property that the Horowitz formula can be used.

In this thesis, we have discussed two types of approximations. In section 1.3, the divergence of the mass function was used to make the approximation $f \sim -2m(V)/r$. In section 5.2 we argued that on small length scales, the two-sphere at the Cauchy horizon can locally be approximated by a plane, so that $r_-^2 d\Omega^2 \sim dx^2+dy^2$. Adopting both of these approximations to the Ori metric (6.15), we arrive at the metric

$$ds^2 = 2drdV + \frac{2m(V)}{r}dV^2 + \frac{r^2}{r_-^2}(dx^2+dy^2) \ . \tag{6.16}$$

The curvature tensor for (6.16) differs from the curvature for the Ori metric by terms of order $1/r_-^2$, which is finite. Compared to the diverging terms in the curvature arising from the mass function these differences are negligible.

Consider the conformal metric

$$\begin{aligned} ds_*^2 &= \frac{r_-^2}{r^2}ds^2 \\ &= 2\frac{r_-^2}{r^2}drdV + dx^2 + dy^2 + 2\frac{m(V)r_-^2}{r^3}dV^2 \ . \end{aligned} \tag{6.17}$$

If we introduce the coordinate

$$U = 2\frac{r_-^2}{r} \tag{6.18}$$

the metric takes the form

$$ds_*^2 = -dUdV + dx^2 + dy^2 + 2L(U,V)dV^2 \ , \ L = \frac{1}{8r_-^2}m(V)U^3 \ , \tag{6.19}$$

which is in the flat plus lightlike Kerr-Schild form [115]. The form of the conformal metric is exactly of the form needed in order to use Horowitz's result.

Before embarking on the calculation of the quantum stress tensor, we should examine the causal structure of the conformal metric. The coordinate U is a retarded null coordinate for the flat metric, but not for the conformal metric (6.19). The correct null coordinate for the conformal metric has the value

$$\bar{U} = \frac{r^2}{r_-} + \frac{2}{r_-}\int m(V)dV \ . \tag{6.20}$$

In the limit $V \to 0_-$, the integral of the mass function vanishes, and the coordinate r approaches r_-. As a result, close the Cauchy horizon, (i.e. $|\kappa_- V| << 1$), the light cone for the conformal metric is approximated very closely by the light cone for the flat metric. Thus, Horowitz's formula can be used to calculate the renormalized quantum stress tensor for fields propagating on the conformal background metric. Page's formula [18] for conformal transformations of the quantum stress tensor can be used afterwards to find the stress tensor in the physical metric.

6.3 The quantum stress tensor

In this section we will apply Horowitz's formula (6.12) to the calculation of the renormalized stress tensor for quantized fields on the classical background of the conformal metric (6.19). Horowitz's formula for $< T_{\alpha\beta} >$ has a non-local component, which is the action of the distribution $H_\lambda(x - x')$ on the conserved tensors $\dot{A}$ and $\dot{B}$. The non-locality of the stress tensor is a reflection of the fact that quantum particles created within the past light cone of the observation point x can travel to the point and contribute to the quantum stress tensor. The manner in which the past fluctuations contribute is controlled by the distribution.

In order for the distribution to be Lorentz invariant and to have the correct dimensions [17], the distribution should be proportional to $\delta'(\sigma(x, x'))$ where $\sigma(x, x')$ is the world function defined as one half the square of the geodesic distance from x to x'. In order to restrict the results to the past light cone a step funtion in the difference between the Minkowski time at x and x' must also be included. Introduce Minkowski coordinates (t, l, θ, ϕ) with origin at the observation point x. Advanced and retarded times are defined by

$$v = t + l \ , \ u = t - l \ . \tag{6.21}$$

In these coordinates the world function is

$$\sigma(x, x') = -\frac{1}{2}uv \ . \tag{6.22}$$

Points x' with $\sigma = 0$ lie on the light cone of x. Consider the action of a distribution $H(x - x') = \delta'(\sigma(x, x'))\Theta(t - t')$ on a test function with compact support

$$\begin{aligned} \int H(x - x')f(x')d^4x' &= \frac{1}{4}\int du\ dv\ d\Omega(v - u)^2 f(u, v, \theta, \phi)\delta'(-\frac{1}{2}uv)\Theta(-t) \\ &= \int dud\Omega\Theta(-t)\left(\frac{2}{u}f(u, 0) - \partial_v f(u, 0)\right) \ . \end{aligned} \tag{6.23}$$

After an integration by parts, this becomes

$$\int H(x-x')f(x')d^4x' = -\int dud\Omega\Theta(-t)\left(2\ln(-u)\partial_u f(u,0)+\partial_v f(u,0)\right) + \int dud\Omega \ln(-u)f(u,0)\delta(u)\,. \qquad (6.24)$$

There are two problems with this expression. First, the coordinate u has dimensions of length, so the logarithms aren't properly defined. As well, the last term is infinite. Horowitz fixes these problems by introducing an arbitrary length scale λ into the integral and by defining the distribution H_λ to be the finite part of (6.24). The result is that the correct distribution has the following action on a test funtion [17],

$$\int H_\lambda(x-x')f(x')d^4x' = \int_{-\infty}^{0} dud\Omega\left(\ln(-u/\lambda)\partial_u f(u,0)+\frac{1}{2}\partial_v f(u,0)\right)\Big|_{v=0} \qquad (6.25)$$

The parameter λ effectively plays the role of a renormalization mass scale [113]. Substitution of the distribution (6.25) into (6.12) yields the formula for the quantum stress tensor which we will now use.

The linearized variations which are to be inserted into the stress tensor formula are given by,

$$\begin{aligned}\dot{A}_{\alpha\beta} &= -2\partial^\mu\partial_\mu\dot{G}_{\alpha\beta}-\frac{2}{3}\partial_\alpha\partial_\beta\dot{G}+\frac{2}{3}\eta_{\alpha\beta}\partial^\mu\partial_\mu\dot{G}\\ \dot{B}_{\alpha\beta} &= 2\eta_{\alpha\beta}\partial^\mu\partial_\mu\dot{G}-2\partial_\alpha\partial_\beta\dot{G}\,,\end{aligned} \qquad (6.26)$$

where $\dot{G}_{\alpha\beta}$ is the linearized Einstein tensor for the conformal metric (6.19), which has the value

$$\dot{G}_{\alpha\beta} = G_{\alpha\beta} = -4L_{,UU}(\partial_{(\alpha}U\partial_{\beta)}V+\eta_{\alpha\beta}). \qquad (6.27)$$

The components of $\dot{A}$ and $\dot{B}$ are

$$\dot{A}_{VV} = \frac{1}{3}\dot{B}_{VV} = \frac{4}{r_-^4}m''(V)U \qquad (6.28)$$

$$\dot{A}_{UU} = \dot{B}_{UU} = 0 \qquad (6.29)$$

$$\dot{A}_{UV} = \frac{1}{3}\dot{B}_{UV} = 2\dot{A}_{xx} = 2\dot{A}_{yy} = -\frac{1}{12}\dot{B}_{xx} = -\frac{1}{12}\dot{B}_{yy} = -\frac{4}{r_-^4}m'(V)\,. \qquad (6.30)$$

In order to evaluate the integrals, we need to relate the flat coordinates (U,V,X,Y) in which the conformal metric (6.19) is written to the coordinates (u,v,θ,ϕ) (6.21). We introduce Cartesian coordinates for the conformal metric, defined by

$$T = \frac{1}{2}(V+U)\,,\ Z = \frac{1}{2}(V-U)\,. \qquad (6.31)$$

The point at which the stress tensor is to be evaluated has coordinates $W_0 = (T_0, Z_0, X_0, Y_0)$. The plane symmetry of the classical problem allows us to choose $X_0 = Y_0 = 0$. Suppose that we introduce Cartesian coordinates (t, z, x, y) with origin at W_0, and $l^2 = x^2 + y^2 + z^2$. The two sets of Cartesian coordinates are related by

$$X = x \tag{6.32}$$
$$Y = y \tag{6.33}$$
$$Z = l\cos\theta + Z_0 \tag{6.34}$$
$$T = t + T_0 \,. \tag{6.35}$$

Given the definitions of the two sets of null coordinates (6.19) and (6.21),

$$V = V_0 + \frac{1}{2}(v+u) + \frac{1}{2}(v-u)\cos\theta \tag{6.36}$$
$$U = U_0 + \frac{1}{2}(v+u) - \frac{1}{2}(v-u)\cos\theta \tag{6.37}$$

We note here that when $v = 0$ and $u = -\infty$, the coordinate V is unbounded below. We have already stated that for our approximation to be valid, $|\kappa_- V| \ll 1$. It is necessary to restrict the range of integration over the coordinate u in (6.25) to $[-|u_*|, 0]$, where $|u_*|\kappa_- \sim 1$. This is equivalent to stating that the vacuum polarization in the region of strong curvature will dominate the contribution to the stress tensor. The introduction of this cut-off alters the formula for the non-local contribution to

$$\int H_\lambda(x-x')f(x') = \int_{-|u_*|}^{0} du d\Omega \left(\ln(-u/\lambda)\partial_u f(u,0) + \frac{1}{2}\partial_v f(u,0)\right)|_{v=0} \tag{6.38}$$

It is now a trivial matter to change from coordinates u and θ on the surface $v = 0$ to coordinates U and V. The integral in (6.38) is transformed to

$$\int_{-|u_*|}^{0} du \int_{-1}^{1} d(\cos\theta) = -2\int_{V_0-|u_*|}^{V_0} dV \int_{U_0-(V-V_0)-|u_*|}^{U_0} dU \frac{1}{u}|_{v=0} \,, \tag{6.39}$$

where

$$u|_{v=0} = V - V_0 + U - U_0 \,. \tag{6.40}$$

The derivatives in (6.38) can be transformed to the conformal coordinates. Their values are

$$\partial_v f(u,0) = \frac{\partial}{\partial U} f(U,V)\frac{\partial U}{\partial v}|_{v=0} + \frac{\partial}{\partial V} f(U,V)\frac{\partial V}{\partial v}|_{v=0}$$

$$= \frac{1}{u}\left((V-V_0)\frac{\partial}{\partial U}f(U,V)+(U-U_0)\frac{\partial}{\partial V}f(U,V)\right) \tag{6.41}$$

$$\begin{aligned}\partial_u f(u,0) &= \frac{\partial}{\partial U}f(U,V)\frac{\partial U}{\partial u}|_{v=0}+\frac{\partial}{\partial V}f(U,V)\frac{\partial V}{\partial u}|_{v=0} \\ &= \frac{1}{u}\left((U-U_0)\frac{\partial}{\partial U}f(U,V)+(V-V_0)\frac{\partial}{\partial V}f(U,V)\right) .\end{aligned} \tag{6.42}$$

As a result, the integrals (6.38) which must be evaluated are of the form

$$\int H_\lambda(x-x')f(x') = I_V(f)+I_U(f) , \tag{6.43}$$

where

$$\begin{aligned} I_V(f) &= -2\pi\int_{V_0-|u_*|}^{V_0}dV\int_{U_0-(V-V_0)-|u_*|}^{U_0}dU \\ &\quad \frac{1}{u^2}\left[(V-V_0)\frac{\partial}{\partial U}f+(U-U_0)\frac{\partial}{\partial V}f\right] \end{aligned} \tag{6.44}$$

$$\begin{aligned} I_U(f) &= -4\pi\int_{V_0-|u_*|}^{V_0}dV\int_{U_0-(V-V_0)-|u_*|}^{U_0}dU \\ &\quad \frac{1}{u^2}\ln(-u)\left[(U-U_0)\frac{\partial}{\partial U}f+(V-V_0)\frac{\partial}{\partial V}f\right] .\end{aligned} \tag{6.45}$$

The evaluation of the integrals required to calculate the stress tensor is straightforward. Intermediate steps for the calculation are listed in appendix C. The leading order behaviour (in the limit $V_0 \to 0_-$) of the quantum stress in the conformal spacetime is

$$< T^*_{VV}(U_0,V_0) > \sim 4U_0\hbar\frac{1}{r_-^4}m''(V_0)\left((a+3b)\alpha_{VV}\ln(\frac{-V_0}{\lambda})+3\beta\right) \tag{6.46}$$

$$< T^*_{UV}(U_0,V_0) > \sim -4\hbar\frac{1}{r_-^4}m'(V_0)\left((a+3b)\alpha_{UV}\ln(\frac{-V_0}{\lambda})+3\beta\right) \tag{6.47}$$

$$< T^*_{UU}(U_0,V_0) > = 0 \tag{6.48}$$

$$\begin{aligned}< T^*_{XX}(U_0,V_0) > &= < T^*_{YY}(U_0,V_0) > \\ &\sim -4\hbar\frac{1}{r_-^4}m'(V_0)\left((a/2-12b)\alpha_{UV}\ln(\frac{-V_0}{\lambda})-12\beta\right) ,\end{aligned} \tag{6.49}$$

where the positive constants α_{UV} and α_{VV} are calculated in appendix C, and the arbitrary constants λ and β were introduced in equation (6.12). Note that the overall sign of $< T^*_{VV}(U_0,V_0) >$ depends on the relative signs of λ and β, which are not fixed by the semi-classical theory.

The stress tensor for the physical metric can be found using Page's formula [18] for the transformation of the quantum stress tensor under conformal transformations.

$$\begin{aligned} < T^{\mu}{}_{\nu} > &= \Omega^{-4} < T^{*\mu}{}_{\nu} > \\ &- \frac{8\alpha'}{\Omega^4}\left[(C^{*\alpha\mu}{}_{\beta\nu} \ln \Omega)^{|\beta}{}_{|\alpha} + \frac{1}{2} R^{*\beta}{}_{\alpha} C^{*\alpha\mu}{}_{\beta\nu} \ln \Omega \right] \\ &- \beta' \left[(4R^{\beta}{}_{\alpha} C^{\alpha\mu}{}_{\beta\nu} - 2H^{\mu}{}_{\nu}) - \frac{1}{\Omega^4}(4R^{*\beta}{}_{\alpha} C^{*\alpha\mu}{}_{\beta\nu} - 2H^{*\mu}{}_{\nu}) \right] \\ &- \frac{\gamma'}{6}(I^{\mu}{}_{\nu} - \Omega^{-4} I^{*\mu}{}_{\nu}) \end{aligned} \tag{6.50}$$

where $\Omega = r/r_-$, the positive constants α', β' and $\gamma' = 2/3\alpha'$ depend on the spin of the field and the bar denotes a covariant derivative with respect to the conformal metric. The tensors $H_{\mu\nu}$ and $I_{\mu\nu}$ are defined by

$$H_{\mu\nu} = -R^{\alpha}{}_{\mu} R_{\alpha\nu} + \frac{2}{3} R R_{\mu\nu} + g_{\mu\nu}\left(\frac{1}{2} R^{\alpha}{}_{\beta} R^{\beta}{}_{\alpha} - \frac{1}{4} R^2\right) \tag{6.51}$$

$$I_{\mu\nu} = 2R_{|\mu\nu} - 2R_{\mu\nu} + g_{\mu\nu}\left(\frac{1}{2} R^2 - 2R^{|\alpha}{}_{|\alpha}\right). \tag{6.52}$$

The formula (6.50) for the transformation of the quantum stress is quite different from that for a classical conformally invariant field. The transformation law for the classical field is simply the first term of (6.50). The presence of the terms in (6.50) which are not conformally invariant shows that the contribution from quantum effects to the stress tensor depends on the length scale that we are probing.

It is not necessary to perform a detailed calculation of all the terms in the tranformation law, if we keep in mind the leading order behaviour of the curvature tensors is

$$\begin{aligned} C^{*\alpha\beta}{}_{\gamma\delta} &\sim C^{\alpha\beta}{}_{\gamma\delta} \sim m(V) \\ R^{*\alpha}_{\beta} &\sim m(V) \\ R^{\alpha}_{\beta} &\sim m'(V). \end{aligned}$$

Since $< T^{*}_{VV} > \sim m''(V)$, and the classical mass function obeys (6.14), it follows that

$$< T_{VV} > \gg mm'(V) \gg m^3$$

so that terms quadratic in curvature are negligible in comparison to the conformal contribution. Only terms involving two derivatives of curvature, (the term proportional to $\ln \Omega$ and I^{*}_{VV}) will contribute. The result of the conformal transformation on the quantum influx is

$$< T_{VV} >= \Omega^{-2}\left(< T^{*}_{VV} > +(8\alpha' \ln\Omega + 2\gamma')\hbar U m''(V)/r_-^2\right). \tag{6.53}$$

Similarly, we find that

$$
\begin{aligned}
< T_{UU} > &= -2\alpha'\hbar U m(V)/r_-^4 & (6.54)\\
< T_{UV} > &= \Omega^{-2}\left(< T^{*}_{UV} > +(2\gamma' + 8\alpha'(1-\ln\Omega))m'(V)/r_-^2\right) & (6.55)\\
< T_{XX} > &= < T_{YY} > & (6.56)\\
&= \Omega^{-2}\left(< T^{*}_{XX} > (4\gamma' + 16\alpha'(1-\ln\Omega))m'(V)/r_-^2\right). & (6.57)
\end{aligned}
$$

Now consider the quantum backreaction equations (6.8) and (6.9). Assume that the term $|\ln(\frac{-V_0}{\lambda})| \gg 1$, so that the contribution from the conformal metric dominates the stress tensor. The size of the correction to r is

$$r^{quant.} \sim -m(V)\ln(\frac{-V_0}{\lambda}). \tag{6.58}$$

Comparing the relative sizes of the terms in (6.8), we find that

$$m^{quant.} \sim m'(V)\ln(\frac{-V_0}{\lambda}) \tag{6.59}$$

The sign of the quantum corrections to r and m depends on the size of λ in equations (6.58) and (6.59). Remember that λ is an arbitrary length scale introduced to the problem and plays the role of a renormalization mass scale. There is no preferred length scale in the theory of a massless quantum field, so the most that we can say is that there are three cases,

$\lambda \sim V_p$ where V_p is the time at which the curvature becomes Planckian. In this case $|V_0| > \lambda$, and the quantum influx is positive and adds to the classical influx. From (6.59), the quantum corrections will make the mass grow faster while the radius will shrink. Both effects will tend to make the Weyl curvature grow faster than predicted by a purely classical analysis.

$\frac{1}{\kappa_-} \gg \lambda \gg V_p$ In this case, while $|V_0| > \lambda$ the backreaction will have a similar effect as in the previous case. However, now it is possible at some critical time $V_{crit} = -\lambda$ (which is still within the range of validity of the semi-classical approximation) for the quantum influx to change sign. After this critical time, the quantum correction to the mass function is negative while the correction to r causes it to grow larger. Both effects tend to weaken the singularity in this case.

$\lambda \geq \frac{1}{\kappa_-}$ Since the solution is valid when $|\kappa_- V_0| \ll 1$, the logarithmic factor is always negative. Thus, in this case, quantum corrections tend to weaken the singularity.

6.4 Conclusion

In the region of strong curvature near a classical singularity, quantum effects due to the creation of elementary particles and vacuum polarization can become very important. It would be remiss to discuss the physical features of a singularity without some estimate of the backreaction of the quantum particles on the spacetime geometry. In this chapter we have made use of the apparatus of the semi-classical to quantum gravity to determine the quantum backreaction on the singular spherically symmetric mass inflation background.

We noted that the mass function, which characterizes the curvature in spherical symmetry, has a special form in which a term linear in the first derivative of the mass function is much larger than terms quadratic in the mass function. In this sense, the mass inflation geometry is similar to linearized gravity. This suggests that Horowitz's formula [17] for the renormalized expectation value of the quantum stress tensor for massless fields propagating in a spacetime which is a linear perturbation of Minkowski space can be applied to our problem.

We find that the quantum stress tensor diverges faster than the classical stress tensor as

$$< T_{\alpha\beta} > \sim \lim_{V \to 0_-} \frac{\hbar}{|r_- V|} T^{class.}_{\alpha\beta} \ln(|V|/\lambda), \tag{6.60}$$

where λ is the renormalization mass scale. We conclude that quantum effects are very important near the singularity at $V = 0$. However, the sign of the stress tensor depends on the magnitude of λ which is not fixed by the theory of a massless scalar field. Hence, we are left with some ambiguity, for if the logarithm is positive, quantum effects will tend to make the singularity stronger, while if the logarithm is negative, the quantum effects will tend to weaken the singularity. It is interesting that the other four-dimensional estimate of the backreaction [99] found a similar ambiguity while lower dimensional calculations [97, 98] found a definite result: quantum effects increase the strength of the singularity.

We should make a remark about the validity of the semi-classical approximation. Simon [112] has discussed the problem of fictitious solutions to the semi-classical equations. He pointed out that the only solutions which should be allowed are those with

$$< T_{\alpha\beta} > \ll T^{class.}_{\alpha\beta}. \tag{6.61}$$

This leads to the condition

$$|V| \gg \frac{\hbar}{r_-} |\ln(|V|/\lambda)|. \tag{6.62}$$

But the Weyl curvature reaches Planckian values at the time V_p, when

$$\Psi_2 \sim |\ln(\kappa_- V_p)|^{-p} \frac{1}{r_- V_p} \sim \frac{1}{\hbar}. \tag{6.63}$$

Substituting (6.63) into the inequality (6.62) results in

$$|V| \gg |V_p| \, |\ln(\kappa_- V_p)|^p \, |\ln(|V|/\lambda)|. \tag{6.64}$$

Since we also require, $|V| \gg |V_p|$ for the semi-classical approximation to hold this suggests that $|\ln(|V|/\lambda)| \leq 1$. However, if this is the case then the logarithmic term doesn't dominate the stress tensor. Instead, it is of the same size as the other (local) terms, including the term proportional to the arbitrary constant β. The sign of the corrections now hinges crucially on the sign of β, which is not fixed by the semi-classical theory. The arbitrariness in β is due to the freedom to add multiples of R^2 to the effective action. This suggests that our calculation reduces to essentially the problem which Balbinot and Poisson [99] considered. They looked at a local effective action which is quadratic in the curvature. When the coupling constant for the quadratic terms is positive, they found that the curvature increases, while a negative constant reduces the curvature. It seems that to say any more would require knowledge of the signs and magnitudes of the coupling constants in the effective action for quantum gravity, which is out of reach of present day theory.

Chapter 7

Conclusion

The physical picture of the non-spherical black hole interior presented in this dissertation is remarkably simple. The main qualitative features are very similar to the description of the charged spherical black hole interior [1, 16, 23, 24]. The gravitational collapse of a star produces a weak gravitational wave tail which decays as an inverse power law in time and enters the black hole's event horizon. Near the event horizon the tail is very weak and its backreaction onto the geometry can be neglected. The propagation of the gravitational radiation into the interior can thus be approximated by the results of scattering on a stationary Kerr background. The scattering of inverse power law radiation tends to create two weakly decaying fluxes (as discussed in chapter 2), a transmitted influx (parallel to CH in figure 7), which is infinitely blueshifted at CH and a refracted "outflux" which crosses CH transversely focusing the generators of CH to smaller radius. As was found for spherical symmetry, the backreaction of this combination of crossflowing gravitational radiation results in a lightlike, observer-independent curvature singularity forming at CH. The divergence is mild, however, since the integral of the Riemann curvature components is finite in a freely-falling frame, so that tidal distortions are finite at the singularity. The lightlike nature of the singularity guarantees that no timelike observer will ever be in causal contact with the singularity, until running into it. Thus we conclude that the black hole interior formed from gravitational collapse is completely predictable as suggested by the strong cosmic censorship conjecture. This does not, of course, prove SCCC, since we have assumed that WCC holds. Since the scattered radiation which transversely crosses CH slowly focuses it to smaller radius, it seems reasonable to make the conjecture that at some later retarded time, the lightlike Cauchy horizon singularity will merge with a stronger spacelike singularity. This conjecture can only be proved with a full four dimensional numerical integration of the Einstein equations. It will be many years before the techniques of numerical relativity will be able to tackle this difficult problem.

Fairly general arguments were raised in chapter 4 which show that Hawking's quasi-local mass and the Coulomb component of the Weyl tensor will always diverge at the Cauchy horizon of a perturbed black hole. In fact, the general solution of the characteristic initial value problem presented in chapter 5 demonstrates that in general *all* components of the Weyl tensor diverge at CH. The leading order divergence of the product of the Weyl components is

$$\begin{aligned}
\Psi_0\Psi_4 &\sim \lim_{v\to\infty} e^{2\kappa_- v} v^{-q/2} \\
|\Psi_2|^2 &\sim \lim_{v\to\infty} e^{2\kappa_- v} v^{-q} \\
\Psi_1\Psi_3 &\sim \lim_{v\to\infty} e^{\kappa_- v} v^{-q/2} ,
\end{aligned}$$

so that the square of the Weyl tensor is dominated by the contribution from the gravitational wave and Coulomb components. The solution presented is very closely approximated (near CH) by a colliding plane gravitational wave metric, which can serve as an easy toy model of the black hole interior. Essentially, the model results from the assumption that on small length scales a curved region of spacetime can be approximated by a plane symmetric metric [116]. However, as we discussed in section 5.2 it should be noted that the boundary conditions for a black hole are completely different than those for a general plane wave spacetime (for example [88]) so that a spacelike singularity doesn't form prior to the Cauchy horizon.

The weakness of the singularity at CH has lead to some speculation [16, 83] that the spacetime could be continued across the singularity. However, the singularity is still a Cauchy horizon for the spacetime, so there is no unique continuation across CH. In our opinion, it is meaningless to speculate on the possibility of a continuation across CH, without taking quantum effects into account. In chapter 6 we showed that the renormalized stress tensor for massless quantum fields diverges exponentially faster than the stress tensor for classical fields on a spherical background. Clearly the magnitude of the quantum corrections is so large that they shouldn't be neglected. Unfortunately, the semi-classical theory doesn't reveal the sign of the correction. If the corrections add to the classical influx, the singularity may be reinforced by quantum effects, resulting in a stronger curvature singularity at which tidal distortions diverge.

However, if the quantum corrections have a negative sign relative to the classical flux. the singularity's strength will be weakened. Only a full backreaction calculation in this case can reveal the resulting spacetime structure.

Appendices

A Curvature calculations

In this appendix we will calculate the Ricci and Riemann tensors. The computation is based on the Gauss-Weingarten equations of section 3.2, which are summarised here. The Gauss-Weingarten equations (3.26) and (3.33) for the covariant derivatives of the tangent vector can be written as

$$e_{a\alpha|\beta} = {}^{(2)}\Gamma^c_{ab} e^a_{(c)} e^b_\beta - e^{-\lambda} K^{A}{}_{ab} \ell^\alpha_{(A)} e_{b\beta} + e^{-\lambda}(K_{Aa}{}^b + \partial_a s^b_A) e^\alpha_b \ell^A_\beta + e^{-\lambda} L_{BAa} \ell^{A\alpha} \ell^B_\beta \ . \quad (A.1)$$

Since $L^A{}_{Aa} = \partial_a \lambda$, the trace of (A.1) is

$$e_a{}^\alpha{}_{|\alpha} = \Gamma^c_{ac} + \partial_a \lambda \ . \quad (A.2)$$

The expression for the derivative of the normal vector follows from (3.27) and (3.34):

$$\ell_{A\alpha|\beta} = K_{Aab} e^a_\alpha e^b_\beta + L_{ABa} \ell^B_\alpha e^a_\beta - L_{BAa} e^a_\alpha \ell^B_\beta + e^{-\lambda} N_{ADE} \ell^E_\alpha \ell^D_\beta \ . \quad (A.3)$$

The trace of (A.3) is

$$\ell_A{}^\alpha{}_{|\alpha} = K_A + D_A \lambda \ . \quad (A.4)$$

Calculation of R_{ab}

The Ricci commutation relation (3.52) for the tangential projection of the Ricci tensor is

$$R_{ab} = -e_a \cdot \nabla(e_b{}^\beta{}_{|\beta}) - e_a{}^\alpha{}_{|\beta} e_b{}^\beta{}_{|\alpha} + \nabla_\beta(e_a \cdot \nabla e^\beta_b) \ . \quad (A.5)$$

The first term follows from (A.2)

$$-e_a \cdot \nabla(e_b{}^\beta{}_{|\beta}) = -\lambda_{;ba} - \Gamma^c_{ab} \lambda_{,c} - \partial_a \Gamma^c_{bc} \ . \quad (A.6)$$

In order to calculate the second term, note that

$$L_{ABa}L^{BA}{}_b = \frac{1}{2}(\lambda_{,a}\lambda_{,b} + e^{-2\lambda}\omega_a\omega_b) . \tag{A.7}$$

It then follows from (A.1) that the second term is

$$-e_a{}^\alpha{}_{|\beta}e_b{}^\beta{}_{|\alpha} = -\Gamma^c_{ad}\Gamma^d_{bc} + 2K_A{}^c{}_{(a}(K^A{}_{b)c} + \partial_{b)}s^A_c) - \frac{1}{2}(\lambda_{,a}\lambda_{,b} + e^{-2\lambda}\omega_a\omega_b) . \tag{A.8}$$

The third term requires the use of (A.1) along with (A.3):

$$\begin{aligned}\nabla_\beta(e_a \cdot \nabla e^\beta_b) &= \nabla_\beta(\Gamma^d_{ba}e^\beta_d - e^{-\lambda}K_{Aba}\ell^{A\beta}) \\ &= \partial_d\Gamma^d_{ba} + \Gamma^d_{ba}(\Gamma^c_{cd} + \lambda_{,d}) - e^{-\lambda}K_{Aba}(K^A + D^A\lambda) \\ &\quad -\ell^{A\beta}\partial_\beta(e^{-\lambda}K_{Aba}) .\end{aligned} \tag{A.9}$$

The last term of (A.9) can be simplified by noting that

$$\ell^\beta_A\partial_\beta(e^{-\lambda}K_{Bba}) = D_A(e^{-\lambda}K_{Bba}) + 2e^{-\lambda}K_{Bd(a}\partial_{b)}s^d_A . \tag{A.10}$$

Substituting (A.6), (A.8), (A.9) and (A.10) into (A.5), yields the result for the tangential component of the Ricci tensor, equation (3.53).

Calculation of R_{AB}

The calculation of R_{AB} proceeds in a similar fashion. The Ricci commutation relation is

$$R_{AB} = -\ell_A \cdot\nabla(\ell_B{}^\alpha{}_{|\alpha}) - \nabla_\beta\ell_A{}^\alpha\nabla_\alpha\ell_B{}^\beta + \nabla_\beta(\ell_A \cdot\nabla\ell_B{}^\beta) . \tag{A.11}$$

The first term follows from (A.4),

$$-\ell_A \cdot \nabla(\ell_B{}^\alpha{}_{|\alpha}) = -D_A(K_B + D_B\lambda) . \tag{A.12}$$

In order to compute the second term in (A.11), we note that

$$\begin{aligned}-2L^D{}_{(B}{}^a L_{A)Da} &= -\frac{1}{2}\eta_{AB}(\lambda^{,a}\lambda_{,a} + e^{-2\lambda}\omega^a\omega_a) \\ N_{ADC}N_B{}^{CD} &= D_A\lambda D_B\lambda - \frac{1}{2}\eta_{AB}D_E\lambda D^E\lambda ,\end{aligned}$$

so that the second term is

$$-\nabla_\beta\ell_A{}^\alpha\nabla_\alpha\ell_B{}^\beta = -K_{Aab}K_B{}^{ab} + \frac{1}{2}\eta_{AB}e^\lambda(\lambda^{,a}\lambda_{,a} + e^{-2\lambda}\omega^a\omega_a) - D_A\lambda D_B\lambda + \frac{1}{2}\eta_{AB}D_E\lambda D^E\lambda . \tag{A.13}$$

The third term is

$$\begin{aligned}\nabla_\beta(\ell_A \cdot \nabla \ell_B{}^\beta) &= \nabla_\beta(-e^\lambda L_{ABa} e^a_\beta + N_{BAD}\ell^D_\beta) \\ &= D_E N_{BA}{}^E + N_{BAE}(K^E + D^E\lambda) \\ &\quad -e^\lambda(L_{AB}{}^a{}_{;a} + 2L_{AB}{}^a\lambda_{,a}) \,.\end{aligned} \tag{A.14}$$

The first term of (A.14) can be simplified to

$$D_E N_{BA}{}^E = D_{(A}D_{B)}\lambda - \frac{1}{2}\eta_{AB}D^E D_E\lambda \,, \tag{A.15}$$

so that substitution of (A.12), (A.13) and (A.14) into (A.11) yields:

$$\begin{aligned}R_{AB} &= -D_A K_B - K_{Aab}K_B{}^{ab} + K_{(A}D_{B)}\lambda - D_{[A}D_{B]}\lambda \\ &\quad -\frac{1}{2}\eta_{AB}\left[\left(D^E + K^E\right)D_E\lambda - e^{-\lambda}\omega^a\omega_a + (e^\lambda)^{:a}{}_a\right] \\ &\quad -\frac{1}{2}\epsilon_{AB}(\omega^a{}_{;a} + \omega^a\lambda_{,a})\end{aligned} \tag{A.16}$$

It appears that this result is not symmetric in AB, as is required for the Ricci tensor. That (A.16) does have the correct symmetry, can be seen by applying the commutation relations (3.42) and (3.44), after which the normal component of the Ricci tensor simplifies to equation (3.54).

Calculation of R_{Aa}

We now turn to the calculation of the mixed components of the Ricci tensor. The Ricci commutation relation is

$$R_{Aa} = -e_a \cdot \nabla(\ell_A{}^\alpha{}_{|\alpha}) - \nabla_\beta \ell_A{}^\alpha \nabla_\alpha e_a{}^\beta + \nabla_\beta(e_a \cdot \nabla \ell_A{}^\beta) \,. \tag{A.17}$$

The first term is

$$-e_a \cdot \nabla(\ell_A{}^\alpha{}_{|\alpha}) = -\partial_a(K_A + D_A\lambda) \,. \tag{A.18}$$

The following equalities are useful for calculating the second term:

$$\begin{aligned}L_{BDa}N_A{}^{BD} &= \frac{1}{2}(\lambda_{,a}D_A\lambda + e^{-\lambda}\omega_a\epsilon_{AB}D^B\lambda) \\ 2K^B{}_{ab}L^b_{(BA)} &= \lambda^{,b}K_{Aab} \,.\end{aligned}$$

The second term is

$$-\nabla_\beta\ell_A{}^\alpha\nabla_\alpha e_a{}^\beta = -\Gamma^c_{ab}K_{Ac}{}^b - \partial_a s^b_B L_A{}^B{}_b - \lambda^b K_{Aab} - \frac{1}{2}(\lambda_{,a}D_A + \epsilon_{AB}e^{-\lambda}D^B\lambda) \,. \tag{A.19}$$

The third term is

$$\begin{aligned}\nabla_\beta(e_a\cdot\nabla\ell_A{}^\beta) &= \nabla_\beta(K_{Aad}e^d_\beta + L_{ABa}\ell^B_\beta) \\ &= K^d_{Aa;d} + \Gamma^b_{ad}K_{Ab}{}^d + K^d_{Aa}\lambda_{,d} + \frac{1}{2}\lambda_{,a}(K_A + D_A\lambda) \\ &\quad +\frac{1}{2}e^{-\lambda}\epsilon_{AB}(K^B + D^B\lambda) + \ell^\beta_B\partial_\beta L_A{}^B{}_a\,, \qquad (A.20)\end{aligned}$$

which can be simplified by noting that

$$D_B L_A{}^B{}_a = \ell^\beta_B\partial_\beta L_A{}^B{}_a - \partial_a s^b_B L_A{}^B{}_b\,.$$

The final result, found by substituting (A.18), (A.19) and (A.20) into (A.17) is the mixed component of the Ricci tensor, equation (3.55).

Calculation of R_{AaBb}

The calculation of the Riemann tensor follows a method similar to that for the Ricci tensor. In this appendix we will calculate the component R_{AaBb} as an example. The Ricci commutation relation (3.51) for the component R_{AaBb} is

$$R_{AaBb} = \ell^\alpha_B\left(\ell_A\cdot\nabla(e_a\cdot\nabla e_{b\alpha}) - e_a\cdot\nabla(\ell_A\cdot\nabla e_{b\alpha})\right) - \mathcal{L}_{\ell_A}e^\beta_a(\ell^\alpha_B\nabla_\beta e_{b\alpha})\,. \qquad (A.21)$$

The first term is calculated with the aid of equations (A.1) and (A.3):

$$\begin{aligned}\ell^\alpha_B\ell_A\cdot\nabla(e_a\cdot\nabla e_{b\alpha}) &= \ell^\alpha_B\ell_A\cdot\nabla(\Gamma^c_{ab}e_{c\alpha} - e^{-\lambda}K^D{}_{ab}\ell_{D\alpha}) \\ &= e^\lambda\Gamma^c_{ab}L_{ABd} - K^D{}_{ab}N_{DAB} - D_AK_{Bab} \\ &\quad +D_A\lambda K_{Bab} - 2K_{Bd(a}\partial_{b)}s^d_A\,. \qquad (A.22)\end{aligned}$$

The second term is

$$\begin{aligned}-\ell^\alpha_B e_a\cdot\nabla(\ell_A\cdot\nabla e_{b\alpha}) &= -\ell^\alpha_B e_a\cdot\nabla((K_{Ab}{}^d + \partial_b s^d_A)e_{d\alpha} + L_{ADb}\ell^D{}_\alpha) \\ &= K_{Bad}(K_{Ab}{}^d + \partial_b s^d_A) - e^\lambda L_{ADb}L^D{}_{Ba} - e^\lambda\partial_a L_{ADb} \quad (A.23)\end{aligned}$$

The third term is

$$-(\mathcal{L}_{\ell_A}e^\beta_a)(\ell^\alpha_B\nabla_\beta e_{b\alpha}) = \partial_a s^d_A K_{Bbd}\,. \qquad (A.24)$$

Before substituting (A.22), (A.23) and (A.24) into (A.21) to calculate the Riemann tensor component, we note that

$$L_{ADb}L^D{}_{Ba} + L_{ADb;a} = \frac{1}{4}\eta_{AB}(\lambda_{,a}\lambda_{,b} + 2\lambda_{b;a} + e^{-2\lambda}\omega_a\omega_b)\frac{1}{2}\epsilon_{AB}e^{-\lambda}(-\lambda_{,[a}\omega_{b]} + \omega_{b:a})\,. \qquad (A.25)$$

The result is

$$\begin{aligned} R^{A}{}_{a}{}^{B}{}_{b} &= -D^{(A}K^{B)}{}_{ab} + K^{A}{}_{bd}K^{Bd}{}_{a} + D^{(A}\lambda\, K^{B)}{}_{ab} - \frac{1}{2}\eta^{AB}D_{E}\lambda\, K^{E}{}_{ab} \\ &\quad -\frac{1}{4}\eta^{AB}(e^{-\lambda}\omega_a\omega_b + e^{\lambda}\lambda_{,a}\lambda_{,b} + 2e^{\lambda}\lambda_{;ab}) - \frac{1}{2}\epsilon^{AB}(-\lambda_{,[a}\omega_{b]} + \omega_{[b;a]}) \\ &\quad -D^{[A}K^{B]}{}_{ab} - \frac{1}{2}\epsilon^{AB}\omega_{(b;a)}\,. \end{aligned} \tag{A.26}$$

The last two terms cancel once the commutation relation (3.43) is applied, yielding the result of equation (3.60).

B The contracted Bianchi identities

In this appendix we compute the projections of the contracted Bianchi identities. We first consider the tangential projection,

$$e^{\alpha}_{a}\nabla_{\beta}R^{\beta}{}_{\alpha} - \frac{1}{2}e^{\alpha}_{a}\partial_{\alpha}R = 0\,. \tag{B.27}$$

Note that the Ricci tensor can be decomposed as

$$R^{\alpha\beta} = R^{ab}e^{\alpha}_{a}e^{\beta}_{b} + 2e^{-\lambda}R^{Aa}e^{(\alpha}_{a}\ell^{\beta)}_{A} + e^{-2\lambda}R^{AB}\ell^{\alpha}_{A}\ell^{\beta}_{B}\,, \tag{B.28}$$

and consider the operation of $e^{\alpha}_{a}\nabla_{\beta}$ on each term of (B.28). The first term is

$$e^{\alpha}_{a}\nabla_{\beta}R^{ab}e^{\alpha}_{a}e^{\beta}_{b} = e^{-\lambda}(R^{ab})_{;b}\,, \tag{B.29}$$

which follows from equation (A.1). The second term is

$$e^{\alpha}_{a}\nabla_{\beta}e^{-\lambda}R^{Aa}e^{(\alpha}_{a}\ell^{\beta)}_{A} = e^{-\lambda}(D_{A}R^{Aa} + K_{A}R^{Aa} + K_{Ada}R^{Ad}) \tag{B.30}$$

where we have made use of the definition of the normal Lie derivative (3.15) and equation (A.3). The third term is

$$e^{\alpha}_{a}\nabla_{\beta}e^{-2\lambda}R^{AB}\ell^{\alpha}_{A}\ell^{\beta}_{B} = -\frac{1}{2}e^{-\lambda}R^{A}_{A}\lambda_{,a}\,. \tag{B.31}$$

Meanwhile, the second term of (B.27) is

$$-\frac{1}{2}\partial_{a}R = -\frac{1}{2}e^{-\lambda}(-R^{A}_{A}\lambda_{,a} + \partial_{a}R^{A}_{A}) - \frac{1}{2}\partial_{a}R^{b}_{b}\,. \tag{B.32}$$

Substituting equations (B.29), (B.30), (B.31) and (B.32) into (B.27) results in the tangential projection of the contracted Bianchi identity (3.71).

The normal projection of the contracted Bianchi identity is

$$\ell^\alpha_A \nabla_\beta R^\beta{}_\alpha - \frac{1}{2}\ell^\alpha_A \partial_\alpha R = 0 \,. \tag{B.33}$$

The first term of (B.33) is

$$\ell^\alpha_A \nabla_\beta R^\beta{}_\alpha \;=\; \nabla_\alpha R^\alpha_A - R^{\alpha\beta}\ell_{A\alpha|\beta} \tag{B.34}$$

The first term of (B.34) is

$$\nabla_\beta(R^a_A e^\beta_a + e^{-\lambda} R^B_A \ell^\beta_B) = e^{-\lambda}(e^\lambda R^a_A)_{;a} + D_B R^B_A + K_B R^B_A \,. \tag{B.35}$$

The second term of (B.34) can be calculated by using equations (B.29) and (B.28). The result is

$$-R^{\alpha\beta}\ell_{A\alpha|\beta} = -K_{Aab}R^{ab} - e^{-\lambda}\epsilon_{AB}\omega_a R^{Ba} - \frac{1}{2}e^{-\lambda}D_A\lambda R^B_B \,. \tag{B.36}$$

Substitution of (B.35) and (B.36) into (B.33) yields the normal projection of the contracted Bianchi identity (3.72).

C Calculation of the quantum stress tensor

In this appendix we will calculate the expectation value of the quantum stress tensor for non-gravitational fields in the conformal background metric (6.17). First we will present a detailed calculation of the component $< T^*_{UV} >$. The calculation of $< T^*_{VV} >$ is very similar, so only the leading order terms will be explicitly derived.

The mixed component of the stress tensor depends on the integral (6.43) of the tensor component $A_{UV} = -\frac{4}{r^4}m'(V)$. Since $\partial_U A_{UV} = 0$, the integral $\int H_\lambda(x-x')A_{UV}$ is fairly easy to evaluate. An algebraic manipulation computer program such as Mathematica or Maple can be used to reduce the required integral to the form

$$\begin{aligned}\int H_\lambda(x-x')m'(V) \;=\;& 4\pi\frac{m_0}{\lambda^2\kappa_-}\int_{\rho_0+|u_*|/\lambda}^{\rho_0} |\ln(\kappa_-\lambda\rho)|^{-p}\ln(\rho-\rho_0)\frac{d\rho}{\rho^3} \\ &+\frac{2\pi}{|u_*|}(m(V_0)-m(V_0-|u_*|))\left(1+\frac{|u_*|}{\lambda}\right)\left(1+\ln\frac{|u_*|}{\lambda}\right) \\ &-2\pi(1+\ln|u_*|/\lambda)m'(V_0-|u_*|),\end{aligned} \tag{C.37}$$

where the positive dimensionless variable $\rho = -V/\lambda$. To find the leading order term in this expression, we need to estimate the size of the integral appearing in the first term of (C.37). The integral is of the form

$$I(V_0, r) = \int_{\rho_0+|u_*|/\lambda}^{\rho_0} |\ln(\kappa_-\lambda\rho)|^{-p} \ln(\rho - \rho_0)\frac{d\rho}{\rho^r}, \tag{C.38}$$

where r is a positive integer and $r \geq 1$. We are interested in the asymptotics of this integral when ρ_0 is very small, which corresponds to points near the Cauchy horizon. It is useful to change coordinates to

$$e^s = \rho - \rho_0 \tag{C.39}$$

where the integral (C.38) takes the form

$$\begin{aligned} I(V_0, r) &= -\int_{-\infty}^{\ln|u_*|/\lambda} ds\, s |\ln \kappa_-\lambda(e^s - V_0)|^{-p} e^{\varphi(s)} & \text{(C.40)} \\ \varphi(s) &= s - r\ln(e^s + \rho_0)\,. & \text{(C.41)} \end{aligned}$$

The behaviour of the integrand is dominated by the term $e^{\varphi(s)}$. At the lower integration limit, $\lim_{s\to-\infty} e^{\varphi(s)} \to 0$, while at the upper limit, the exponential term is a finite number. Inspection of the form of the integrand leads one to suspect that $I(V_0, r)$ will diverge, so the value of the integrand at $s = \ln|u_*|$ will not make a significant contribution to the value of the integral. The integral can be evaluated using the method of stationary phase. The phase $\varphi(s)$ is stationary at the point s_0,

$$s_0 = \ln\frac{\rho_0}{r-1} \tag{C.42}$$

which is a maximum since

$$\varphi''(s_0) = -\frac{r-1}{r}\,. \tag{C.43}$$

The result is that $I(V_0, r)$ is approximated by

$$I(V_0, r) \sim -\sqrt{2\pi}\frac{(r-1)^{r-1}}{r^r}\sqrt{\frac{r}{r-1}}\,|\ln|\kappa_-V_0||^{-p}\left(\frac{\lambda}{-V_0}\right)^{r-1}\ln(\frac{-V_0}{\lambda})\,. \tag{C.44}$$

The integral in equation (C.37), is

$$I(V_0, 3) \sim -m'(V_0)\ln(\frac{-V_0}{\lambda})\,. \tag{C.45}$$

The result (C.45) can now be used to estimate the leading order behaviour of the stress tensor. First consider the terms in (C.37). Since $V_0 << 1$, $m'(V_0) >> m(V_0)$, so that terms in the first two lines will dominate over the terms in the third line. Since

$|V_0| < |V_0| + |u_*|$, it follows from the form of the mass function that $m'(V_0 - |u_*|) < m'(V_0)$. The initial surface $u = u_*$ is taken to be finite, so that $|\ln|V_0|| >> \ln|u_*|$. Hence for small V_0,

$$\int H_\lambda(x - x')m'(V) \sim -\alpha_{UV} m'(V_0) \ln(\frac{-V_0}{\lambda}) . \tag{C.46}$$

where the positive constant α_{UV} is

$$\alpha_{UV} = (2\pi)^{3/2} \sqrt{\frac{3}{2} \frac{2^2}{3^3}} \tag{C.47}$$

Substituting this result into Horowitz's equation for the stress tensor (6.12), we find that

$$< T^*_{UV} > \sim -4\alpha_{UV} \hbar \frac{a + 3b}{r_-^4} m'(V_0) \ln(\frac{-V_0}{\lambda}), \tag{C.48}$$

where $a + 3b$ is a fixed positive number for all quantum fields. The sign of this component depends on the size of λ as is discussed in the main body of the text.

A similar calculation yields the component $< T^*_{VV} >$. The leading order contribution to the appropriate integral is

$$\int H_\lambda(x - x')U\, m''(V) \quad \sim \quad -12\pi U_0 m_0 \frac{1}{\kappa_- \lambda^3} I(V_0, 4) . \tag{C.49}$$

Substituting (C.44) into (C.49) we find that

$$\int H_\lambda(x - x')U\, m''(V) \sim U_0 \alpha_{VV} m''(V_0) \ln(\frac{-V_0}{\lambda}) , \tag{C.50}$$

where we have defined

$$\alpha_{VV} = 3(2\pi)^{3/2} \sqrt{\frac{4}{3} \frac{3^3}{4^4}} . \tag{C.51}$$

The quantum stress tensor has the lightlike flux component

$$< T^*_{VV} > \sim 4U_0 \alpha_{VV} \hbar \frac{a + 3b}{r_-^4} m''(V_0) \ln(\frac{-V_0}{\lambda}) . \tag{C.52}$$

Bibliography

[1] E. Poisson and W. Israel, Phys. Rev. **D41**, 1796 (1990).

[2] S.W. Hawking and G.F.R. Ellis, *The large scale structure of space-time*, (Cambridge University Press, Cambridge, 1973).

[3] R. Penrose, Riv. Nuovo Cimento **1**, 252 (1969).

[4] P. Yodzis, H.J. Seifert, and H. Müller Zum Hagen, Commun. Math Phys. **34**. 135 (1973); **37**, 29 (1974).

[5] K.S. Thorne, in *Magic without magic*, Ed. J.R. Klauder, (W.H. Freeman, San Francisco, CA, 1972) p. 231.

[6] S.L. Shapiro and S.A. Teukolsky, in *Directions in General Relativity, Volume 1*. Eds. B.L. Hu, M.P. Ryan Jr. and C.V. Vishveshwara, (Cambridge University Press, Cambridge, UK, 1993) p. 320.

[7] J.R. Oppenheimer and H. Snyder, Phys. Rev. **56** 455 (1939).

[8] R. Penrose, in *Battelle Rencontres*, eds. C.M. De Witt and J.A. Wheeler, (W. A. Benjamin, New York, 1968), p. 222.

[9] W. Israel, Phys. Rev. **164**, 1776 (1967); Commun. Math. Phys. **8**, 245 (1968).

[10] B. Carter, Phys. Rev. Lett. **26**, 331 (1971).

[11] R.H. Price, Phys. Rev **D5**, 2419 (1972); **D5**, 2439 (1972).

[12] A.G. Doroshkevich and I.D. Novikov, Zh. Eksp. Teor. Fiz. **74**, 3 (1978).

[13] E. Poisson and W. Israel, Class. Quant. Grav. **5**, L201 (1988).

[14] W. Hiscock, Phys. Lett. **83A**, 110 (1981).

[15] V.A. Belinskii, E.M. Lifshitz and I.M. Khalatnikov, Adv. Phys. **19**, 525 (1970); C.W. Misner, Phys. Rev. Lett. **22**, 1071 (1969).

[16] A. Ori, Phys. Rev. Lett. **67**, 781 (1991).

[17] G.T. Horowitz, Phys. Rev. **D21**, 1445 (1980).

[18] D.N. Page, Phys. Rev. **D25**, 1499 (1982).

[19] E.W. Leaver, Phys. Rev. **D41**, 2986 (1990).

[20] W. Krivan, P. Laguna and P. Papadopoulos, Phys. Rev. **D54**, 4728 (1996).

[21] C. Gundlach, R.H. Price and J. Pullin, Phys. Rev. **D49**, 883 (1994).

[22] C. Gundlach, R.H. Price and J. Pullin, Phys. Rev. **D49**, 890 (1994).

[23] A. Bonanno, S. Droz, W. Israel and S.M. Morsink, Phys. Rev. **D50**, 7372 (1994).

[24] A. Bonanno, S. Droz, W. Israel and S.M. Morsink, Proc. R. Soc. Lond. **A 450**, 553 (1995).

[25] M. Simpson and R. Penrose, Int. J. Theor. Phys. **7**, 183 (1973).

[26] J.M. McNamara, Proc. R. Soc. Lond. **A 358**, 499 (1978).

[27] S. Chandrasekhar and J.B. Hartle, Proc. R. Soc. Lond. **A284**, 301 (1982).

[28] V. de la Cruz, J.E. Chase and W. Israel, Phys. Rev. Lett. **24**, 423 (1970).

[29] R.A. Isaacson, Phys. Rev. **160**, 1263 (1968); **160**, 1272 (1968).

[30] U. Yurtsever, Class. Quant. Grav. **10**, L17 (1993).

[31] M.L. Gnedin and N.Y. Gnedin, Class. Quant. Grav. **10**, 1083 (1993).

[32] P. R. Brady and J. D. Smith, Phys. Rev. Lett. **75**, 1256 (1995).

[33] Y. Gursel, V.D. Sandberg, I.D. Novikov, and A.A. Starobinsky, Phys. Rev. **D19**, 413 (1979); Y. Gursel, I.D. Novikov, V.D. Sandberg, and A.A. Starobinsky, Phys. Rev. **D20**, 1260 (1979).

[34] R.A. Matzner, V.D. Sandberg and N. Zamorano, Phys. Rev. D **19**, (1979) 2821.

[35] S. Droz, Ph.D. Thesis, (1996).

[36] D.N. Page, in *Black Hole Physics: Proceedings of the NATO Advanced Study Institute on Black Hole Physics*, eds. V. De Sabbata and Z. Zhang, (Kluwer Academic Publishers, Dordrecht, 1992), p. 185.

[37] C. Gundlach and J. Pullin, gr-qc/9606022.

[38] A. Lichnerowicz, J. math. pures et appl. **23**, 37 (1944).

[39] R. Arnowitt , S. Deser and C. W. Misner in *Gravitation: an Introduction to Current Research*, Ed L. Witten (Wiley, New York, 1962), Chap. 7.

[40] P.R. Brady, S. Droz, W. Israel and S.M. Morsink, Class. Quant. Grav. **13**, 2211 (1996).

[41] P.A.M. Dirac, Rev. Mod. Phys. **21**, 392 (1949).

[42] F. Rohrlich, Acta Phys. Austriaca Suppl. **8**, 277 (1971).

[43] J. Kogut and L. Susskind, Phys. Rep. **8**, 75 (1973).

[44] R. Geroch, A. Held and R. Penrose, J. Math. Phys. **14**, 874 (1973).

[45] R. A. d'Inverno and J. Smallwood, Phys. Rev. **D22**, 1233 (1980).

[46] J. Smallwood, J. Math. Phys. **24**, 599 (1983).

[47] D. McManus, J. Gen. Rel. Grav. **24**, 65 (1992).

[48] S.A. Hayward, Class. Quant. Grav. **10**, 779 (1993).

[49] R.A. d'Inverno and J.A.G. Vickers, Class. Quant. Grav. **12**, 753 (1995).

[50] W. Israel, Il Nuovo Cimento **44B** 1 (1966).

[51] W. Israel, Can. J. Phys. **64**, 120 (1986).

[52] C.W. Misner, K.S. Thorne and J.A. Wheeler, *Gravitation*, (Freeman, San Francisco, 1973).

[53] R.K. Sachs, J. Math. Phys. **3**, 908 (1962).

[54] E. Newman and R. Penrose, J. Math. Phys. **3**, 566 (1962).

[55] S.B. Edgar and G. Ludwig, Gen. Rel. Grav. **24**, 1267 (1996); gr-qc/9605006.

[56] H. Bondi, M.G.J. van der Burg and A.W.K. Metzner, Proc. R. Soc. **A 269**, 21 (1962).

[57] S.A. Hayward, Class. Quant. Grav. **10**, 773 (1993).

[58] S.A. Hayward, Ann. Inst. Henri Poincare, **59**, 399 (1993).

[59] C.G. Torre, Class. Quant. Grav. **3**, 773 (1986).

[60] G.T Horowitz and B. Schmidt, Proc. R. Soc. Lond. **A 381**, 215 (1982).

[61] R.M. Wald, *General Relativity*, (University of Chicago Press, Chicago, 1984).

[62] T. Regge and J.A. Wheeler, Phys. Rev. **108**, 1063 (1957).

[63] R.A. Hulse and J.H. Taylor, Ap. J. Lett. **195**, L51 (1975).

[64] D.M. Eardley in *Sources of Gravitational Radiation* edited by L. Smarr, (Cambridge University Press, Cambridge, 1979).

[65] S.W. Hawking, J. Math. Phys. **9**, 598 (1968).

[66] S.A. Hayward, Phys. Rev. **D49**, 831 (1994).

[67] A. Ashtekar and R.O. Hansen, J. Math. Phys. **19**, 1542 (1978).

[68] R. Schoen and S.T. Yau, Commun. Math. Phys. **65**, 45 (1979); Phys. Rev. Lett. **43**, 1457 (1979).

[69] A. Ashtekar and G.T. Horowitz, Phys. Lett. **89A**, 181 (1982).

[70] A. Ashtekar and A. Magnon-Ashtekar, Phys. Rev. Lett. **43**, 181 (1979).

[71] T.A. Morgan and A. Peres, Il Nuovo Cimento **27**, 1266 (1963).

[72] R. Schoen and S.T. Yau, Phys. Rev. Lett. **48**, 369 (1982).

[73] R.K. Sachs, Proc. Roy. Soc. **A 270**, 103 (1962).

[74] G.T. Horowitz and M.J. Perry, Phys. Rev. Lett. **48**, 371 (1982).

[75] C.W. Misner and D.H. Sharp, Phys. Rev. **571** (1964).

[76] S.A. Hayward, Phys. Rev. **D53**, 1938 (1996).

[77] G. Bergqvist, Class. Quant. Grav. **11**, 3013 (1994).

[78] R.J. Epp, gr-qc/9511060.

[79] S.A. Hayward, Class. Quant. Grav. **11**, 3037 (1994).

[80] M. Cure and N.A. Zamorano, in *Relativity, Cosmology, Topological Mass and Supergravity*, C. Aragone ed., (World Scientific, Singapore, 1983).

[81] U. Yurtsever, Phys. Rev. **D40**, 329 (1989).

[82] A. Bonanno, Phys. Rev. **D53**, 7373 (1996).

[83] A. Ori, Phys. Rev. Lett. **68**, 2117 (1992).

[84] P.R. Brady and C.M. Chambers, Phys. Rev. **D51**, 4177 (1995).

[85] A. Ori and E. Flanagan, Phys. Rev. **D53**, R1755 (1996).

[86] R.A. Matzner and N.A. Zamorano, Proc. Roy. Soc. Lond. **A 373**, 223 (1980).

[87] N. Zamorano, Phys. Rev. **D26**, 2564 (1982).

[88] U. Yurtsever, Phys. Rev. **D38**, 1706 (1988).

[89] S.A. Hayward, Class. Quant. Grav. **6**, 1021 (1989).

[90] S. Droz, gr-qc/9608034.

[91] R. Penrose, in *Battelle Rencontres*, eds. C.M. De Witt and J.A. Wheeler, (W. A. Benjamin, New York, 1968), p. 122.

[92] G.A. Vilkovisky, Class. Quant. Grav. **9**, 895 (1992).

[93] L.H. Ford and L. Parker, Phys. Rev. **D17**, 1485 (1978).

[94] V. Frolov and G.A. Vilkovisky, Phys. Lett. **106B**, 307 (1981).

[95] W.G. Anderson, P.R. Brady and R. Camporesi, Class. Quant. Grav. **10**, 497 (1993).

[96] S.A. Fulling, *Aspects of Quantum Field Theory in Curved Space-Time*, (Cambridge University Press, Cambridge, 1989); N.D. Birrel and P.C.W. Davies, *Quantum Fields in Curved Space*, (Cambridge University Press, Cambridge, 1982).

[97] R. Balbinot and P.R. Brady, Class. Quant. Grav. **11**, 1763 (1994).

[98] A. Steif, Phys. Rev. **D49**, 585 (1994).

[99] R. Balbinot and E. Poisson, Phys. Rev. Lett. **70**, 13 (1993).

[100] W.G. Anderson, P.R. Brady, W. Israel, S.M. Morsink, Phys. Rev. Lett. **70**, 1041 (1993).

[101] P. Ramond, *Field Theory: A Modern Primer*, (Addison-Wesley, Redwood City, CA., 1990).

[102] I.D. Novikov and V.P. Frolov, *Physics of Black Holes*, (Kluwer, Dordrecht, 1989).

[103] W.G. Unruh, Phys. Rev. **D14**, 870 (1976).

[104] D.G. Boulware, Phys. Rev. **D13**, 2169 (1976).

[105] S.W. Hawking, Commun. Math. Phys. **43**, 199 (1974).

[106] W.A. Hiscock, Phys. Rev. **D15**, 3054 (1977); Phys. Rev. **D21**, 2057 (1980); N.D. Birrel and P.C.W. Davies, Nature **272**, 35 (1978).

[107] R. Balbinot, P.R. Brady, W. Israel and E. Poisson, Phys. Lett. **A161**, 223 (1991).

[108] M.A. Markov and V.P. Frolov, Teor. Mat. Fiz. **3**, 3 (1970).

[109] R. Herman and W.A. Hiscock, Phys. Rev. **D49**, 3946 (1994).

[110] R.M. Wald, Commun. Math. Phys. **54**, 1 (1977).

[111] R.M. Wald, *Quantum Field Theory in Curved Spacetime and Black Hole Thermodynamics*, (University of Chicago Press, Chicago, 1994).

[112] J.Z. Simon, Phys. Rev. **D41**, 3720 (1990); Phys. Rev. **D43**, 3308 (1991).

[113] P.R. Brady, Ph.D. Thesis, (1994).

[114] A. Campos, R. Martin and E. Verdaguer, Phys. Rev. **D52**, 4319 (1995).

[115] M. Misra, Proc. R.I.A. **69A**, 39 (1970).

[116] R. Penrose, in *Differential Geometry and Relativity*, ed. M. Cahen and M. Flato. (D. Reidel Pub. Co., Dordrecht, Holland, 1976).