Russell on Logical Form

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Abstract

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This thesis discusses the impact of Russell's paradox on Russell's thinking on metaphysics and epistemology. The period covers Russell's work on Logicism in the *Principles of mathematics* in 1900 to the *Theory of Knowledge* manuscript written in 1913. Russell's Logicism was the thesis that mathematics and logic presuppose a common stock of abstract entities. Russell saw his task as identifying these objects and demonstrating their role in the generation of mathematics. Unfortunately, a number of these entities were implicated in paradoxes. This thesis provides an account of these paradoxes and Russell's attempts to avoid them. Russell's method consisted in regimenting the use of expressions designating entities. This method involved the use of 'incomplete symbols', the idea being that ontological complexity was to be reflected in the use of contextually defined expressions. This method would have a profound influence on the evolution of Anglo-American philosophy. Russell's considered view saw the entities required for the logical reconstruction of mathematics organized into a hierarchy. While this solution dealt effectivley with the paradoxes I argue that it comes into conflict with Russell's realism. Russell held that we come into contact with entities existing independently of our mind's apprehension of them. These entities are grasped in a kind of direct contact, and knowledge of this kind is called knowledge by 'acquaintance'. Knowledge by acquaintance is knowledge of individuals. Russell's solution to the paradoxes was to posit different types of things. However, the type of thing an individual is cannot be known by

acquaintance. Knowledge of this kind is knowledge of a truth. I argue that this difficulty undermined the *Theory of Knowledge* manuscript and led Russell to abandon it.

Dedication

This thesis is dedicated to the memory of Louise Holtved.

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Chapter 1

Introduction

It has been said that a philosopher is never refuted, just ignored. Perhaps an even worse fate would be to end up regarded as someone who would have been a better philosopher had he only been Wittgenstein. In some sense this seems to have been Bertrand Russell's lot, and attempts to redress this verdict have often only exacerbated the problem. And this is unfortunate because Russell did more than any of his contemporaries to usher in the profound change in concerns that would be the mark of what has come to be known as the Analytic school. In measuring the extent of Russell's impact on that school, it will be helpful to bring out a variety of themes Russell took to be of paramount importance during the period that runs from his work on Leibniz at the start of the 20th century to the abortive attempt in 1913 to systematize epistemology in the *Theory of Knowledge* manuscript. I hope to make clear to the reader the decisive nature of the paradoxes in shaping Russell's philosophical concerns. And this I propose to do concentrating on Russell's own words.

At the heart of the matter was Russell's desire to isolate what, if anything, we can be said to know for certain. Like others before him, Russell went looking for answers in mathematics. What he found there were entities presupposed by any sort of reasoning whatsoever. The trouble was that access to these objects was frustrated by a poor understanding of the underlying structure of the propositions of mathematics. Russell felt he had inherited a framework that treated all propositions as being of subject-predicate form. The problem with this account of form was that it forced a reading of the entities required by mathematics as being attributes abstracted by the mind and so, in some sense dependent on the mind. The grammatical structure of the subject-predicate logic had as ontic correlate the inherence of attributes in substances. That is, attributes, properties and the like were occult entities having no existence apart from those substances they were dependent on. Philosophers had offered a number of explanations of this ontological dependence. One possibility was that the mind supplied the missing ingredient. Properties and relations were added on by the intellect and without this contribution the real state of things was outside of our understanding. This philosophical attitude is known as *Idealism* and Russell regarded it as the dominant current in British universities during his day. But if knowledge was fashioned by and mediated through the mind this fact, if it is one, yielded doubts as to the objectivity of the known.Knowledge was always qualified by whatever contribution the intellect supplied.

Russell was troubled by this account. However, he was not out to purge nature of the mental. Russell was not a materialist in that sense. Russell was concerned rather to show how the knower could know with confidence. Russell simply took for granted that knowledge could not begin without assuming an independently existing world. Indeed that is why the entirety of the *Principles* is devoted to arguing the case for admitting a very broad number of things as part of our ontology, such as entities subsisting in a world outside of space and time. Russell took it as a given that human beings were engaged with entities exisiting independently of our apprehension of them. That is, the world comes to us in such and such a way: What then, can we come to know about the nature of that world? For British philosophy at the turn of the century, this would prove to be a radical idea indeed. Russell thought that the world and our place in it could be understood through the painstaking analysis of concepts. And this analysis would be subject to Occam's razor as its guiding methodological principle. The idea was not to refrain from adding entities unnecessarily, but to pursue analysis as far as it would go. Russell's methodology was distinctively Cartesian in inspiration. Reason would be our guide in discovery. The end result of this process would yield an understanding of the fundamental ontological constituents presupposed by all others. It was the conviction that reason could deliver this kind of understanding that would be the locus of Russell's dispute with F. H. Bradley.

Although Idealists did not speak with one voice, Russell devoted particular attention to the thought of F. H. Bradley. Bradley certainly disputed the notion that the world could be broken down into discrete, self-contained constituents. In particular, in *Appearence and Reality* he argues that relations cannot be known independently of their terms. His argument there is particularly difficult and obscure, but because of its importance and particularly its impact on Russell the argument will require some discussion. Russell accepted the essential point of Bradley's that propositions were not composed of subjects and predicates. Both philosophers rejected this traditional presupposition for similar reasons. They thought that such an account was incoherent. The traditional view either made subjects empty loci of attributes, in which case it was unclear what properties were the properties *of*, or subjects were made up of properties with no substantial residue left over, in which case a subject's status as a unified entity became problematic. Russell's way out of this impasse consisted in locating unity and order in independently existing relations. Bradley countered that relations could not enjoy this sort of independence. Chapter 1 of this essay lays out the historical background and states the problem in terms Russell would have clearly recognized. I think that this historical detour is essential in avoiding any risk of anachronism.

Despite not finding a knockout blow to Bradley's arguments, Russell pursued the project of reconstructing mathematics, and relations continued to occupy a prominent place in this reconstruction. This project is generally known as logicism and is explored in Chapter 2. Accounts of logicism often claim that the goal was to reduce mathematics to logic. It is not clear that *this* is what Russell (or Frege for that matter) was after. Russell's work reflected the state of scientific enquiry during this time. Mathematics, and the sciences generally, were the focus of efforts at unifying and tying together the heterogeneous fields of chemistry, biology, psychology and physics. The belief had emerged that these various disciplines were at some level understandable in the same terms and that the differences among them were more apparent than real. Russell shared this new outlook and was after an explanatory framework that would unify our understanding of those general principles underlying the structure of reality. The project was to uncover the entities necessarily presupposed in all reasoning whatsoever. The discovery of such entities could only proceed from given truths. With a correct account of logical form in place came the task of isolating a variety of entities and providing them with nominal definitions. As the account unfolds in the *Principles of Mathematics* these entities are endowed with complete autonomy from any mind. Propositions are complex entities with a variety of constituent entities as parts. Any informative proposition was not of subject predicate form and the propositions of mathematics are revealed to be synthetic a priori. So the first problem was to provide a proper account of the logical form of mathematical propositions. In Russell's account, propositions are structured entities consisting of things and relations among these things. Properties and attributes are independently real and not somehow products of the mind. Of particular importance are relations. It was Russell's singular achievement to show the importance of these for an understanding of mathematics and he attributed their neglect as stemming from a traditional prejudice against abstract entities.

What would prove to be a major obstacle to Russell's project was the discovery of a cluster of paradoxes, discussed in Chapter 3, which caused a crisis for philosophers and mathematicians concerned with foundational issues. This chapter does not pretend to be a technical work on the paradoxes. Formal symbolism is used only where it could not be avoided so as not to intrude on my objective of bringing out the profound impact these paradoxes had on Russell's thinking on metaphysics and epistemology. Because Russell thought that the nature and order of things were amenable to a unified theory, he naturally expected to find some common explanation for the paradoxes. Paradoxes do not occur 'out there' in the world, so it seemed that the problem lay in how such things were designated, specified, picked out. While this led Russell to devote considerable attention to questions around syntax and language he remained convinced as ever that reality was ordered through a set of subsistent entities and that all of this occurs independently of minds.

Unfortunately, this remarkable account would slowly unravel under pressure from the paradoxes. These appeared to be generated when certain entities were made into logical subjects. The difficulty then became one of specifying a way to dismiss such entities without violating a fundamental principle of Russell's, which was that logic did not merely govern the laws of thought, but rather, that its reach was *universal*. Eventually, suspicion would settle on certain designatory expressions, and a method was found to eliminate these through contextual definitions in suspect cases. The role syntax played in contributing meaning was not something Russell devoted any attention to. It is true that Russell spoke of particular expressions having meaning only in context and not in isolation. But the contribution these syntactic elements made to content was only via a ground in something either existing or subsisting, some sort of ontological correlate. The fundamental problem posed by the paradoxes was how certain expressions that seemed to pick something out in fact did not. If, for example, 'The round square' or 'the set of all things in the universe' led to antinomies, Russell's conviction that there could be no such entities led him to search for some principled way of treating such expressions as syntactically ill-formed.

Chapter 4 discusses Russell's 'syntactic turn' and the results of this shift taking the form of the theory of definite descriptions and his substitutional theory. Russell's theory of definite descriptions is best understood as the centerpiece of his efforts to deal with the paradoxes. The theory was certainly not part of a broader commitment to a nominalistic pruning of abstract entities from ontology. Russell did not view functions as Frege did, as somehow unsaturated or incomplete entities. Whatever cause for disagreement there may have been between Frege and Russell, it was certainly not over the ontological status of senses as platonic entities. What Frege and Russell did disagree about concerned the treatment of such entities. Frege worked within a Leibnizian understanding of entailment. He spoke of 'content' that, as he put it in metaphorical terms, could be 'carved out' in different ways. This is how 'senses' were understood, as different routes to a referent. However, Frege alsodrew a distinction between 'objects', which were complete or 'saturated', from 'concepts', which were incomplete, or 'unsaturated'. Russell thought that this distinction revealed a deeper commitment to the subject-predicate logic than was desirable. Frege's views were open to the same sort of criticism which Russell himself had directed at Leibniz. If attributes were dependent on subjects, what kept everything said about a subject from being necessarily true of it?

Russell's diagnosis of the paradoxes focused on the way expressions comprehended properties. The solution consisted in requiring that properties attributed to entities be subject to ranges of application. In Russell's account, the world is clearly organized into an ontological hierarchy, with individuals at the base and finely grained properties built from the ground up. In chapter 5, Russell's definitive solution to the paradoxes will be discussed in the light of a propositional paradox which was known to him as early as the *Principles*, but whose decisive importance was not appreciated until later. The paradox in question resulted from propositions being taken as individuals. The problem is that Russell's propositions are subject to very fine-grained criteria of individuation. Identical propositions will contain the same constituents. To avoid such 'intensional' paradoxes Russell was led to distribute properties in a hierarchy determined by the Vicious Circle Principle. The result was the ramified theory of types. Once these foundations had been secured, axioms were provided yielding the extensional objects which are the direct concern of the mathematician. The role played by the axiom of reducibility is particularly difficult to understand and has long been regarded with considerable suspicion. The axiom works to loosen the constraints imposed by the ramification of properties by providing coextensive attributes at the base. This allows coextensive functions to be substitutable for each other in extensional contexts. Most philosophers have followed Quine and Ramsey in claiming that the axiom simply undid the effects of ramification. This charge is certainly mistaken. In intensional contexts coextensive predicates will be prevented from being substituted.

Russell's treatment of properties led to Gödel's accusation that in so doing Russell succumbed to constructivism. For Gödel, Russell's decision to circumvent the paradoxes by restricting the ranges of properties was unjustifiable. Properties, as any realist should insist, just *are*. In chapter 5, I argue that the charge of constructivism leveled at Russell misses its target. Russell would have undoubtedly agreed that properties are given, but he would have insisted that they are given in certain ways. At no point does Russell suggest that elements of his ontology are mind-dependent let alone constructed through acts of understanding. In fact, Russell's difficulties owe more to a realism poorly served by his epistemology than to constructivism. At the heart of the matter is Russell's view of the individuals at the base of the hierarchy as including a variety of intensional entities, in particular, relations in intension. When Russell describes the relations individuals enjoy as external to them he wants to drive home the point that the relations individuals have cannot be read off from the individuals involved. However, if relations are themselves individuals, how can they function in this way?

Russell has a very serious problem here. There is a constant blurring of the distinction in the *Principles* between *things* and *concepts* both of which are interchangeably described as *terms*. Russell recognized that some sort of distinction had to be drawn between them. But he insisted on conferring individuality on relations. This stemmed from his adopting the anti-Idealist view explaining thing-hood in terms of independence from minds. Russell labours to secure an independence from minds for these contents. What emerged from this period was a conviction that expressions used to designate complex entities had to reflect that complexity syntactically. But at no time did this conviction shake Russell's commitment to the existence of mathematical entities as mind independent. Russell also wished to rid philosophy of explanations that seek to make the world a function of mind. But Russell did not want to banish intuition. On the contrary. The universe was made up of discrete entities, some existent. some subsistent. Minds took all these elements and with them fashioned an understanding of the world. Nothing in Russell's account of the understanding found in *Theory of Knowledge* would be out of place in *Principles.* But it is obscure how the status of these entities as things is to be reconciled with the roles they are to play. The explanation that makes sense is that the epistemological account has minds filling in the ontological gaps. That is. minds act on subsistent entities in entertaining propositions. Propositions are then juxtaposed to facts, and truth is yielded when a correspondence holds. But this epistemological account tended to hide the fact that if relations functioned in this way, they could certainly not be the discrete isolated entities Russell insisted they be. Relations are entities not easily accounted for in isolation.

The problem is that Russell included concepts, or intensional entities. among individuals. Individuals are known directly through acquaintance, as all knowledge must rest on entities known in this way. But how can concepts be known in this way? Concepts are different kinds of things than 'individuals'. Individuals can be known in isolation. It is difficult to see how intensional entities are like this. It is difficult to reconcile Russell's realism about concepts and his doctrine of acquaintance. However, it should emerge from this discussion that there are good reasons to assume he was a realist about quite a number of such entities.

The sixth and final chapter deals with the difficulties he had in completing the *Theory* *of Knowledge*. What emerges in *Theory of Knowledge* is a tension between Russell's intensional hierarchy and his thinking about epistemology. In the rush to see Wittgenstein's hand in the demise of this work, not enough attention has been paid to understanding Russell's purposes here. I will try to show that the difficulties Russell encountered owe more to internal pressures. The work clearly marks the end of an era, of the grand ambitions of the *Principles of Mathematics*. It is after this work that Russell abandons the account of mathematics as a synthetic a priori science. But even more importantly was the decisive blow dealt to a certain vision of philosophy as a method for guiding intuition to the discovery of a unified account of the world and what makes that world the way it is. This kind of grand project would not survive Russell's decision to leave it behind.

Chapter 2

The subject predicate logic

2.1 Introduction

Russell regarded the traditional view that every proposition was composed of a subject and a predicate as an obstacle to philosophical understanding. The metaphysics presupposing this view held that particular substances provided support for attributes. Russell would argue that this entire framework was mistaken. On the one hand the notion of substance fulfilled no useful function, while on the other the ontological status of attributes and their objects remained shrouded in obscurity. Before one can assess Russell's conclusion, it would be helpful to begin by briefly examining the origins of the traditional view and how it evolved into the doctrine known as Idealism. For Russell was particularly concerned to reject a certain philosophical cast of mind that made knowledge dependent in any essential way on the mind of the knower.

2.2 The Subject-Predicate Proposition

Russell begins by rejecting the traditional analysis of the proposition as composed of a subject and a predicate (Russell 1903, page 16). One way of understanding the traditional view is as an answer to what could be described as Moore's Question. G. E. Moore (Moore

1899) once remarked that philosophers have traditionally taken the diversity of things in the world as a given, and framed for philosophy the 'problem' of explaining how common threads can be woven into the fabric of nature. That is, given particulars, how were general features of the world to be accounted for? This dualism between particulars on the one hand, and the more general features of the world was reflected in differences in the kind of knowledge we could allegedly attain of such things. Knowledge of particular things was, by its very nature, infused with subjectivity. It was generally assumed that our knowledge of particulars could not be readily transmitted to others because such knowledge was, by its very nature, confined to individual knowers. Yet the existence of a shared understanding seemed to demand, by way of explanation, some sort of content in the world which transcended the confines of particular experiences. So would begin the peculiarly philosophical enterprise of discovering what features of the world, if any, were untainted by subjectivity. Among such features of the world was certain knowledge to be found.

One presupposition shared by those who undertook this enterprise was that picking out some thing and then characterizing it in a certain way is an activity fundamental to human thought. This activity was expressed through the use of propositions of subject-predicate form. The grammatical distinction between subject and predicate was virtually taken as a model for a dualistic ontology consisting of universals and particulars. Anything said about the world, i.e., by means of a proposition, took the form of a subject (a particular) qualified by some predicate (a universal). According to this tradition each proposition was the attribution of a property to a subject. Not surprisingly, one of the perennial problems in philosophy has been to account for how a collection of properties can be attached to a single subject. Identifying each property seems to leave us with just a list. The problem is then to show how such a collection of properties could amount to something more, to count as a thing. Adding more members to the list would not do the job. The problem (if it is one) is not strictly tied to the more familiar issue of the relations between universals and particulars. Grains of sand can make up a single pile. How can a collection of things yet be a single thing?

The historical origins of this philosophical problem are undoubtedly remote, but appear with unmistakable clarity in the works of Plato and Aristotle. For Plato, what the senses deliver to us are particular appearances of *Ideas* (or Universals). *Ideas* alone were real, as opposed to the imperfect copies of them (particular things) which human beings have contact with in ordinary experience. For Plato, thought and talk about the world proceeded by establishing relations among, and a hierarchical ordering of, these *Ideas*. Plato's analysis in the *Sophist* was that a sentence constituting a statement was one made up of two parts: an onema (noun) and a rhema (or verb).

However, the nature of the copula to effect the uniting of the subject and predicate in the proposition, has been a topic of some dispute. One possibility is to regard nouns and verbs as combining on their own without the need of any other sort of device to make the link. According to this view, Plato's essential insight is that the *onema* and *rhema* are heterogeneous entities, with the *rhema* possessing an essentially predicative nature (Geach 1972, page 47). Here, the copula is merely a word connecting names standing for the Forms, but which is not itself a name for a Form. On this reading Plato's copula is ontologically inert. It acts, not as a name, but rather to register the ontological 'communion', of one Form with another. According to another view, the copula must be more than mere marks on paper. It serves, rather, to indicate the relation the subject bears to the predicate in an attributive statement. Presumably, the thought here is that whatever effects the communion of Forms must itself be something.

Aristotle incorporates and expands on this treatment of a sentence as the combination of a noun and a verb in the *Categories* and in *De Interpretatione*[1]. There, whatever is picked out by the predicate expression can enter into two fundamental and distinct relations with what is picked out by the subject expression. On the one hand the predicate

[1] *Categories* I a 16ff and *De Interpretatione* 16 a 1-17.

expression may or may not say something of the subject. On the other, what is picked out by the predicate expression may or may not be inherent in the subject. Aristotle then goes on to distinguish those beings which are neither said of nor inherent in a subject[2] . Such a being is said to be a 'primary substance'. Primary substances are the most basic constituents. Aristotle seems to have in mind here a concrete individual, persisting thing - such as a rock or a horse. Primary substances possessed a unity which allowed them to persist identically through qualitative change.

Individual properties, such as this patch of white, while they are not said of a subject, are inherent in the subject[3]. Finally, general properties, such as the attribute 'Whiteness', are both said of, and inherent in, a subject. Aristotle implies that the existence of items in these last two categories somehow depends on the existence of substances, but not vice versa - the dependency is one sided or asymmetrical. Thus qualities like whiteness and circularity only exist because there are individual substances that are white or circular. This ontological asymmetry is reflected in the grammatical fact that qualities are predicated of substances, but primary substances themselves are not predicated of anything else.

The extent to which Aristotle was intent on reversing Plato's ontological order by placing particulars at the foundation and endowing universals with a derived existence in the particulars that instantiate them, has been a long standing matter of dispute. It is certainly clear that what leads Aristotle to call individuals the primary substances is the suggestion that concrete individuals constitute in some sense the ground of all real existence. However, there is certainly a sense in which the secondary substances predicated, or said of other substances, exist over and above the individuals that instantiate them, much as Plato's *Ideas*. This encourages a certain realist attitude towards universals (universals both *in re* and *ante rem*), in that the species *man* is a separate entity, for

[2] *Categories*, 5, 2^a12-2^b26.
[3] *Categories*, 5, 2^a28-33.

example, and not inherent in any individual man. Predication in this sense is said to be *essential* predication. Attributes, on the other hand, are distinguished by not being detachable, separable entities: they depend for their being on the things in which they are said to inhere. On this conceptualist view of universals (*Universalia post rem*) color, say, just exists on a body's surface. To claim, as Aristotle does, that *attributes* are inherent, is to deny the Platonistic construal of them as separate detachable entities. Thus, the abstraction of secondary substances is different from the abstraction of simple attributes. Predicating, or saying something of an inherent abstract is to mark its entityhood as dependent on its being discerned by an intellect; such entities are fundamentally conceptual. Predication of this kind is thus described as *accidental*. Secondary substances possess, on the other hand, the property of being separate and independent regardless of their apprehension by any intellect.

The treatment of universals as substances has been described by P. T. Geach as the greatest calamity to afflict man since The Fall (Geach 1972, page 47) . If a predicate is only attributable to a self-subsistent subject, and only particulars enjoy independent existence, predication is possible of particulars alone. This view of predication entails that only particular substances can validly be subjects of predication. Now, it is clear that Aristotle allows predications with universal terms as subjects[4]. In Aristotle's account, a simple statement is made up of a pair of terms combined with a third expression, the copula, whose role is to link the two terms to form a statement. Thus, a proposition had the form: *subject-copula-predicate*. And it is clear that from the modern point of view this marks a difference. It is a fundamental tenet of the contemporary, analytic, view, that universals, or concepts, are 'incomplete', or 'essentially predicative'[5] . Any phrase predicating something of them can be transcribed into a canonical notation where they are eliminated as logical subjects. According to Geach, the problems begin when Aristotle

[4]Post Anal II, 13, 97b 15-23

[5]Frege, Translations from the philosophical writings of Gottlob Frege. (Ed P. T. Geach and M. Black, Oxford University Press) pp.48-51

relinquishes certain entities as being essentially predicative and the concomitant blurring of the distinction between the saturated and unsaturated constituents of a proposition.

What is important for our immediate purposes. however, is the Aristotelian doctrine of *Form*. These were secondary substances and as such, autonomous entities. According to Aristotle's physics, Primary substance was the basic matter making up the world. This matter was extended throughout space and was formless. Individual things required contact with *Forms*. That is, these *Forms* acted as stamps that carved out individual entities from a homogeneous bath of prime matter. As such it was a kind of unifying principle with each *Form* providing an *entelechy*, or principle of individuation. This was Aristotle's answer to the problem of the one and the many. This problem of explaining unity amidst diversity appears throughout the history of philosophy and it will be argued through the course of this essay that it was *the* problem Russell thought he had to solve in order to explain the nature of mathematics. By the time Russell encountered this problem in the work of F. H. Bradley it had undergone a number of transformations. and it will be worth our while to follow its philosophical trajectory, as it were, up to the latter half of the 19^{th} century.

2.3 The Moderns

Philosophers of the 17^{th} century inherited an account of propositions according to which they were of subject predicate form and that in some way this grammatical distinction reflected the ontological make up of the world. 17^{th} century philosophy was of course profoundly affected by surrounding scientific developments. Descartes looked for mechanistic explanations of nature, arguing for a plenum filled with matter. Descartes was opposed to action at a distance and he felt that empty space or the void, required explanations of an occult nature to account for the motion of bodies. Rather, according to contact mechanics, things moved through direct physical contact. Such mechanistic explanations conflicted with Galileo's law of inertia, the latter requiring empty space or a vacuum to account for motion. The mechanical philosophy encouraged the view that necessary connections in nature could be 'read off' via this contact between entities. Belief in a vacuum, on the other hand, led to mathematical descriptions of phenomena such as gravitation, where agnosticism towards an underlying reality was more natural. So realists about the plenum drew on the notion of contact between bodies to explain unity amidst diversity. On the other hand scepticism about the plenum led to an emphasis on mathematical descriptions of phenomena and to the opinion that the source of this unity, if there were one, was opaque to human understanding.

Leibniz was averse to the idea of a void, or gaps, occurring in nature. And this aversion was reflected in his theory about the structure of propositions, and how such propositions could say things about the world. The exact nature of particulars and universals was a matter of contention but the debate remained within an essentially Aristotelian framework until Leibniz. Leibniz also held that every proposition was composed of a subject and a predicate, and that this grammatical distinction corresponded to an ontological one between substance and attribute. But Leibniz also insisted that certain syntactic constituents (the syncategoremata) were ontologically inert and a sentence was formed when the subject and predicate 'fit' together, securing sentential unity without the need for any additional ingredient. This marked a break with the classical tradition, because, for the Aristotelian, terms can be connected only by means of an expression designed for that purpose, some sort of term functor.

Just as predicates were 'in' subjects, attributes were said to 'inhere in' substances. Here substance (or subject) and attribute (predicate) are of fundamentally different categories, with particulars providing substantial support for attributes. A subject stood for a substance and was never a predicate. This restriction denied substantial value to predicates, be they qualities represented grammatically by adjectives or relations indicated by verbs. In effect, Leibniz rejected the Aristotelian notion of predication as involving entities (forms) existing *out of* a subject yet *said of* that subject. Rather, by *Predicatum inest subjecto* Leibniz means that predication is of accidents. Ontologically, predicates are abstract entities dependent on their subjects, and not enjoying an existence independent of any mind:

> The ratio or proportion between two lines L and M may be conceived in three several ways; as a ratio of the greater L to the lesser M; as a ratio of the lesser M to the greater L; and lastly, as something abstracted from both, that is, as the ratio between L and M, without considering which is the antecedent, or which the consequent; which the subject and which the object... In the first way of considering them, L the greater is the subject, in the second, M the lesser is the subject of the accident which philosophers call *relation* or *ratio*. But which of them will be the subject, in the third way of considering them? It cannot be said that both of them, L and M together are the subject of the accident; for if so, we should have an accident in two subjects, with one leg in one, and the other in the other; which is contrary to the notion of accidents. Therefore we must say that this relation, in this third way of considering it, is indeed *out of* the subjects; but being neither a substance nor an accident, it must be a mere ideal thing, the consideration of which is nevertheless useful. (Russell 1900, pages 12-13)

The difficulty here, according to Leibniz, is that this notion of a property existing independently of the subjects possessing it goes against the 'law of accidents'. An accident was a single determinate thing ontologically dependent on the subject containing it. So if one wished to attribute some property to an aggregate, such as its number, one would have to use an entity that could be shared by more than one subject. And this would imply that it could somehow detach itself and move about from entity to entity, and attributes are, by definition, not endowed with this sort of autonomy.

The gap between subject and predicate that Aristotle had bridged in the *Analytics* with the logical copula, was transcended in Leibniz's account by making Universals the mere modifications of substances. Thus all of a subject's predicates are contained within it. Instead of an unconnected patchwork of sensations, things do indeed hang together, as it were, because they are designed to do so. All of a substance's attributes unfold from it in a necessary fashion. Thus, Leibniz eventually replaced the term 'substance' with the term 'monad' (meaning a unit). Each monad reflected an external plurality, and this resolved the problem of the one and the many.

Two possibilities would emerge. One was that there is no underlying unity to a collection, just the individuals that are in that collection. Here, attributing a number to it reduces to establishing some sort of correspondence between it and other collections. This is Hume's principle, which says that collections have the same number if their members can be put in a one to one correspondence. This principle is congenial to the empiricist in that no appeal is made to any sort of unitary 'thisness' underlying the collection. Such a correspondence could be established amongst collections presumably without construing them as unitary things. For the Rationalist, on the other hand, such an attribution can be made of a plurality only by presupposing such an underlying unity, which was presumably provided by the mind, and as such, viewed as ideal. This was Leibniz' position: a shared property was an ideal being constituted through acts of perception, be they of God, or by finite minds:

> Aggregates, not having unity, are nothing but phenomena, for except the component monads, the rest (the unity if the aggregate, I suppose) is added by perception alone... And so these beings by aggregation have no other complete unity but that which is mental; and consequently their unity is also in some way mental or phenomenal, like that of the rainbow.(Russell 1900, pages 115-116)

Russell's thesis about the nature of mathematics is best understood as an attempt to transcend the unsatisfactory nature of these accounts. Against the empiricists Russell felt that establishing relations between aggregates required that the latter be taken as logical subjects, as entities in their own right. But against the Rationalists Russell felt that this unitary content was more than some figment of the mathematician's imagination. Indeed, if it were so, all objective grounding for our understanding of generality would be lost. Russell's problem with Rationalism was not its emphasis on an intuitive inner light of reason, as much as the thought that without contact with objects (both concrete and abstract) external to minds, reason lacked the resources to navigate the world in a disciplined way. And it was this lack of a 'robust sense of reality' that led rationalists to offer an account of necessity that Russell, and many others, would find wanting. So it is worth our while to pause and explore this theory as it appears in Leibniz.

As a result of his account of substance and attribute, and of subject and predicate, Leibniz proposed a thesis about identity maintaining that a subject was uniquely determined by its set of attributes. Logically, a definition such as $3 = 2+1$, asserts the identity between the definiendum and the definiens, and the function of such an analysis is to decompose a complete concept into its constituent concepts. Ontologically, such an analysis suggests the identity of a substance with its attributes. By identifying predication and inherence, all a subject's predicates became contained within it and universals made the mere modification of substance. For Leibniz, (with the exception of tautologies such as A is A), the subject is a complex concept, a collection of attributes unified by an active principle of unity, and the predicate is a part of this collection. In an analytic proposition (Leibniz did not use this term, but rather, spoke of truths of reason), the notion of the subject contains all its predicates. So all propositions were analytic and necessary, following from the Law of Contradiction.[6] However, particular existents

[6]This struck Russell as unreasonable, and so in his book on Leibniz he attacked Leibniz's theory of necessary propositions. In his critique of Leibniz, Russell argues that many of Leibniz's analytic propositions are not really analytic, or that they are tautologies, and thus are not really propositions. Russell's argument consists in showing that these analytic judgements presuppose judgements which are synthetic. Now according to Leibniz, the subject concept has to be possible, i.e., the attributes out of which it is composed have to be logically compatible. The trouble is that such compatibility is not itself analytic:

did not exist of necessity but only by the will of God,should He determine some reason for this existance. The existence of a particular was, to this extent, contingent. Thus, while propositions about particulars were of subject predicate form with each particular containing its attributes necessarily, they remained contingent propositions to the extent that existence could not be deduced from a particular being's attributes. In other words, once a particular's attributes are given, they can be deduced from each other necessarily, but the attributes they have are not theirs of necessity.

Leibniz maintained every substance contained its 'complete' concept, or every concept that could be truly predicated of that substance (though, of course, there being infinitely many such true predications, no finite human mind can be expected to grasp such a complete notion in its entirety). A subject contained all its past, present and future attributes. All attributes of a subject were already contained in the notion of the subject but in a form 'confusing' to finite minds. This made all of a thing's properties essential to its being what it is (and a fortiori, that all its relations are internal to it). Idealists and monists held that the connections between each of a thing's properties, (including its relational properties) was so intimate that the change of a single property rendered the thing no longer what it was. In this way, any extrinsic characterization was in fact an intrinsic affection of the subject. For example, if the relation of A to B is deducible from A's complete notion, the relation to B must be a modification of its internal nature. However if a particular is dissolved into a congeries of properties these properties also have intrinsic natures to be discovered by inquiry and such an inquiry would, in principle, discover relations of entailment between all possible properties of all possible particulars. The obvious conclusion is that a particular stands in a necessary

Thus to return to arithmetic, even if 2+1 be indeed the *meaning* of 3, still the proposition that 2+1 is possible is necessarily synthetic. A possible idea cannot, in the last analysis, be *merely* an idea which is not contradictory; for the proposition itself must always be deduced from synthetic propositions (Russell 1900, page21).

So Leibniz's necessary truths cannot be 'wholly' analytic, in that they presuppose that certain synthetic propositions are available to determine which of a subject's possible constituents are compatible.

relation to all its properties.

Leibniz also held that analysis and synthesis simply reflected the two directions of movement along the same deductive chain. All proof depended on the principle of identity. A proposition expressed 'identity' if the predicate was either explicitly identical with the subject, or somehow otherwise included in it. Similarly, Leibniz treated hypotheticals, or compound sentences, as having categorial form (Leibniz 1966). Categoricals had the form: A contains B, where A is the subject and B is the predicate. So relations between propositions, including entailment, will share in this form. This treatment of compound sentences as categoricals required that a proposition could be conceived as a term: "That a proposition follows from a proposition is simply that a consequent is contained in an antecedent, as a term in a term. By this method we reduce inferences to propositions and propositions to terms" (Leibniz 1966, page 87). This treatment of the relation between antecedent and consequent in a conditional as one of entailment would make all such relations analytic and necessary. And these connections between attributes remain knowable a priori.

The most crucial objection to this account would evolve from the common sense Aristotelian view that there are both particulars and properties of particulars, and that each particular stands in a necessary relation to some of its properties and a contingent relation to others. There is clearly a sense in which given the laws of nature and the past history of the universe, it is contingent that a given particular be located at a given point in space at a given time. Some sort of distinction had to be drawn between physical and logical necessity. But how was one to detach properties from particulars in a way that made sense of this distinction between the contingent and the necessary?

One line of thought emerged from Locke. The problem was cast within the framework he inherited of predication as the inherence of accidents in a subject. Now, the assumption that a subject is simply defined by its predicates made all propositions about a subject analytic, and necessary. Some way had to be found to sever this tight connection between the two and allow for those propositions that were known only contingently. Locke would look to the Aristotelian conception of substance as a subject of predication which is not itself predicable of anything. Thus a particular stripped of all its properties was granted a sort of ontological status, the details of which Locke 'knew not what'. The notion of a bare particular seems to have its origins in Locke's views on substance. Particulars were reduced to an unknowable substratum.

Particular things were known only through contact with their secondary qualities. Contact with objects was mediated through such qualities impinging on our sense organs. The intellect then worked on these ideas, bringing them together or breaking them down into constituent units, as needs warranted. Because contact with the sources of these ideas was mediated, knowledge of them was at best second hand. Locke did not wish to deny that necessary connections held among the constitutive properties of objects and between objects themselves. Rather, Locke was concerned to deny that such connections could be transparent to human reason.

For Locke, we must find a ground for what we know in what we experience, and universals subsist in our minds as what he called concepts. Universals were thought of as what remains after what is individual to diverse particulars has been abstracted away by the mind. Now, the line of thought leaving particulars bare suggested that there are no analytic propositions ascribing qualities to particulars. That is, none of a thing's phenomenal properties are essential to it (and thus, a fortiori that no relations are internal to it) and that a distinction had to be made between the thing itself and a description of it. The conclusion naturally followed that "logical necessity" only characterized relationships between universals, and the notion of logically necessary relations between particulars and their properties was dropped.

Thus while Locke regarded the substratum as an anchor binding properties to form a unity, Berkeley and Hume would soon dispense with it, leaving the sense impressions from which we abstracted 'ideas' to float freely about as mere flotsam on a sea of habit and intentions. There emerges a characteristically empiricist account of objects, where discrete isolated sense impressions are combined to form what we encounter as a unified whole. The result was a skepticism that denied knowledge of anything beyond our sense impressions. And it would be a problem to explain how disparate and isolated sense impressions could be combined to form an external world with the continuity and structure it had. Phenomenal particulars must be 'synthesised' in some way, and as Hume had demonstrated, all we have at the phenomenal level is the constant conjunction of ideas. While the implausibility of the notion of an unknowable substratum drove the empiricists to account for the world in terms of ideas as the correlates of sense impressions, these were discrete isolated items whose interconnections could not be known. So with Hume we are left with items of sensation as the only things we can be said to know for certain. And the only necessary connections between these items are those supplied by our habits.

What emerged was a theory about the informative content of propositions. An informative proposition was restricted to one involving the association of ideas and was synthetic. Empiricists were not as opposed to the notion that certain propositions could only be justified by an appeal to something lying beyond the confines of experience, as they were to the idea that propositions of this type could extend our knowledge. One of the central tenets of empiricism is that any proposition whose truth does not depend on experience is analytic, or true by convention or definition. It is this rejection of the synthetic a priori which is the sine qua non of modern empiricism.[7] Leibniz had tried to transcend the dualistic ontology through a monism that made properties the form of a substance. This in turn made a thing's properties necessary to it. Hume and Berkeley would insist that such necessity was not an item of sensation but this response left us

[7]The Logical Positivists had taken Wittgenstein in the *Tractatus* as having established the tautological nature of the propositions of logic. Furthermore, when Russell and Whitehead 'reduced' mathematics to logic in the *PM*, they therefore demonstrated that mathematical propositions were analytic, or tautologies. Russell himself would come to adopt this thesis. However, from the *Principles of Mathematics* to his encounters with Wittgenstein, Russell would insist on the synthetic a priori character of mathematical knowledge, and at the same time, the synthetic character of the propositions of logic.

with no account of why our world came with the structure it had.

2.4 The Emergence of Idealism

One presupposition shared by the Rationalists and the Empiricists was the subject predicate tie as one of inherence. This common assumption led, interestingly enough, to the conclusion that abstract particulars were the work of the mind. For the Rationalist, however, the structure given to our experience by the mind was knowable in principle. The Empiricists, on the other hand, were drawn to deny that the underlying principle of unity could ever be known. Kant and his Idealist followers felt they had an answer to this skepticism: unity is the outcome of an intuitive act whereby these elements are combined according to the Categories. Intuitions must be brought under concepts and there is no experience without judgment. Kant's idealist successors took a proposition as constituted by an act of judgement. Reality became a kind of function of the mind, where the mind was seen as a sort of stamp acting to shape and structure the phenomenal stream. Objects did not contain blueprints of their representations, taken up by passive minds, rather, our minds are what structured our experiences. The view became that what is empirically 'given' does not exhaust all that can be known about the world, and as in Locke, a distinction between the necessary and the contingent is bought at the expense of having to posit an unknowable source of our sensible intuition.

According to the Idealists only *ideas* had what was called 'meaning' and in each judgement there was some subject of the judgement, which was not an idea, and had no meaning. Expressions used to directly pick out some portion of reality did nothing to advance our understanding. Rather, knowledge came with the characterization of reality through concepts. Understanding of particular bits of the world was incomplete, with knowledge increasing incrementally as particular subjects were brought under concepts. A proposition attributed a meaning, a property, an idea, to a designated reality, making propositions dependent for their being on an act of judgement. In this conception a presented object depended on the form our intuition took in the apprehension of it as well as through the judgement of the judging subject.

For Bradley, however, judgements were not just the association of ideas. When we make a judgment, we are in some sense bringing reality under concepts. In judgment, while a predicate is asserted of a subject, subjects and predicates are very different kinds of things. That part of reality brought under an idea, Bradley calls an individual. However, there is no such thing as a pinpoint particular, which is somehow the subject of the judgment. The inner core of particularity, that which marks individuals out as distinct from others, *this*, we cannot have direct unmediated contact with. In this sense, for Bradley, particulars do not exist. An individual was regarded as a kind of universal and called a concrete universal (Bradley 1883). Individuals could only possess 'identity in difference'. Bradley's point was that the identification of particular things was only achieved through some sort of conceptual mediation. In support of this, Bradley pointed to asymmetrical relations that implied 'diversity of their terms'. To say that 'This is to the left of that' immediately yields 'this' and 'that' as diverse meanings.

Bradley would make this the centerpiece of a complex, and at times obscure, argument against the notion of identity: Assume that 'A is A' is of subject predicate form. Bradley's contention is that either such a statement is mistaken, in that *assigning* a predicate presumes that the subject and the predicate are already given as *distinct* entities. Or that such a statement is trivial, for the predicate is not marking out anything distinct about the subject, and one is claiming 'mere' identity (there would be no distinct feature by which the judger could determine in what respect they were so related). Thus, preserving the principle of identity seems to require giving up the subject-predicate form.

The argument is spelled out in some detail in *Appearance and Reality*. A thing is somehow the locus for a plurality of attributes. Qualities and properties are taken to be abstract particulars. Thus when Bradley says 'This lump of sugar is white and hard and sweet', (Bradley 1893, page 16) there is a single thing (the lump of sugar) qualified by a number of adjectives which in turn stand for individuals, and are said to 'inhere' in the lump of sugar. Different qualities are taken to be numerically distinct, and in this sense are many. Without a substantial medium to anchor attributes, predication becomes the attribution of identity. In a statement such as 'This lump of sugar is white, and hard, and sweet', the function of the copula is to mark the qualities said to 'inhere in' the subject. The subject is said to contain such qualities, which are abstract particulars. What then remains to be explained is how the subject, as a single unitary thing, can yet be identical with its properties, which are many. Bradley marshals a series of arguments designed to show that such identity is illusory, an 'identity' in appearance only.

The sentence 'This lump of sugar is white, and hard, and sweet' asserts that a single thing, i.e., this lump of sugar, is identical with three individual things; the qualities White, Hard, and Sweet. But what did the inherence tie consist in? One such possibility was to construe the identity as involving the individual thing and each of its qualities, taken as individuals. Yet the thing cannot be any one of its accidents, for these are, after all, distinct individuals. Another possibility is that the thing is just its qualities taken all together, or as many. The trouble here is that such unity is presupposed by the attribution of a property, and so cannot be conferred upon it by such a property. A lump of sugar is not identical with the sum of its attributes because it is one and not merely many. It is, after all, a substance, or a unitary thing, not merely a collection of such things. It has to be the sort of thing possessing enough unity that it makes sense to call it the subject of a predication. Under the first possibility, then, to claim that the lump of sugar is identical to any one of its attributes is false. In the second case, the attribution is question-begging in that the very unity being defined is already presupposed in the definition.

Having eliminated these two possibilities, Bradley then considers a third option, and this is the possibility that the unity of a collection of qualities arises from their being combined in a certain way. Thus a single thing is identical to its qualities in some relation:

> But it is our emphasis, perhaps, on the aspect of unity which has caused this confusion. Sugar is, of course, not the mere plurality of its different adjectives; but why should it be more than its properties in relation? When 'white', 'hard', 'sweet', and the rest coexist in a certain way, that is surely the secret of the thing. The qualities are, and are in a relation. (Bradley 1893, page 17)

Bradley's suggestion is that no sense can be made of a *relation*, construed as an independent entity binding the members of a collection into a unity, and his attack will center on what such a relation could be said to unify. Bradley urges us to consider, then, the possibility that the relation is with each quality taken individually. Thus in establishing such a tie one is establishing the identity between each quality and the relation. Yet this attempt at turning the one into the many is simply incoherent, for one is presupposing that each quality is numerically distinct. But the relation apparently yields the identity only of each quality (in the sense that each quality is being combined into a single thing), which is absurd. So the notion that each quality could be identical with some relation is false. That is, the falsity arises because one is attributing numerical identity to what are, supposedly, numerically diverse things. The other possibility, then, is to take the relation not as with each abstract particular taken singly, but, rather, a relation involving abstract particulars in the collection as a plurality. Here, the problem is that the relation is said to be identical with a plurality. But the plurality has no unity, it is a many. As a plurality, this collection cannot be identical with the relation, which is a single thing.

The problem Bradley begins with seems to arise when predication is treated as inherence. If one construes the unity as being inherent in each quality taken individually or in the many, the claim is false. On the other hand, construing the unity as a single thing inherent in the qualities of the thing taken as a plurality is question begging in that the very unity is presupposed in identifying the plurality of qualities as qualities of the same thing. Bradley's argument is then that should one attempt to locate that unity in a relation (some configuration of the qualities), it will similarly be impossible to state what the relation could be a *relation of.* Bradley would contend that to locate the source of unity in an independently real entity would lead to a regress:

> Let us abstain from making the relation an attribute of the related, and let us make it more or less independent... the problem is not solved by taking relations as independently real. For, if so, the qualities and their relation fall entirely apart, and we have said nothing. Or we have to make a new relation between the old relation and the terms; which, when it is made, does not help us. It either demands a new relation, and so on without end, or it leaves us where we were, entangled in difficulties. (Bradley 1893, pages 17-18)

The relation cannot be identical with each quality individually. Each quality is a distinct entity, and their combination through a relation does not collapse them into one, as it were: this is just false. So is the tie the relation could be thought to have with the qualities taken as many. For the relation is one, and the qualities are many. The only other possibility seems to be to take the relation as identical to whatever unitary thing underlies the plurality of qualities, and this is vacuous. The idea here is that a complex is a single thing when its parts lie in a relation that makes them one. Thus the relation must be identical to its terms 'as many'. But this is impossible since the terms are many, not one. Another attempt to locate a candidate, to isolate some abstract particular responsible for the unifying relation will only presuppose that very entity.

Bradley's criticism of the traditional distinction between subject and predicate amounts to a dilemma, which shows that any attempt to explain this unity amidst diversity is unintelligible. On the one hand a thing is not the substantial residue left behind after its attributes have been stripped away. On the other hand, treating a thing as the sum of its properties leads to absurdity or a regress. What the regress argument purports to show is that whatever it is that bestows order and structure on a collection of properties, 'it' cannot be a property like the others. And whatever that thing is which properties belong to, to 'that', no one can have direct access. Rather, it seems something reached as a kind of limit, something always slipping from our conceptual grasp, always just beyond the borders of our comprehension, access to which is made by ever greater degrees of generality. Bradley clearly thought that part of the problem lay in the subject predicate logic's notion of the inherence tie. The moral the idealists drew was that the principle of identity had to go. They retained the subject predicate form with the proviso that knowledge of a subject was always incomplete and attained only by enveloping the subject in a web of ideas.

2.5 Russell and Moore's Critique of Idealism

Russell agreed that the inherence model of the subject predicate relation is incoherent but he would draw very different conclusions. Russell and Moore shared with Bradley the assumption that there is a tension between the logic of substance presupposed by the subject predicate form, and the principle of identity. By traditional lights, the subject of predication, i.e. substance, is devoid of meaning. Thus, assuming the traditional subject predicate logic, an analysis of the proposition 'This is heavy' is not forthcoming because the referent of 'this' cannot be specified. It has no meaning. As Russell again puts it:

> A substance is not properly speaking *defined* by its predicates. There is a difference between asserting a given predicate of one substance, and asserting it of another. The substance can only be *defined* as "this". Or rather-and this is where the doctrine of substance breaks down-the substance cannot be defined at all. To define is to point out the meaning, but a substance is, by its very nature, destitute of meaning, since it is only the predicates which give a meaning to it. Even to say "this", is to indicate some part of space or time,

> or some distinctive quality; to explain in any way which substance we mean, is to give some substance some predicate. But unless we already know which substance we are speaking of, our judgement has no definiteness, since it is a different judgement to assert the same predicate of another substance. Thus we necessarily incur a vicious circle. (Russell 1900, pages 59-60)

At bottom the charge of vicious circularity Russell is making here is of a kind with the charge the Idealists made. If subjects are to be identified through the predicates inhering in them, the ascription of a predicate seems to presume that one has already locked on to that subject, holding it up, as it were, for inspection. The question is, then, how do we achieve this? Clearly it cannot be supposed that reference to this subject is achieved through some predicate because this is circular. On the other hand, assuming the traditional subject predicate model, it isn't evident what other means we have at our disposal to secure this reference. So in what the predication relation can amount to, Russell confesses that he shares Locke's wonder (Russell 1900, page 50). [8]

The moral Russell drew was that the subject predicate tie could only be a relation among individuals. Russell would say that, to begin with, particular things are just there, given to us independently of any conceptual scheme. Universals are also entities given independently of our grasp of them. Together, these combine to form complex unities. Russell did not regard relations as intrinsically related to their terms. The entities entering into a complex through some relation remained unaffected by that relation, and would remain the same entities when reconstituted into another complex by a different relation:

[8]This would be a recurring theme for Russell: "'Substance' when taken seriously, is a concept impossible to free from difficulties. A substance is supposed to be something distinct from all its properties. But when we take away the properties, and try to imagine the substance by itself, we find that there is nothing left. To put the matter in another way: What distinguishes one substance from another? Not difference of properties, for, according to the logic of substance, difference of properties presupposes numerical diversity between the substances concerned. Two substances, therefore, must be *just* two, without being, in themselves, in any way distinguishable. How, then, are we ever to find out that they *are* two? (Russell 1946, page 211)

> What I admit is that no *enumeration* of their [complex unities] constituents will reconstruct them, since any such enumeration gives us a plurality, not a unity. But I do not admit that they are not composed of their constituents; and what is more to the purpose, I do not admit that their constituents cannot be considered truly unless we remember that they are their constituents. The view that I reject holds that the fact that an object x has a certain relation R to an object y implies complexity in x and y, i.e., it implies something in the 'natures' of x and y in virtue of which they are related by the relation R. (Russell 1910)

Moore and Russell would attack idealism on two fronts. On the one hand they objected to the psychologism inherent in Idealist epistemology. What Russell found objectionable was the obscurity inherent in an account of objects as constituted by minds. For Russell, this sort of view reflected a confusion between an object itself and the thought of the object. When the object in question is empirically given this kind of confusion is usually not caused. However the confusion frequently surfaces when the object in question is apprehended only in thought The object is then assimilated with its mode of presentation, becoming an 'idea', a mere creation of the mind. For Russell this served to compromise the objectivity of truth. Russell insisted that objects were 'out there', existing independently of our conceptual schemes and not somehow the work of the mind. For Russell these objects were not created, they were discovered. If the understanding is not creative, but rather discovers its objects, then these must possess in themselves an autonomous reality. They must have being.

Thus the attack on psychologism develops into an attack on the holistic construal of necessity inherent in the Idealist doctrine of internal relations. Knowledge of isolated, detachable bits of the world can be had. The so called "internal modifications" borne by subjects reflected nothing more than a change in the factual combinations of terms: "The notion that a term can be modified arises from neglect to observe the eternal self-identity of all terms and all logical concepts, which alone form the constituents of propositions" (Russell 1903, page 448). Objects are independent of the understanding. The cognitive relation is like any other relation in that the relation tying the subject to the known object is purely external. That is, the lump of sugar before me remains unaffected by my thinking about it. The object does not depend in any way on the subject. The mind here is merely a passive receptacle, which enables Russell to compare knowing with sense perception. This insistence on a passive and direct relation to the objects of knowledge would later be called 'knowledge by acquaintance'.

2.6 The New Logic

This general critique of the traditional theory of the proposition was extended to the conventional view of logic that it supported. Now, Kant had defined an analytic proposition as one which consisted in some sort of inclusion, or inherence, between notions; i.e., between representations. When Kant identified analytic propositions with the propositions of logic what emerged was a view of logic as sterile and uninformative. The propositions of mathematics, on the other hand, were said to be synthetic a priori. The philosophical battle lines drawn over this issue would not be recast in any marked way until Frege's critique of Kant.

Frege maintained that the laws of arithmetic were analytic as were the propositions of logic from which they are derived. According to Frege, (Frege 1884, section 88) Kant gave too narrow a definition of an analytic judgement. For Kant, an analytic judgement was a universal affirmative judgement. In this sort of judgement, one asks whether the attribute concept is included in the subject concept. But it ignores existential propositions and any judgement where the subject is a simple particular (i.e., not composed of parts (Frege 1884)). Of greater consequence, however, was Kant's propensity to tie analyticity to the content of a judgement. Frege had a somewhat different understanding of analyticity.

The mark of an analytic judgement was that it could be given a logical demonstration; if it followed from logical laws and definitions. An analytic judgement was a method of proof. This was the case with the truths of arithmetic, so that each one of them was a "law of logic, albeit a derived one" (Frege 1884, section 87) Frege, in the *Grundlagen* saw himself as having "achieved an improvement" on Kant's views (Frege 1884, section 109) by showing that arithmetic was an extension of logic. On the other hand, propositions of geometry are synthetic and a priori, depending as they do on axioms that are irreducible to pure logic (Frege 1884, section 89). Frege's conclusion was that mathematics was, like logic, analytic.

Russell, on the other hand, accepted the Kantian and Leibnizian characterisation of analyticity, according to which an analytic judgement is one which makes explicit the inclusion of the predicate in the subject concept. Or, again, if the judgement cannot be denied without contradiction. According to Russell, Kant had been right to hold that the judgements of mathematics were necessary and synthetic (Russell 1900, page 24). Indeed, Kant should have gone further and reject the notion that the propositions of logic were analytic (Russell 1903, page 457). So Russell's strategy was to take up the Kantian notion of the synthetic a priori, define analyticity as Kant and Leibniz understood it, and then try to contain the range of such 'propositions'. In effect, the presupposition was that the informative content of the propositions of logic could be displayed only by denying that they were analytic. This in turn would open an account of the fruitfulness of logic and mathematics as stemming from some sort of direct contact with the structure underlying reality itself.

The result was a thesis to the effect that at some fundamental level logic and mathematics are the same:

> All pure mathematics deals exclusively with concepts definable in terms of a very small number of fundamental logical concepts, and that all its propositions are deducible from a very small number of fundamental logical princi-

ples. (Russell 1903, page iv)

If mathematics possessed irreducibly logical content, there was no longer any need to account for its truths in terms of our a priori intuitions of space and time. Rather, mathematics was a priori through its intimate connection with logic (Russell 1903, page 8). So Russell was in need of an account of a priori knowledge, which he proceeds to spell out in *Problems of Philosophy*. There, a priori knowledge is accounted for in terms of a direct epistemic contact with a world of subsisting universals. The world had the structure it had due to a variety of logical and mathematical entities. This is why Russell regarded it as a mistake to suggest that the principles of logic represented the so called 'laws of thought'. These logical structures governed our thoughts, but also reality as a whole. Thus, while one may be thinking of the law of contradiction, the law itself is not a thought, but is rather an objective feature about things in the world (Russell 1912, page 89).

According to the new doctrine attributes, relations, and things are equally real. Metaphysically, the universe is made up of a plurality of such distinct, independent entities, whose relations which each other are not 'internal' to their objects. This ontology was crucial to ruling out any propositions being analytic. In the first place, relational propositions couldn't be interpreted as the inclusion of a predicate in a subject concept. Nor could a relational proposition be seen as an assertion of some sort of relation among representations. Rather, the entities involved were either concrete or abstract objects whose natures were completely independent of any sort of representation. So the notion of analyticity failed to account for the structure of relational propositions. Nor could predicative propositions of subject predicate form. Propositions such as these had to be distinguished. There were, on the one hand, general propositions such as "All men are mortal". Logical analysis of such propositions revealed an underlying structure which was relational rather than predicative, i.e., *for all x, if x is a man, then x is mortal.* While these propositions asserted a relation between predicates, this relation did not signify that *Mortality* was somehow 'contained' in *Humanity* (Russell 1984, page 94). Then there were singular propositions such as *Socrates is mortal*, which were of subject predicate form. While these sorts of judgements seemed to be of a non-relational nature (Russell 1903, page 49), they presupposed an underlying relation[9] . And this relation remained external as well in that the predicate was never part of the subject. So the conclusion was that no true subject predicate proposition was analytic. Indeed, practically all propositions were synthetic. The only exceptions to this rule were tautologies, and these propositions were dismissed as trivial.

Frege and Russell both thought that linking mathematics with logic would close the gap through which Kant had introduced a 'transcendental subjectivity'. And both philosophers rejected the notion that the propositions of logic were incapable of fruitfully extending our knowledge. But given Russell's position that the propositions of mathematics and logic are both synthetic rather than analytic[10], just how logic could be synthetic remained to be shown (Russell 1903, page 457). This task would be the object of Russell's labours in the *Principles of Mathematics*, to which we will now turn.

[9]In discussing the Identity of Indiscernibles Russell would remark "that there is no essential difference between subjects and predicates..." (Russell 1903, page 451). Specifying the nature of this relation would be problematic, and as a result of the paradoxes he would come to think that the relation of predication involved a fundamental logical difference of type between the terms involved.

[10]Moore drew similar conclusions in his paper on necessity (Moore 1900). Moore argued that a non-circular definition of analyticity was impossible. Analyticity could not be defined in terms of the principle of non- contradiction or the law of excluded middle, because such principles are not analytic truths. Moore concluded that the only analytic truths were tautologies, but that these were not genuine propositions.

Chapter 3

Russell's Propositions

3.1 Introduction

Russell's logicism was the thesis that the propositions of mathematics were complexes composed of logical entities. The object of philosophical analysis was to discover these entities and reveal their natures. This presupposed an understanding of logic as the synthetic a priori science of generality. This chapter will explore Russell's analytical method and the sorts of entities it would uncover in propositions. Of particular importance to him are intensional entities called denoting complexes that were presupposed in any proposition about number. Number would turn out to be a property of classes, and the isolation of this property would require what Russell would call his Principle of Abstraction.

3.2 Propositions and their Constituents

Russell was in need of an account of propositions. While the proposition was the primary object of philosophical analysis, it must be understood that for Russell the analysis of a proposition was not a syntactic analysis of linguistic items:

> *Words* all have meaning, in the simple sense that they are symbols which stand for something other than themselves. But a proposition, unless it happens to be linguistic, does not itself contain words: it contains the entities indicated by words. Thus meaning, in the sense in which words have meaning, is irrelevant to logic (Russell 1903, page 47).

There is thus a distinction between a sentence and a proposition. Propositions, though apprehended through a linguistic form, are not linguistic. A sentence is merely a particular linguistic expression for a proposition which itself belongs to an extra linguistic reality. The French sentence "Socrate est humain" and the English sentence "Socrates is human" express in different ways the same proposition. Words are just symbols, signs for things other than themselves. So a sentence is a complex sign for a proposition.

Meanings join to constitute a proposition, acting as the building blocks in the propositional complex. In the *Principles of Mathematics* Russell regarded a proposition as a structured entity with individuals, predicates and relations as parts. While Frege found it peculiar to say that a mountain or Napoleon could literally be a part of what one thinks or says, Russell insisted that Mont Blanc itself is a component part of what is actually asserted in the proposition 'Mont Blanc is more than 4 thousand metres high'. There is a certain complex in which Mont Blanc is itself a part. If 'Desdemona' and 'Cassio' are proper names, their Russellian meanings are the very people Desdemona and Cassio, and the Russellian meaning of 'loves' is the concept "Love", which is for Russell something as much a part of the external world as are Desdemona and Cassio.

The propositional complex was a structure of objects displaying a peculiar sort of unity in which the concepts and all other Russellian terms appeared as constituents. Thus what we believe, assume or deny when we believe, assume or deny that Desdemona loves Cassio is a single thing: a complex entity that has for its constituents the meanings of the words that make up the sentence in which we express the belief, assumption or denial. Propositional constituents were part of the ultimate furniture of the world.

There is then the question as to how we grasp the semantic units that lie at the heart of Russell's construal of the propositional complex. The function of the theory of meaning is to ground our discourse empirically by establishing a direct relation between a proper name and an effectively present reality.

G. E. Moore gave the essentials of the view in his paper *On the Nature of Judgement* (Moore 1899, page 179) according to which a judgement such as "This rose is red" does not attribute a quality to some substantial reality. Rather, a judgement is the assertion of a conjunction of concepts. Concepts combine to form propositions and the world is simply made up of all true propositions, independently of their being judged. The adjectives, nouns and verbs making up a sentence designated independent and self-sufficient entities. Understanding a proposition involved standing in a direct relation to the meanings of those expressions. Such meaning relata were called terms. Every word indicated a term and our contact with that term is unmediated. Indeed, what is distinctive about Moore's view of concepts is that they were not regarded as mind-dependent. What this meant was that a term could be apprehended regardless of the relations it had to other terms. In contrast to the idealist view, then, the picture was one of a plurality of independent terms making up the world as we know it. But what gave the particulars of that world a structure are the concepts that form part of its ontological fabric.

Russell took over much of this picture from Moore. The world was composed of objectively existing entities similar in nature to Moore's concepts, which Russell also called *terms*:

> Whatever may be an object of thought, or may occur in any true or false proposition... I call a term... A term is, in fact, possessed of all the properties commonly assigned to substances or substantives. Every term, to begin with, is a logical subject. Again every term is immutable and indestructible. What a term is, it is, and no change can be conceived in it which would not destroy its identity and make it another term (Russell 1903, page 44).

Terms are of two kinds, things and concepts. And these combine in one form or another as the constituents of propositions. Proper names pick out terms and general names pick out concepts. Things and concepts are the ontological correlates of single words, and are to be thought of as individuals (Russell 1903, page 497), although complex expressions may also designate single entities. Russell always associated a unique semantic entity with each grammatical unit, with these symbols referring to that reality. Particulars exist in space and time while universals are said to subsist. Concepts, in turn, are intensional entities in that two concepts can be coextensive and not be identical.

This crucial distinction between proper names and general names presupposes the following two theses:

1. A proper name is a name which can only function as a grammatical subject (i.e., as a substantive)

2. A general name is a name which can appear in two guises (as adjective/substantive) and be used in two ways (as predicate/subject).

Proper names stood for things (particulars), and general names stood for universals. That is, meaning was a relation between linguistic symbols and extra linguistic elements of reality. A thing is whatever is indicated by a proper name with the sole logical function of appearing as a subject. A thing may only occur as a constituent of a proposition, as a logical subject. A concept, which is signified by a general name, is a term with two functions. Its primary function is one of qualifying when it functions as a predicate, (or of linking when it is a relation). However, it may also function as a subject when it is nominalised. The term designated by the adjective "human", for example, could be mentioned by the substantive "humanity" (for that matter, substantives, adjectives, and verbs all indicate terms that could be nominalised into logical subjects). When a predicate is designated by the substantive expression, the concept picked out by that general name functions as a subject of the proposition. Concepts enjoy a dual role in that they may occur as constituents in a proposition as either concept or logical subject. They may be named and made into logical subjects without restriction. Properties can be meaningfully predicated of them. The occurrence of a concept in a proposition as a concept is not the same as its occurrence as a term. A concept is the logical subject of a proposition in which it occurs only when the proposition is about that concept, i.e., only when the concept occurs in the proposition as a term.

Russell's rejection of the traditional subject predicate logic had consequences for his ontology. The terms of traditional logic came to possess characteristics traditionally assigned to substances, such as being unities, besides being the logical subject in propositions. But now concepts, as terms indicated by general names, could become the subjects of assertions. So predicates and relations became endowed with all those characteristics the classical tradition had seen fit to attribute to substances. Furthermore, a particular term possessed a numerical identity with itself and numerical diversity with respect to all other terms, and terms that were concepts possessed a conceptual diversity. This made a universal an entity, or individual, thus cutting against a long philosophical tradition. This would be the source of constant problems leading up to the *Theory of Knowledge* manuscript.

The first part of the *Principles of Mathematics* is devoted to an "explanation of the fundamental concepts that mathematics accepts as indefinable" (Russell 1903, page xv). Access to these concepts is not reached through some sort of immediate intuitive insight but rather through the painstaking analysis of mathematical propositions. At the heart of this approach is the doctrine that propositions in general are complexes composed of real entities and endowed with a structure. Russell's task is to find a principled way of isolating the mathematical content of propositions in a way presupposing neither some sort of epistemic property (such as a proposition's 'obviousness') nor some metaphysical property (say, it being necessary as opposed to contingent). Rather, what marks a mathematical proposition is its generality, in the sense that they do not contain as constituents any particular existents. Thus, a propositional series such as *Socrates is a man, all men are mortal, therefore Socrates is mortal*, would not be counted as a proposition of logic, precisely because *Socrates*, *Man* and *Mortal* are empirically real existent terms. Mathematical propositions are all general formal implications. So mathematics is only concerned with the *forms* of propositions.

As the propositions of mathematics were formal implications, or propositions having the form: *for all x, if x is such that ... then x is such that....* Russell regarded it as crucial that *for all x* ... be understood as meaning for *all possible entities whatsoever*. A mathematical proposition is not *about* any particular subject. So a formal implication will contain a formal or unrestricted variable as a constituent. But to attain a completely general proposition the analysis of a formal implication must be pursued by the elimination of any particular class concept, until nothing remains but the form of the proposition, consisting of formal variables and logical constants. What remains when every non-logical constant is replaced with a variable is a logical proposition. Such a proposition is logically necessary and self evident, because it is neither provable nor disprovable by empirical criteria, and because it can be apprehended without the experience of any particular thing or its qualities and relations. Logical constants are thus purely formal concepts (Russell 1903, page 7). These are primitive and indefinable entities, and the propositions of mathematics derived from logical propositions are dependent in the end on these basic logical constants. These logical constants were entities with a being extending throughout all possible worlds and thus different from that of contingent particulars existing in space and time. The propositions of logic are thus devoid of any factual content of an empirical nature.

It is these entities -the logical constants- which are presupposed by *any* proposition. Here are to be found the primitives of mathematics. The level of generality and abstraction displayed by the propositions of mathematics is due to the fact that these formal concepts are indispensable to the construction of any concept whatsoever. So the entities that are the subjects of mathematical propositions are contained in mathematical propositions as constants, and the indefinable primitive notions of mathematics are either these entities, or entities presupposed by these. Logicism is thus the outcome of an analysis revealing how the various kinds of proposition, as well as the fundamental constituents underlying them, turn out to be the very concepts presupposed in mathematical propositions.

3.3 Denoting Concepts

A predicate is a concept that is the meaning of a general name when the latter appears as an adjective, such as in "Socrates is human". The class concept, on the other hand, is the concept meant when the predicate has been made into a substantive, such as in "Socrates is a man". Both predicate and class concept here are the same entity (Russell 1903, pages 54-56). So "Socrates is human" and "Socrates is a man" are equivalent propositions. Now, according to the Russellian analysis, when a general name forms part of an expression beginning with certain logical words, this indicats an unusual sort of propositional constituent called a *denoting concept.* Most terms perform one of two clearly defined functions as propositional constituents. They can either link other terms to form a unified complex, or they can just serve as the objects of this activity. Denoting concepts, on the other hand, function as terms whose role is to point to things beyond the proposition. So these constituents were endowed by Russell with an ability to point to, or indicate, other entities. It is this relation, called 'denoting', which is the expression of generality. Russell's task was to uncover the constituents of the propositions of mathematics that were responsible for the expression of generality. And it is through the denoting concept that that which is not a term and which cannot occur in a proposition as a single entity can be a logical subject in the sense of being that which a proposition is about.

What Russell, called "denoting phrases" have terms called denoting concepts as their meanings. Such terms denote –that is, for their part, they can have other terms (specifically, what were called objects) as their meanings. Denoting concepts were complex concepts derived from a class-concept (i.e., one ordinarily represented by a common noun) by means of the operations associated with the words 'all', 'every', 'any', 'a', 'some', and 'the'. Class-concepts are either predicates, i.e., concepts indicated by adjectives, or are obtained by the application of the device "such-that" to a propositional function, such as is indicated e.g. by 'x such that φx'. Take, then, the proposition expressed by the sentence "Socrates is a man". When the adjective "human" is replaced, or replaced by the substantive "man", this is accomplished by linking the substantive "man" with the word "a". In this way the adjective is transformed into a complex denoting phrase, consisting of a substantive, or class concept, preceded by some logical particle. A denoting concept functioned to mediate the referential link between a general name and an object (in Russell's technical sense). That is, while meaning is a direct relation between a word and a term (or collection of terms if these form a unity), denotation is more complex in that a denoting concept is an entity that allows for reference in a discursive or descriptive way. Take the proposition *I met a man*. This proposition differs from the singular proposition *I met Socrates*, the latter being about the entity Socrates, which it contains as a constituent. The former proposition, on the other hand, is not about the concept *a man* (which it also contains as a constituent). The subject of this proposition is not the concept *a man*, but some flesh and blood individual. And this individual is the subject of this proposition (*I met a man*) because of the symbolic nature of its conceptual constituent:

> But such concepts as *a man* have meaning in another sense: they are, so to speak, symbolic in there own nature, because they have the property I call *denoting*. That is to say, when *a man* occurs in a proposition..., the proposition is not about the concept *a man*, but about something quite different,

some actual biped denoted by the concept. Thus concepts of this kind have meaning in a non-psychological sense (Russell 1903, page 47).

According to Russell, then, in order for a predicate such as *man* (which does not denote anything on its own) to fulfil the function of denoting, the predicate must be prefixed with a logical word so as to constitute a denoting expression. For Russell in the *Principles of Mathematics* meaning was a relation between a linguistic expression and an entity. It served to indicate the term meant by the expression. Denotation, on the other hand, was a relation between a term, and various kinds of entities.

This discursiveness of the denotation relation is a function of the logical words which can be prefixed to the class concept. That is, each logical particle characterizes how the denoting phrase and denoting concept single out objects. To paraphrase Frege (Frege 1892, page 100), if the class concept is a lens, the logical words act as beams of light which the lens then projects as various objects. Thus "all", "any", "every", "some", "a(n)", and "the", evoke different objects through their relation with the class-concept "man". The denotation relation will yield a distinct object for each denoting phrase constructed from a logical word and a class concept. It is what allows us to go beyond the mere naming of a thing while retaining the extensional nature of reference (Russell 1903, page 53).

This is how propositions can be about things that are not constituents of propositions: for example, in the sentence "All men are mortal" mortality is ascribed not to the concept *men* but to the set of all mortal men taken as an object, i.e., a class as many. The theory of denoting concepts explains how this is possible. A denoting concept 'denotes'; that is, points to a particular combination of terms (which is not to exclude a 'combination' of one: a denoting concept formed by 'the' and a class concept, i.e., a definite description). Such combinations are called 'objects' (Russell 1903, page 58) and Russell was very much aware that being able to refer to combinations of terms which were not unified complexes was a problem[1]. So any analysis of generality will involve a study of the relation of denotation: specifically, the nature of these concepts and the objects they denote.

Of fundamental importance was the denoting concept *any (term)*. The *Principles* discusses two ways in which expressions of generality are underwritten. On the one hand are denoting concepts such as *some man*, *any man*. *any term*; it is the presence of such concepts in propositions which enable these to have the objects or the aggregates denoted by these concepts as their subjects. But generality is also expressed by means of formal implications, which contain variables instead of denoting concepts. Indeed, variables are the only entities used in mathematics to express generality. Furthermore, general propositions containing denoting concepts as constituents are equivalent to some formal implication. So, for example: *All men are mortal* is equivalent to the formal implication: *for all x, if x is a man, x is mortal.* The corresponding logical proposition would be *if x is a member of the class α and if every member of α is a member of β then x is a member of the class β, for any x, α and β* or its translation in terms of propositional functions (Russell 1924, page 111).

This raises the question to what extent the notion of a denoting concept could be defined in terms of more primitive notions, such as those of a variable and a propositional function, that is, the notions that are involved in the analysis of formal implications. The difficulty in defining *any* by using the notion of the variable and formal implication is that because it is a concept presupposed by the very concept of a variable and by the notion of formal implication (along with that of a propositional function), *any* is an irreducible notion. Unlike other denoting concepts, the denoting concept *any* cannot be defined, even contextually, by the notion of the variable, because of its being presupposed by the variable. It is this denoting relation that is presupposed in mathematical symbolism, and accounts for how a mathematical proposition can express generality.

[1] "I shall use the word *object* in a wider sense than *term*, to cover both singular and plural, and also cases of ambiguity, such as 'a man'. The fact that a word can be framed with a wider meaning than *term* raises grave logical problems." (Russell 1903, page 55) footnote *

Whenever classes as many were treated as a single logical subject, they were called Wholes. Wholes came in two forms. On the one hand were what were called Aggregates (i.e., a class). The other type of Whole was called a 'true unity' of which propositions were the clearest example. With the exception of those objects denoted by definite descriptions (more will be said about these in a moment), objects are terms combined in various ways, but are themselves neither particulars nor concepts. The logical word "all", for example, when conjoined to a class concept, will denote a complex object that is a numerical conjunction of terms. The complex object obtained in this way is called a class, and the terms so conjoined are to be thought of as neither individuals, nor as a whole, but rather, collectively. Similarly, the word "every" indicated a relation involving a combination of terms, but marking what Russell described as their propositional conjunction. And determining the ontological characteristics of this sort of ambiguous object, with one foot in unity and the other in plurality, was to prove very difficult.

3.4 Classes

The basic relation of denoting was crucial to the logicist program because this relation had classes as one of its possible objects. Any combination of terms can form a collection. A class was defined as the numerical conjunction of terms; it was a "collection defined by the actual mention of the terms, and the terms are connected by *and*" (Russell 1903, page 69). Russell regarded any such collection of terms as an *object* and called it a 'class as many'. The immediate problem was to explain how such a plurality could be made into the logical subject of some proposition. The adjective 'ten' as in 'ten men' is not attributed to a single object but, rather, to a totality. Yet a class as many could not be picked out either by using a proper name nor by employing a general name[2] . Now if the

[2]There were, however, situations where such a numerical conjunction of terms can be singled out through the use of a singular referring expression, such as, for example, the word "collection". For Russell this was evidence that a numerical conjunction can form a single whole: "The distinction of a class as many from a class as a whole is often made by language: space and points, time and instants,

class as many could not be treated as a single logical subject (Russell 1903, page 70), some way had to be found to treat such collections, given that they were logical subjects in mathematical propositions. Any statement about number presupposes a collection being a logical subject, yet retaining its status as a multiplicity. This seems to require an account of multiple logical subjects (Russell 1903, page 77, page 132, and pages 516-17), an account of the logical status of the class as many. How could a complex object be treated as a logical subject, which would seem to be required in any proposition about a number? Any assertion about a number is about a plurality of terms; a class as many. So some sense had to be made of classes as entities.

Traditionally, the objects of mathematics were regarded as extensional entities. Philosophers have always been somewhat loose as how best to understand this. Presumably, whatever sorts of entities the objects of mathematics are, mathematicians want them to have clear criteria of identity. A class, for example, is to be identified with the objects that make it up. To draw on anything other than its members as an aid to its identification would only encourage obscurantism. Russell, however, saw difficulties in this requirement of extensionality. Certain cases seem to belie an exclusively extensional account of a class as merely the numerical conjunction of its terms. First of all, such a treatment makes it difficult to see how a distinction could be drawn between the unit class, the class as one, and the class as many when it is a "collection" consisting of a single term. So some sort of distinction had to be drawn between the unit class and its only member, a distinction, that is, between x and $\{x\}$. In addition there is the difficulty of providing an account of infinite classes through only extensional means. If a class is the numerical conjunction of terms, such a process would not be practicable for infinite classes. Finally, there is the question of the ontological status of the null-class: is the

the army and the soldiers, the Cabinet and the Cabinet Ministers, all illustrate the distinction" (Russell 1903, page 68). When such a numerical conjunction of terms formed a whole (and Russell thought it axiomatic that it did), such a whole was called a *class as one*. A class as one was a term, and could be made into a logical subject.

null class an entity? That is, is there some sort of entity underlying the notion of the null set? Could the null set even be an object? The extensional viewpoint seems inadequate for determining the kind of entity the null class might be, and it is not even clear how such a thing could even be an object.

One answer to this last problem congenial to a philosopher would just be to abandon the extensional viewpoint and say that the null class is what is picked out by a property with no instances. Accordingly, even though there is assumed to be no such thing as the null class, there certainly are expressions such as "all a's" which have no terms falling under the expression. The idea then would be that even though such an entity was not given in extension, it was given intensionally by grasping the concept that determined it[3] On this account, the intensional viewpoint turns out to be indispensable.

Traditionally, this sort of intensional approach had been subject to a number of criticisms, the most common being that classes are not uniquely specified by entities such as properties (Quine 1941, pages 147-148). The latter are, after all, intensional entities, and any number of them can be correlated with the same collection of terms. A set could be characterised in any number of ways, depending on what sorts of properties are possessed by the objects making up that set. In Russellian terms, the problem here is that a number cannot be uniquely specified by means of a class-concept, because a collection is characterized by any number of class-concepts. Class concepts may have the same extensions and yet not be identical (Russell 1903, page 115). So, in order to provide for an extensional viewpoint in the theory of classes, some distinction had to be made between a class and the properties which could be attributed to it. The problem, then, was to find some way of isolating which of those properties was the particular property of being that sets' number. Furthermore, in order to attribute some mathematical property

[3]Frege was also very clear about the limitations of a purely extensional account of classes. For Frege, the extension of a concept (*Begriffsumfang*) was called its value range (*Wertverlauf*), and the latter could not be completely detached from its concept. It contained its arguments (a concept was a kind of function) as well as the truth values corresponding to them (Frege 1891). See also (Frege 1893).

to a set, such as its number, that set had to be considered as a single thing, or in other words, as a logical subject. So, a definition of number had to accomplish two very important tasks. On the one hand it had to secure the set as a unitary thing and, having done that, it had to isolate among its properties the one which was its number.

However, it was clear to Russell that access to a number of objects could only be had via an intensional medium. Classes were objects of this sort. But how could reference to such objects be made in a way allowing for univocal designation of their properties? Russell felt he had an answer:

> The equality of class-concepts, like all relations which are reflexive, symmetrical, and transitive, indicates an underlying reality, *i.e.* it indicates that every class-concept has to some term a relation which all equal class-concepts also have to that term-the term in question being different for different sets of equal class-concepts, but the same for the various members of a single set of equal class-concepts.(Russell 1903, page 75)

A class is, in fact, an intensional object in that a class is the object denoted by the concept of the class, and access to this object can only be made via some sort of conceptual medium, or content (Russell 1903, page 73). There were times when the mere extensional enumeration of their terms failed to determine them as objects. In such cases reference had to be secured through denotation by a concept of the class. This required such concepts as autonomous entities, whose status was guaranteed independently of whatever collection of entities they were said to be the property of.

A class, then, is the object denoted by a denoting expression built up by using a class concept word preceded by the word ≪all≫ . This, of course, comes down to saying that the theory of number depends on the theory of denoting. Now, a class for Russell was an object formed from the numerical conjunction of those terms satisfying a propositional function, and determined equally by any equivalent propositional function. The denoting concept "all men", for example, denotes the class (as many) of men. Those denoting concepts which denoted classes as many are indicated by expressions of the form 'all u's', where 'u' itself stands for a class-concept. A class as many is not an individual or term and cannot be treated as a single logical subject, but only as a "numerical conjunction of terms". The plural denoting concept that contains a class-concept as a component, and denotes the class as many of objects satisfying the propositional function, is the concept indicated by x's such that ϕx'. This concept is synonymous with both 'all (the) x's such that ϕx' and 'the x's such that ϕx'. A class is the object denoted by a denoting concept, which is itself constructed from a predicate (the class of men is the object denoted by the concept *men* or *all men*, which is formed using the predicate *man*). In other cases a class is determined by a propositional function: the x's such that φx, where φx is the propositional function.

This reveals the importance of the theory of denoting concepts, because, according to Russell, it was through such concepts that collections displayed a unity of a sort that seems to 'go beyond' their mereological sum and which enabled them to be treated as single logical subjects. Access to the class was to be mediated by the conceptual content contained in the propositional function or, more accurately, in the denoting concepts which it presupposed. Specifying the nature of this unifying element, while maintaining a commitment to an extensional pluralist atomism, would be the thorniest problem Russell would wrestle with during this period. Russell's suggestion would be that a class was an object denoted by a denoting concept; it was a particular kind of complex object denoted by the concept of the class. The conflict between the extensional and intensional viewpoints is only apparent, due to a faulty logic (Russell 1903, page 66). Our ability to refer to classes presupposes some intensional medium and is revealed by abandoning the traditional subject predicate logic. Once relations are incorporated into the logical repertoire, reference to classes will be secured without sacrificing the sort of extensional features that are fundamental in mathematics.

3.5 The Principle of Abstraction and the Definition of Number

With the fundamental framework in place we can now turn our attention to how Russell deployed it to provide a rigorous definition of number using only logical principles. Since the foundational work of Cantor it was generally agreed that any definition of number should ignore whatever particular features distinguish those individuals brought together in a collection and merely take into account their common membership in that set. In this way one generated the concept of an abstract set. The second step consisted in ignoring the order of the elements in the set. Two sets were then said to have the same number if there was one-to-one correspondence between their members. This is known as Hume's principle and it was attractive because, given a number of collections, one could establish which of them had the same number in a way which did not presuppose numbers. And it was understood that number was a property shared (held in common) by all those sets related in this way by Hume's principle. That is, the strategy was to take sets related by Hume's principle and to give a contextual definition for that abstract entity (the number) they had in common. In short, if it was possible to establish a one-to-one correspondence between their members, classes were said to have the same number.

This approach makes it possible to offer a contextual definition of number. The basic idea was simple enough: find a criterion of identity for the new entity by using a relation between other entities whose status was unproblematic. Examples of the technique were very familiar to mathematicians. Say one wanted to define *direction of a line*. This concept could be abstracted from the simpler notion of *parallel line* in the following way. Because a *new* kind object was to be abstracted from some given entity it couldn't be identified or picked out by the use of any name already at hand. Peano (Peano 1897, page 45) suggested using the following definition to this end:

$$h_{u,v} \supset .\varphi u = \varphi v. = .P_{u,v} \ Df.$$

Here $h_{u,v}$ is just the hypothesis that u and v are lines. Under this assumption, $\varphi u = \varphi v$ is the identity marking our new property, namely that u and v are lines having the same *direction*. The strategy is simply to have this identity signify the same thing as the relation denoted by $P_{u,v}$, which, in keeping with our example, is the relation of being parallel. This latter relation is what is called an equivalence relation and it holds between the lines u and v.

Again, the idea here is that by reconstruing 1.

1. Line u is parallel to line v

with the help of the concept of identity, we get:

2. The direction of line u is identical to the direction of line v.

In this way we can obtain the concept of direction. This is, of course, Frege's suggestion in the *Grundlagen*: "we replace the symbol // by the more general =, by distributing the particular content of the former to a and b. We split up the content in a different way from the original way and thereby acquire a new concept." All classes which were equinumerous which each other had something called 'content', and the strategy suggested by Frege, is to isolate it by 'carving' it (the metaphor is Frege's) i.e., abstracting it from known content.

So in defining numerical equality between classes by Hume's principle one proceeded by establishing two classes as *similar*. The relation of similarity is transitive, reflexive, and symmetric. These properties define an equivalence relation. The crucial move in the case of number was to deduce from an equivalence relation between classes the fact that the classes have the same number, which is precisely what Russell assumes when pointing out that this sort of definition reformulates a transitive, symmetrical and reflexive relation as the possession of a common property by those terms. This move went beyond Peano's definition by abstraction in that an equivalence relation is understood as possession of a property in common.

Russell saw his task as one of specifying what is meant by saying that certain entities are bound together by an identity of content. Russell proceeded by defining numerical equality as a property held in common by all similar classes. However, in the *Principles*, he directs a number of criticisms at Peano's definition by abstraction. The substance of the charge is that such definitions do not yield a unique property because they don't effectively yield, or isolate, the property being introduced. From the fact that two terms possess a property in common one can validly conclude that these two terms are linked by an equivalence relation. But the converse doesn't hold: two terms linked by an equivalence relation can possess many properties in common. To any equivalence class their might correspond any number of properties. At best, Peano's definition by abstraction allows one to define not the number of a class, but rather, a class of such numbers:

> In order to make this point clear, let us examine what is meant, in the present instance, by a common property. What is meant is, that any class has to a certain entity, its number, a relation which it has to nothing else, but which all similar classes (and no other entities) have to the said number. That is, there is a many-one relation which every class has to its number and to nothing else. Thus so far as the definition by abstraction can show, any set of entities to each of which some class has a certain one many relation, and to one and only one of which any given class has this relation, and which are such that all classes similar to a given class have this relation to one and the same entity of the set, appear as the set of numbers, and any entity of this set is *the* number of some class. If, then, there are many such sets of entities-and it is easy to prove that there are an infinite number of them-every class will have many numbers, and the definition wholly fails to define *the* number of a class (Russell 1903, pages 114-115).

Russell sees his principle of abstraction as a way of correcting this "absolutely fatal defect".

His diagnosis laid the fault at the door of the traditional subject-predicate logic. On the one hand, Russell was unhappy with this framework because by assuming it the ascription of a property to a subject invariably failed, given that the subject had no meaning. So a way had to be found to avoid making such an ascription in subject predicate form. Russell's strategy was to devise a way in which the ascription of number was relational, which meant that the subject (in this case the class involved) had to be an entity, and the property (i.e., the number) also had to be an entity. Russell's method was to take the property they had in common to be the relation each term had to the set it belonged to taken as a whole. This involved treating that set as an entity:

> When there is any relation which is transitive, symmetrical and (within its field) reflexive, then, if this relation holds between u and v, we define a new entity $\phi(u)$ which is to be identical with $\phi(v)$. Thus our relation is analyzed into sameness of relation to the new term $\phi(u)$ or $\phi(v)$. Now the legitimacy of this process, as set forth by Peano, requires an axiom, namely the axiom that, if there is any instance of the relation in question, then there is such an entity as $\phi(u)$ or $\phi(v)$ (Russell 1903, page 220).

Peano's definition by abstraction requires an additional axiom implying the existence of the new entity without providing us with a criterion for identifying it. Russell's 'principle of abstraction' endeavors to provide just such a criterion. He provides an analysis of what Peano took to be an ultimate datum: the attribution of a common property to terms linked by an equivalence relation. The attribution of a common property is analyzed in terms of a common many-one relation to a new element. The relational analysis allows the logical reduction of the initial relation of similarity to this new relation. In the article *On the Logic of Relations* this reduction is the object of the following theorem:

$$*6.2 \qquad R \in \mathrm{rel} \,.\, R^2 \supset R \,.\, R = \breve{R} \,.\, \exists R \,.\, \supset \,.\, \exists Nc \rightarrow 1 \cap S \ni (R = S\breve{S})$$

This translates in terms of relations Peano's definition by abstraction. It says that a one-one, reflexive, symmetric and transitive relation R is identical to the relative product of a many one relation S with its converse. The elusive common property is then *defined* in the course of specifying the relatum of the relation S; that is, it is defined as the class of all the classes linked by R. In Russell's words: "If R be a symmetrical and transitive relation, and a term a is in the field of R, then a has, with the class of terms in which it has the relation R taken as a whole, a many-one relation, which multiplied by its converse, is equal to R." (Russell 1903, page 167).

To see what Russell has in mind, consider a set E of elements $x, y, \ldots$; define on this set an equivalence relation S (a relation which is reflexive, symmetrical and transitive). This relation induces a partition of that set into equivalence classes K_i and abstract is a property common to some given element of K_i. For Russell, the criterion which identifies that property is given by the univocal relation $T \in Cls \rightarrow 1$ which applies the elements of each equivalence class K_i onto that class; this surjective application assures the univocity of the definition. The existence of that property as an entity is then inferred when each unique entity corresponding to each equivalence class is posited as a relatum of the relation T. While the elements of E are simply grouped into mutually exclusive subsets by S, the consideration of these subsets (equivalence classes) as entities allows one to construct a transitive and asymmetrical relation between these K_i and their organization into a series. In this way we can construct the absolute series of cardinals and ordinals.

This definition of the common property solves all the difficulties inherent in a definition by abstraction. It is an important tool for making inferences to entities such as numbers, magnitudes and points in space and time, understood in absolute terms. Definitions by abstraction, without the addition of Russell's definition of a common property, lend themselves to relativistic characterizations of these entities. In definition by abstraction, time is the ordered succession of events; that is, it recognizes three fundamental relations between events: *simultaneity*, the *before than* relation and the *later than* relation. This last relation being the converse of the second, time is nothing outside events ordered by these relations. To say that a number of events took place at the same time is just to assert that there exists between them a relation S which, being reflexive, symmetric, and transitive, is an equivalence relation. Thus a moment in time is a common property of the equivalence class of simultaneous events. Russell's criticism is that this is not a univocal definition given the infinity of entities satisfying these equivalence conditions.

To see this, consider what is asserted by Peano's definition by abstraction: suppose that $x\mathrm{S}y$ and $y\mathrm{R}z$ implies that $x\mathrm{R}z$ and that $x\mathrm{R}y$ and $y\mathrm{S}z$ implies that $x\mathrm{R}z$. In such a case all the terms can be arranged in a series; but in the series all those terms related by the relation S will occupy the same position. This position is nothing more than the relation had between one term and numerous others: so there is no absolute position, and instead what we have is the relation of simulteneity between events. According to Russell's principle of abstraction, on the other hand, there exists a relation T, such that, if $x\mathrm{S}y$, there exists an entity t— the moment or instant— for which $x\mathrm{T}t$ and $y\mathrm{T}t$. The object of analysis is to discover the different entities t corresponding to those groups of terms originally forming a series. But this series, unlike the first, is such that any two different given terms will always be related by an asymmetrical transitive relation — never the symmetrical relation S. These terms form absolute temporal positions.

The Principle of Abstraction was the main instrument of Russell's realism, yielding nominal definitions of the entities presupposed in all mathematical reasoning. It was this principle that transcended the gap between intension and extension by isolating the sort of entity underlying the attribution of number: such an underlying reality is an intensional entity, a property, accessible only through a denoting concept. The Principle of Abstraction would in turn presuppose relations as autonomous entities. Russell regarded relations as irreducible, and having the sort of autonomous existence enjoyed by other Platonic universals. Russell felt that any account of the property of number in terms of containment or inherence required that relations be treated as predicates, and inevitably to their being regarded as merely ideal. And we will now be in a better position to take the full measure of Russell's realism about abstract entities, and how the objectivity of the propositions about mathematics is founded on their being independent of minds. By objecting to the Kantian picture of some kind of representative medium lying between the mind and an external reality Russell was also rejecting idealism. As a result, philosophers went casting about for something 'particular' about mathematical reasoning, for an intuitive grounding for arithmetic and geometry in a priori forms of sensibility. While idealists accounted for the propositions of mathematics in terms of an intuition peculiar to mathematics, Russell tries to derive them using only the concepts available to logic. However, this account would unravel because of the paradoxes. Russell's discoveries of these and his efforts to resolve them are the focus of the following chapter.

Chapter 4

The Contradictions

4.1 Cantor's Theorem and Russell's Paradox

What threatened to unravel this logicist construction of cardinal numbers were the Paradoxes. Cantor had proved a theorem showing that the cardinal number of the set of all subsets of a given set is greater than that of the original set. Take the set $M : \{1, 2, 3\}$. Then $\wp(M)$, the power set of M is $\{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$. M's cardinal number is 3 and the cardinal number of $\wp(M)$ is 8. Clearly $\text{card}(\wp(\text{M})) > \text{card}(\text{M})$; that is: $\text{card}(\wp(\text{M})) = 2^{\text{card}(\text{M})}$, and it is straightforward that $2^n > n$. Cantor extended this result to include infinite sets. Now this innocuous looking theorem immediately generates a contradiction once n is given as the cardinal number of the set of all sets. Let E be this set of all sets. By definition, its cardinal number should be at least as great as that of its power set $\wp(E)$, because each member of this set is itself a set. However, Cantor's theorem requires that the cardinal number of the subsets of the set of all sets be greater than the set of all sets: $\text{card}(\wp(\text{E})) > \text{card}(\text{E})$. So we can't define the greatest cardinal.

In the theorem's proof, Cantor defines a denumerable set as one that can be put in a one-one correspondence with the set of natural numbers. The cardinal number of such a set's power set is $\aleph_0$. Now take any arbitrary denumerable set $x_1, x_2, x_3, \ldots$. The infinite number of elements of this set can be arranged in a variety of ways giving rise to different series E_n. Take then the set M of all these series E_n . Because this set is the set of all denumerable series one would expect that it is itself a denumerable set. What Cantor proves is that in fact this is not the case, for we can always construct some new series E_0 that will be added to the set of denumerable series E_n. The result is that the set M has a cardinal number greater than $\aleph_0$. The procedure used to construct this additional set E_n is relatively straightforward. Take the set $N : \{1, 2, 3, 4, 5, \ldots\}$ and the following subsets of N:

E_1 = the set of even numbers: $\{2, 4, 6, \ldots\}$

E_2 = the set of all odd numbers: $\{1, 3, 5, \ldots\}$

E_3 = the set of squares: $\{1, 4, 9, \ldots\}$

...

Each series in the set E_n is denumerable given that it can be put into a one-to-one correspondence with the set N. Take, for example, the set E_1 :

$$\begin{array}{ccccccccc} N & = & \{ & 1, & 2, & 3, & 4, & 5, & \ldots \} \\ & & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ E_1 & = & \{ & 2, & 4, & 6, & 8, & 10, & \ldots \} \end{array}$$

Now define the property m as the property that characterizes each member of N having a correspondent in the series E_n. Then define the property w as that property characterizing each element of N not having a correspondent in E_n. We can now construct a tableau putting the set N into a one-to-one correspondence with each of the subsets of E_n each element of N possessing either property m or w being shown by a 'Yes' or 'No', respectively

N	1	2	3	4	5	...
E_1	No	Yes	No	Yes	No	...
E_2	Yes	No	Yes	No	Yes	...
E_3	Yes	No	Yes	Yes	Yes	...
E_4	Yes	No	No	Yes	Yes	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

In order to construct a series E_0 that can be added to the denumerable set E_n all we need do is turn our attention to the diagonal of the above tableau. Reading off the values given in the diagonal we have: No, No. Yes, Yes. When these values are inverted we have: Yes. Yes, No, No. These values in turn correlate with the sequence of the numbers 1. 2, 3,..., that is, the beginning of a series made up of elements of N which possess the property w; in other words, which do not have a correspondent in E_n. And this series is the series E_0. Indeed it will differ from E_1 at least in its first member, from E_2 on at least in its second member, and so on. In this fashion a new series is added to the set of denumerable series E_n. So we have proved that the power set of the set of subsets of N is greater than the power set of N: card($\wp$(N)) > card(N) .

The concept of class proved to be problematic, all the more so given Russell's definition of numbers as classes of equivalent classes. The generation of Russell's paradox stems from applying the diagonal method to the object that is the class of all classes. Call M the class of all classes. We then put each class in M into correspondence with itself. This relation generates a partition of M separating the classes into those that possess the property m and are members of themselves and those that possess the property w and are not members of themselves. In this way we can construct the class W of all classes possessing the property w: the class of all classes not having themselves as members. Since this class belongs to M, it seems reasonable to ask whether this class possesses the property m or w. A contradiction arises here. If the class W possesses the property

m, it is a member of itself and must possess its defining property: the property of not belonging to itself. If, on the other hand, the class W possesses the property w, it is not a member of itself and so does not possess the property defining it. But, not possessing the property w, it necessarily possesses the converse property m and is a member of itself. Each alternative leads to a contradiction:

$$W = \mathrm{cls} \cap \mathrm{x}(\mathrm{x} \notin \mathrm{x}). \supset\!: \mathrm{W} \in \mathrm{W}. \equiv\!: \mathrm{W} \notin \mathrm{W}$$

What this says is that if W is the class of x's such that x does not belong to x, then W is a member of W is the same as $W$is not a member of W. Given the above definition of W, for every class x we have:

$$(x)[(x \in W) \equiv (x \notin x)]\ .$$

Recalling that W itself is a class, by merely substituting W for the class variable x we can obtain the following contradictory formula:

$$(W \in W) \equiv (W \notin W)\ .$$

The source of the difficulty is that classes, like all constituents of propositions, are treated as terms. Hence the power set of the class of all terms can be mapped into the class of terms by simply mapping each class in the power set onto itself. Conversely the class of terms can be mapped into its power set by mapping each term onto its unit class. According to Cantor's theorem, there are more classes of objects than objects. But this cannot be, according to Russell's account of an object, since, according to that conception, every member of the power set of the class of all objects is itself an object. The trouble seems to stem from treating a class as a single entity that could then be taken as a value of a variable. When taken as a whole, that is, as a unity or term, the class w is paradoxical. The antinomy arises the moment we suppose that there is a certain class w that we are talking about when we ask the question whether or not w is or is not a member of itself as many. In other words, the variable x may take on any value in the definition of the class w:

$$x \in w \text{ iff } x \notin x$$

What the antinomy shows is that the class as one doesn't always exist. Prior to the discovery of his paradox, Russell took it as obvious that the class as many could also be viewed as a whole composed of the members of the class as many (Russell 1903, page 141) . A class as many, when viewed in this way was called "a class as one", and a class as one was to be found wherever there was a class as many. Because it was a whole, the class as one was always a single logical subject, i.e. a term or individual, and in fact it was taken by Russell to be "an object of the same type as its terms" (Russell 1903, page 104) . After the discovery of his paradox Russell came to believe that it was this assumption that was the cause of the contradiction.

So the effect of the Paradox on Russell was to shake his confidence in the existence of classes as unities or terms. Now, it was essential to Russell's project to be able to treat certain collections as terms, to regard them as entities that can be named.[1] One traditional answer allowing for an aggregate to be the object of referential attention (to have the class as many play the role of logical subject in a proposition), required an intension i.e., some sort of property held in common by the entities making up the aggregate. This was, in effect, the traditional answer to the problem of the one and the many: the many become one by partaking in a universal. However, for the purposes of mathematics, it was crucial that the theory of classes be extensional, because classes may be picked out using any number of properties. A class has to be distinct from the concept used in its definition. Take, for example, the concept *provincial capital of Canada*. This

[1] As Russell puts it: "It remains to prove –what is a purely logical point– that every identical content among a number of terms is analyzable into possession of one and the same relation to one and the same entity-that, in short, the supposed floating adjectives are not merely mental, and are not nothing apart from their ascription to substantives, but are rather like Platonic ideas, real entities, which, even if they do not live and move, have yet their being among the constituents of the universe." (Russell 1902, page 229)

concept, *qua* concept has being and is one; yet there are 10 such provinces in Canada. It is natural to conclude that there is some sort of 'object', distinct from the concept of a Canadian political entity that is somehow *10*, that is, of which it makes sense to predicate of it a number, and which is also the extension of this concept. This extension is precisely the nine objects in question, Toronto, Halifax, etc.. - what Russell calls the numerical conjunction of the terms.

But in another sense this class is also a unity, something which can be talked about, can be a possible logical subject (as in 'Toronto is a provincial capital'). What Russell saw as the great advantage to his Principle of Abstraction was that it allowed a class to be uniquely specified intensionally without obscuring the extensional view of it. An intension was necessary to ensure the class's unity, while its number remained an extensional property of that class (the number of a class being understood as the class of all classes similar to this given class). This required that classes be the terms of a relation[2] So even though a class as many was not a possible logical subject, successful reference to it was achieved by some sort of proxy standing in its place. This proxy was the class as one, which itself was the result of having the members of the class as many being brought together through a relating relation.

4.2 Functions and Classes

As mentioned above, Russell assumed that it was perfectly acceptable to take any formula $F(x)$ containing a free variable x, form the class-concept expression "x such that $F(x)$" from it, and then the denoting expression "the x's such that $F(x)$" whose extension was in turn called "the class as many". Abstraction of classes can appear natural, once abstraction of functions is accepted. If there is such a class satisfying a given propositional

[2]It is essential for the reader to bear in mind Russell's insistence that the entity unifying the class not be inherent, or contained, in that class. The first reason being that it would have made the attribution of a number analytic. The second reason being to ensure the class's extensionality: a class's various properties, including the unifying entity, must be kept distinct from the class.

function, then, according to Russell, the very condition expressed by 'x is an element of u' represents a predicate of x. So a propositional function was predicative (and defining a class as one), on the condition that it satisfied the unrestricted comprehension principle. The comprehension principle supposes that functions be treated as entities; that is, the abstraction of what in the *Principles* was called the functional part of the propositional function. So Russell attached the possibility that a class could exist as one, i.e., that the class could exist as "a new single entity", with the possibility of there being some propositional functions that are separable, and therefore, single, entities.

However, the assumption that functions are subsistent entities generates a paradox. If one assumes that functions are entities, then it would make perfect sense to write $\phi(\phi)$ and $\sim\phi(\phi)$. Treating ϕ as a variable, one can then define a function F, say, 'not being predicable of oneself'. But from the function $F(F)$ a contradiction can be derived. A contradiction arises here as a consequence of treating functions as entities capable of being made the argument of any function without restriction:

> The reason that a contradiction emerges here is that we have taken it as an axiom that any propositional function containing only one variable is equivalent to asserting membership of a class defined by that propositional function. Either this axiom, or the principle that every class can be taken as one term, is plainly false, and there is no fundamental objection to dropping either. But having dropped the former, the question arises: Which propositional functions define classes which are single terms as well as many, and which do not? And with this question our real difficulties begin (Russell 1903, page 103).

So Russell realized that particular doubts about the nature of classes were not peculiar to a proper diagnosis of the paradoxes. The logical fault lay at a deeper level, in functional abstraction.

There were then two intimately connected problems[3] x is a u' is faulty, is that it speaks of 'a function ϕ' without any argument. Now a function, as Frege himself has rightly urged, is nothing at all without some argument; hence, we can never say, of a formula containing a variable function, that it holds 'for some value of ϕ' or 'for all values of ϕ', because there is no such thing as ϕ and therefore there are no 'values of ϕ'. (Russell 1906c, page 171). In Russell's earlier correspondence with Frege he argues against Frege's saturation principle. Russell had to wrestle with: the first was the notion of a class as a logical subject, i.e., of a class as one as opposed to a class as many. The second problem was the notion of a propositional function as a single and separate entity (with the alternative being that a propositional function existed only in the many propositions that are its values). If, as Russell suspected, the problem lay with the use of an unrestricted comprehension principle a way had to be found to determine when a propositional function determined a legitimate concept and when it did not, such as the property of being a class which was not a member of itself.

For Russell, however, the lesson to be drawn from the paradoxes was that the notion of a function was more general than that of a predicate or concept. Predicates and concepts are included among the stock of primitive entities making up the world. They are data, as Russell would say. However, functions (assuming they exist) could no longer be contained in this basic stock. In any event, functions are complex and isolated by a form of abstraction from propositions. Whatever entities functions are, even though their logical role would be comparable to that of concepts, they would have a lower degree of reality than concepts. So the existence of functions is a priori more doubtful than that of concepts.[4]

[3]Just how intimate Russell came to take the link to be is evident in his remark that: "The point where, as it seems to me, the above definition of '

[4]Interestingly, the different forms the Contradiction took reflected a difference in the kinds of entities concepts were as opposed to functions. There is, in fact, a version of the contradiction in terms of concepts (predicates): certain concepts are predicable of themselves, others are not: then suppose that there exists a predicate 'not predicable of itself'. Now the conclusion Russell drew from this contradiction was simply that the apparent predicate is not a predicate, not that predicates in general don't exist.

Why should Russell emphasize this difference in treatment between functions and concepts, given that both are associated with a contradiction? The crucial difference is that functions are *assertions.* Functions were the result of a certain analysis whereby a sentence was divided into a proper name/grammatical subject on the one hand, and what was left over on the other. What remained included the verb, which had the job of relating the various parts of the sentence. It was the verb acting as a verb (as a relating relation) which endowed functions with their assertive character. In Russell's eyes, functions had a feature corresponding to the affirmative force contained in the verb:

> The assertion is everything that remains of the proposition when the subject is omitted: the verb remains an asserted verb, and is not turned into a verbal noun; or at any rate the verb retains that curious indefinable intricate relation to the other terms of the proposition which distinguishes a relating relation from the same relation abstractly considered (Russell 1903, pages 83-84).

Thus the paradoxes would open an investigation into the nature of functions.Was it possible to take some declarative sentence, isolate what remains when all proper names designating those terms have been omitted, and treat what remains as some sort of unity? Needless to say, this remainder would be a complex entity formed out of the simpler constituents indicated by the remaining words, but would it contain enough unity to be regarded as a full fledged entity?

In the *Principles*, Russell considered an analysis comparable to Frege's of the proposition in terms of 'subject and assertion'. This meant that given the proposition 'Socrates is a man', one distinguished besides the term Socrates, the assertion (taken as a whole)

However, the contradiction in terms of functions or assertions had graver ontological consequences. Here we are not led to merely conclude that some particular function does not exist, but rather that no functions exist. The contradiction expressible in terms of functions separate from their arguments seems to lead to the conclusion that the contradiction can only be avoided by denying that the functional part of a proposition is an independent entity. (Russell 1903, page 88) *is a man*, that is, the sort of entity that Frege would call a function. The problem with conceiving of functions as assertions in this way is that it seems as though they cannot be named, or transformed into logical subjects, precisely because when nominalised their assertive value is lost. Thus what exposes functions to a contradiction is that they seem to be the sort of 'entities' that cannot be made logical subjects. Russell thought it contradictory to say that an entity existed yet could not be named. The principle that any term can be made a logical subject held without exception because its denial is contradictory. The most fundamental of the contradictions is arguably that of some entity not being able to be a logical subject (Russell 1903, page 501). It is important to understand why the denial of the principle is contradictory; it is simply that in naming an entity, in the course of stating that it can't be made a logical subject, one is treating that entity as a logical subject. So, because functions cannot be logical subjects, they do not exist.[5].

4.3 Assertions and the Unity of the Proposition

Thus, the question over whether propositional functions were entities was also a matter of determining whether assertions are detachable entities. Russell posits *assertions* as entities in order to account for the unity of a proposition (Russell 1903, pages 83-84).

[5]In the correspondence between Russell and Frege, Frege makes it clear he doesn't think much of this argument. The following exchange concerning the nameability of classes is revealing:

Russell to Frege 10.7.1902

Dear Colleague

Concerning the contradiction, I did not express myself clearly enough. I believe that classes cannot always be admitted as a proper name. A class consisting of more than one object is in the first place not one object but many...I believe that I can therefore say without contradiction that certain classes (namely those defined by quadratic forms) are mere manifolds and do not form wholes at all. This is why there arise false propositions and even contradictions if they are regarded as units.

Frege to Russell 28.7 1902

Dear Colleague

You write that a class consisting of more than one object is in the first place not one object but many. While an ordinary class forms a whole, certain classes do not form wholes but are mere manifolds, and what gives rise to contradictions is that they are nevertheless regarded as units...If a class name is not meaningless, then, in my opinion it means an object. In saying something about a manifold, or set, we treat it as an object (Frege 1980)

The question is, do assertions exist as well as concepts? Russell alludes to his notion of an assertion being comparable to Frege's notion of a concept (Russell 1903, page 505). I think that the question of whether functions exist and whether assertions are single entities is one and the same (i.e., the same putative entity is involved). Propositions are unities, and not merely collections of terms. The origins of the difficulty lie in Moore's account of the proposition as a complex made up of concepts, which would make such an entity the mereological sum of its parts. Whence does it derive its unity?

Russell's understood propositions to be complex combinations of terms called 'true unities'. Propositions were not indicated by a simple name but were designated by a complex phrase and could be subjects of other propositions. In this, Russell was again following Moore, who denied any significant difference between a proposition and a concept, propositions merely being complex concepts (Moore 1899, page 180). Ontologically speaking, each proposition was a unity, a term, and as such, an entity. Once the unity characterizing concepts or terms is attributed to propositions, these are also recognised as entities. For Russell, because the predicative function of the concept came from an original process of reference to objects called *denotation*, it was necessary to pursue analysis further in order to find an additional constituent that would assure this indispensable function of providing the cement; of unifying the proposition. If Russell could not appeal to some sort of 'unsaturatedness' of concepts as providing this 'cement', their dual natures did suggest a plausible alternative. When the concept was named by a substantive, the concept was being used *as* a term. Yet the same concept could be used *as such* i.e. predicatively when its general name was used as an adjective. According to Russell's analysis, verbs differed from basic substantives and they resembled adjectives in that they could occur either as verb or as derived verbal noun. Like adjectives, this derivation of a verbal noun could take many forms. So putting the verb into its infinitive form, we move from "A differs from B" to "The difference of A and B". The fact that a verb, when nominalised (the use of a general name indicating a relation) can be placed in the position of a substantive and take on the role of logical subject was proof of its entity-hood. The thesis that any term could be made a logical subject meant that regardless of the form or function of the general name designating it, a relation would not be changed by being made such a subject, and it would remain an independent self sufficient entity. The difference between a conjugated verb and a verbal noun merely reflected a difference in the logical function of the relation.

In contrast to what occurred when adjectives were nominalized, nominalised verbs involved the whole proposition. While "A differs from B" expresses a proposition. "The difference of A and B" on the other, expresses what Russell calls a *propositional concept*, or an unasserted proposition (Russell 1903, page 503). Transforming the verb into a verbal noun in this way produced what could be called a nominalization of the proposition itself, allowing it in turn to take on the role of logical subject of a new proposition, as in "'The difference of A and B' is a propositional concept". Thus the propositional concept and the proposition represented the same logical reality- indeed both are unities or terms. Yet there remains an irreducible difference between the concept of the proposition and the proposition itself. From "A differs from B" to "The difference of A and B" something seems to be lost. In the sentence "A differs from B", the verb "differs" indicates a *relating relation* which effectively relates the terms A and B, generating a complete whole, a unit. In contrast, in "The difference between A and B", the substantive form "difference" still indicates the relation of difference, but this relation no longer functions as relating. Indeed, there seems to be no way of accounting for the dictinction.

Russell accounts for the unity of the proposition through the notion of assertion, which, by taking into account the specificity of the conjugated verb, explains how the unity of the proposition could arise out of its constituents. Any proposition, such as "A differs from B" can account for its unity from the assertive character of the verb used as such. On the other hand, a verbal noun, "The difference between A and B", loses this assertive function and figures in the proposition as merely *considered*. Russell regarded relations as well defined terms, i.e., as independent entities which were given the crucial task of acting as the 'cement' or 'glue' linking other terms, which also were independent, and the 'glue' is to be found in examining the specific role of the verb in the proposition.

The distinction between the concept of the proposition and the proposition itself was further invoked to explain how propositions could be brought together to form inferences without making those connections necessary. The view that all propositions were necessarily connected flowed inevitably from the monistic assumption that their relations were governed by the analytic relation of part to whole. Rather, propositions are related via the relation of material implication. This logical operation relates propositions regardless of their particular content. Because the relation of implication was external to the propositions involved, the generation of an infinite number of inferences was not evidence of a vicious regress, but reflected an informative contribution to our understanding.

Like any other object of knowledge, a proposition for Russell did not depend for its being on the judgement of a knowing subject. Far from being the result of such an act, it is rather presupposed by that act. So the truth of a judgement would depend not on the subject, but on the proposition, specifically, on the sorts of connections between the concepts constituting the proposition. This makes its truth an intrinsic property. Russell says that a proposition is true like a rose is red (Russell 1904, page 75). Russell's account leaves Truth as an irreducible residue found in asserted propositions. That is, truth is a logical property that intrinsically characterises some propositions. What is true is true absolutely and completely, independently of us. A true proposition is a complex that stands in a certain relation to the concept of truth; a false proposition is a complex that stands in that same relation to the concept of falsehood; and the concepts truth and falsehood are simple and indefinable.[6] In short, then, Russell's problem was

[6]There has been a tendency to see this as opening up a problem for Russell over false propositions. The suggestion is that if the unity of the proposition is provided by the verb as verb, by relations actually relating, then it is difficult to see how a proposition could be false without positing the existence of objective falsehoods. And some will argue that in a false proposition a relation cannot actually relate, that it loses its source of unity and cannot be a proposition.

now to find a principled way of isolating whatever constituent or entity was responsible for a proposition's unity. In the case of our contact with such entities as propositions, such contact is possible because the constituents are brought together in some way. How can the independent constituents making up the proposition effectively combine these constituents to form a unified whole? If the relation R is defined solely as a term, how do we account for the relational fact that xRy? On the other hand, if the proposition xRy is to be considered as a whole, how is the verb which picks out the relation as an autonomous entity to be picked out? Years earlier, Frege had dealt with a comparable problem by his doctrine of the 'unsaturatedness' of concepts. This will now be examined up close.

4.4 Frege and the Unsaturated Function

Frege found in *concepts* the sort of 'cement' needed to assure the cohesion and unity of the proposition.

> In the sentence 'Two is prime' we find a relation designated: that of subsumption ... This ... creates the impression that the relation of subsumption is a third element supervenient upon the object and the concept. This is not the case: the unsaturatedness of the concept brings it about that the object, in effecting the saturation, engages immediately with the concept without the need for any special cement. Object and concept are fundamentally made for each other, and in subsumption we have their fundamental union (Frege 1984).

For Frege, taking concepts as general names ignored their predicative dimension, and was evidence of a deep confusion between concept and object. Rather, concepts were quite distinct from objects: "Not all the parts of a thought can be complete; at least one must be 'unsaturated' or predicative; otherwise they would not hold together" (Frege

1984).

The distinction between concept and object arose from Frege's use of function-argument analysis in logic. Its root lay in the idea that any functional expression contains 'gaps' where the names of the arguments are inserted. He said that, when an expression such as '$2 \cdot x + x$' is used to designate a function, the function is actually designated by what is present over and above the letter 'x'. Frege suggests that this might be indicated by writing '$2 \cdot (\ ) + (\ )$'. He calls functions, and signs for functions, incomplete or 'unsaturated'. The result of completing a function with an object (or argument) gives us the value of the function for the argument. Just as a function was called 'incomplete' and in need of 'supplementation', a concept was simply a special kind of function for Frege– a function whose value was always a truth value. Frege's early writings frequently refer to the argument or value of the function as the content (inhalt) of an expression.

The process of concept formation was dependent on that of judgement; "I start from judgements and their content, not from concepts...I allow the formation of concepts to proceed only from judgements" (Frege 1984). That is, he assumed that content is something contained within the object(s) of a judgement. Concepts are generated by making judgements about these objects, as we direct judgement toward this or that object. It is function-argument analysis that allows a propositions' 'conceptual content' to be 'carved up' in different ways. Frege's 'carving' metaphor should not obscure what was the fundamental point to his theory of judgement; and this was that judgement was directed at what was given independently of any psychological attitude we may have towards the objects concerned. Judgement is not creative in the sense that concepts are dependent on mental acts. Concepts were not, in Frege's picture, some sort of occult 'content' coming and going with each act of judgement. Judgement is more akin to rearranging the furniture than assembling that furniture from scratch. But if Frege did not view concepts as products of the mind, he did view them as a function of, and contained within, the kinds of objects that were the focus of judgments.

Objects were designated by complete expressions (which may be names, pronouns and/or definite descriptions). and these were used to fill the gaps in incomplete expressions which stood for concepts. Once the gap(s) of the incomplete expression is (are) filled, the completed expression is in turn available to fill the 'gaps' contained in an expression (function) found at a higher level. These higher-level predicates are propositional functions and quantifier expressions, which are in turn saturated by sentences, rather than names. Frege then established a hierarchical ordering of these higher level predicates (quantifiers and propositional functions) so as to make some sort of distinction between which expressions are functions and which are names for saturated entities.

Frege felt that allowing nominalised concept expressions opened the possibility of confusing concepts of different levels. The strategy he employed to counter this problem was to draw a distinction between formal implication and the hierarchy of concepts. Just as objects were said to 'fall under' concepts, a similar relation held between concepts: one concept could "fall under" another concept. This relation is to be seen in the way that concepts can be ranked in a hierarchy, in the sense that only an $n-1$ level concept could fall under an n-level concept. Letting F be a 1^{st} level concept. then by $x : Fx$ we would mean the extension, i.e. the class of objects falling under F. Similarly, letting ϕ be a 2^{nd} level concept, then by $F : \phi(F)$ we mean the class of concepts falling under ϕ. Classes were objects, i.e. no stratification of extensions was allowed that corresponded to the way in which concepts were stratified. All objects, including classes, were of the same level: the level 0

Frege thought it was a mistake to analyse the following two propositions;

1. The number 20 can be represented as the sum of four squares

and

2. Every positive integer can be represented as the sum of four squares

as consisting of the function "is represented as the sum of 4 square numbers", and as arguments "The number 20" and "Every positive whole number":

> We can discern the error of this view from the observation that "the number 20" and "every positive integer" are not concepts of the same rank. What is asserted of the number 20 cannot be asserted in the same sense of [the concept] "every positive integer"; though, of course, in some circumstances it may be asserted of every positive integer. The expression "every positive integer" by itself, unlike [the expression] "the number 20", yields no independent idea; it acquires a sense only in the context of a sentence (Frege 1879).

In other words, while the expression "the number 20" designates an object and is saturated, the expression "every positive integer" was incomplete and received meaning only in the context of a proposition. Its grammatical form notwithstanding, proposition (2) does not concern objects such as the positive whole numbers. Rather, it concerns the inclusion of the concept characterising those objects within the concept "represented as the sum of 4 square numbers'. And this was the case with every universal proposition or formal implication. That is, a quantifier phrase such as *everything* stood for a second level concept within which first-level concepts fall, rather than as an operation that transforms the subject concept into a denoting expression, as Russell would see the matter in *Principles*. To do otherwise was to think of concepts *as terms*, and this was to ignore their predicative (unsaturated) dimension. He writes:

> The behaviour of the concept is essentially predicative, even when something is being asserted about it; consequently it can be replaced there only by another concept, never by an object. Thus the assertion that is made about a concept does not suit an object. Second-level concepts, which concepts fall under, are essentially different from first-level concepts, which objects fall under. The relation of an object to a first-level concept that it falls under

> is different from the (admittedly similar) relation of a first-level to a second level concept. (To do justice at once to the distinction and to the similarity, we might perhaps say: An object falls under a first-level concept; a concept falls within a second-level concept.) The distinction of concept and object thus still holds, with all its sharpness (Frege 1892, §201).

The problem was then to explain how we move from second level concepts within which first level concepts fall to those objects denoted by singular terms as represented by such statements as 'The number 20 is represented as the sum of four square numbers'. Frege's solution was to interpret these nominalised predicates as denoting the extension of the concept that the predicate otherwise stood for. It was with this purpose in mind that Frege incorporated a device for the representation of the extension of a concept. By applying the smooth breathing operator to a symbolic term,

$$\acute{x}\ \phi(x)$$

a new expression was formed which denoted the extension of the concept ϕ, i.e. the class of things that are ϕ. [7]

One well known difficulty with Frege's proposal concerns the paradox of the concept *horse*. While the distinction between concepts and objects keeps the proposition from being a mere list, this sharp distinction prevents concepts from being referred to by a proper name. For proper names pick out objects. The expression 'the concept horse' is a proper name, and as such may only refer to an object, not a concept. Can one name an unsaturated entity? This is a problem in that in second order objectual quantification we wish to quantify over concepts, which is hard to do if such 'things' have an unsaturated nature. Here was the heart of the disagreement between Frege and Russell.

[7] There remained for Frege to explain how a complex name such as "The number 20" could be formed from a concept. In order to solve this problem Frege introduced the function ξ which mapped the extension of that concept onto that object itself when the description was proper. For those cases where the description is improper, that is, where the description lacked a referent or failed to uniquely determine a single referent, the value of the function was the argument itself.(Frege 1893, §18).

According to Frege's context principle, the meaning of separate, detachable constituents of a proposition can only be found through an understanding of their role in larger units. So, understanding a concept and an object requires understanding the role each plays in a complete thought. Russell on the other hand regarded a constituent of a complex as independent of the complex in which it is found. Russell would argue that Frege's hypothesis that concepts cannot be logical subjects is self-refuting. A concept is first of all the meaning (Frege would say the *Bedeutung*) of a grammatical predicate, of an adjective such as 'human' or 'mortal'. The question then is whether or not the names of the corresponding abstract objects designate these same concepts or something else. If you hold that *human* is a concept, and as such cannot be a logical subject, you seem thereby to treat *human* as a logical subject. For Russell, any entity whatsoever can be argument for the unrestricted variable in a propositional function, regardless of context, and any entity could be the logical subject of any proposition.

4.5 Conclusion

Russell regarded the ontological status of all sorts of entities as unproblematic. The general test determining what was an entity from what was not was the idea that *any* term could be made into a logical subject. This principle comes into conflict with the idea that some terms have the role of unifying collections of other terms. So, for example, in the *Principles* Russell regarded propositions as entities containing an assertion, the element responsible for the unity of the propositional complex. Yet the unity inherent in a proposition was somehow lost when this proposition was made a logical subject. It was certainly paradoxical that this unifying element could not itself be made into a logical subject. So there seemed to be a threshold which, when crossed (as when a proposition was made into a logical subject), generated a very elusive entity. The search for this paradoxical entity ensured that propositions would remain at the forefront of Russell's efforts to find a remedy for the paradoxes without sacrificing such a fundamental first principle of logic as that of the unrestricted variable.

So the paradoxes lead to doubts as to the status of certain objects. Among these were propositional functions, entities that expressed assertions, binding together propositional constituents into a unity. But Russell also had a conception of propositional functions as denoting complexes representing generality. The idea was that the entities satisfying a propositional function formed a class, which could in turn be taken *as one*, as a unity. The link between these conceptions was that propositional functions provided a binding element, a relation which brought together independent objects to form unified wholes. This explains why Russell would begin looking for a solution to the paradoxes here. How then are propositional functions to be understood given the paradoxes? Russell was well aware of the problem he faced:

> But in general it is impossible to define or isolate the constant element in a propositional function, since what remains, when a certain term, wherever it occurs, is left out of a proposition, is in general no discoverable kind of entity. Thus the term in question must be not simply omitted, but replaced by a *variable* (Russell 1903, page 107).

Because propositional functions are said here to contain variables, one possible interpretation has been to construe them as linguistic[8].Nevertheless, it remains that in the *Principles*, the variable is an entity, albeit very complex (Russell 1903, page 93) . This can be seen by describing in linguistic terms what Russell took propositional functions to

[8]Russell says as much: "Whitehead and I thought of a propositional function as an expression containing an undetermined variable and becoming an ordinary sentence as soon as a value is assigned to the variable" (Russell 1959, page 92). But in a paper written shortly after the *Principles* he would say: "As regards [what remains of the said unity when one of its terms is simply removed, or when (if it has more than one term) two or more of its terms removed], what remains can in some cases be regarded as a single object, of the sort which I call an *assertion*; in other cases we get a mere aggregate of objects, and the original unity cannot be regarded, in any straightforward sense, as compounded of this remnant and the removed term. When the remnant is an assertion we may regard the assertion as the ϕ in the notation ϕx. But in other cases, the ϕ has by itself no assignable meaning." (Russell 1993, page 687)

be. Accordingly, a propositional function is what remains of a sentence after all proper names have been removed. This remaining expression should be seen as a kind of schema representing generality. So the letters for variables are abbreviations for the denoting phrase "any term", and this expression has as meaning a denoting concept. This concept in turn denotes the totality of entities (or some sort of combination of such entities). All of this is to say that the only kind of class defined by such a function is the class as many. And of course this meant a revision of the kind of thing which could function as a logical subject. By admitting plural logical subjects into his ontology (Russell 1903, page 516) the question then became this: to what extent does this change in the kinds of things which can be logical subjects require a corresponding change in the kinds of things which imposed some sort of ordering of them?

Philosophers have generally been in agreement that Russell's paper, *On Denoting*, would mark a profound shift in his views. According to this consensus, Russell rejects the analysis of definite and indefinite descriptions which is given in the *Principles* (according to which such expressions have denoting concepts as their meaning). In this paper, Russell provides detailed contextual definitions of these denoting expressions through the use of propositions containing variables and propositional functions, claiming they are equivalent to those propositions. But the connection between the ontological reductions undertaken in *On Denoting* and the paradoxes has been underplayed in the secondary literature. The result has been a certain interpretation congenial to the analytic school's nominalism, whereby the ontological reduction at work springs from a desire for ontological economy, in particular, an attempt to prune that ontology of 'abstract entities'. Thus the theory of descriptions is seen as an account of semantic content, of a study of how information about the world can function independently of the objects that may be found there. Such a view takes language to be used not to pick things out in the world, so much as to find different ways of characterising them.

Russell's theory of descriptions, however, did not emerge from a concern with language per se. Rather, the theory arose as a response to the paradoxes. That is, Russell's program of eliminating denoting concepts as the meanings of denoting phrases is best understood as part of a broader attempt to avoid the paradoxes. Indeed, to the extent that *On Denoting* is seen to mark a shift in Russell's thinking about the variable and denoting concepts, the textual evidence is somewhat ambiguous. The following chapters will argue that Russell was aiming at a kind of reduction of denoting concepts to the denoting concept *any*, or perhaps even more fundamentally, the denoting concept *anything*. This will require an analysis of a number of papers written shortly after the publication of the *Principles*, where Russell attempts to draw out the distinction between meaning and denotation, and then is led to reject this distinction primarily due to problems over paradoxes. What does emerge from a study of these papers is the sense that this entity must include a representation of complexity. The focus of the problem became one of determining whether an entity that denoted could represent this complexity and yet remain autonomous and independent.

Chapter 5

Incomplete Symbols

5.1 Introduction

The standard account of the philosophical significance of Russell's paper *On Denoting* maintains that Russell inaugurated a sort of linguistic, perhaps even nominalist, turn. Russell's advance is taken to be the recognition that syntactic units in a sentence do not all have ontological import. By providing contextual definitions for definite descriptions, for example, Russell showed that they need not be taken as logically significant in and of themselves. The standard account goes on to claim that Russell intended to prune philosophy of a bloated ontology and that what emerged from his work at the time was a set of eliminative techniques. Russell is then credited with a critique of the Fregean doctrine of Sinn und Bedeutung, although with the result that he became a victim of systematic "use-mention" confusions. This view ends with attacks on the foundational work in the *Principia* as incoherent, and better understood when any commitment to properties is abandoned (and the propositional functions in that work treated as open sentences).

The trouble with this account is that it obscures the role of the paradoxes in driving these developments: what is lost sight of is the intimate connection between Russell's concerns regarding the foundations of mathematics, along with the emergence of the paradoxes, and the resulting work on denoting complexes. Until these connections are made explicit, much of what transpires in *On Denoting* and what follows after it can only be veiled in obscurity, and the historically inaccurate picture of Russell as an empiricist bent on a nominalist reduction of abstract entities will persist.

5.2 Meaning and Denotation

In the *Principles*, Russell drew a distinction between things and concepts.[1] Concepts were terms with a two-fold nature. At times they occurred in a propositional complex as entities, but at others they occurred as meanings. Now, when these concepts occur as meanings, their function seemed to be to combine other terms into a unified complex. And in this role they were not constituents of this complex.[2] On the other hand, denoting phrases occurring in the subject or term position indicate denoting concepts, which are constituents of the proposition, denoting objects that are not constituents of the proposition. These denoted objects were themselves often complexes, called plurals, but not always, i.e., definite descriptions picked out individuals. As an entity in a complex this concept's role was to designate an object. Denoting concepts are peculiar in that they are meanings taken to occur as entities in a proposition.

Denoting concepts were very important for Russell's foundational project in that they secured reference to infinite complexes through finite means. This sort of reference is achieved via an intension, that is, some property that the members of this infinite class have in common. We are not 'acquainted' with an infinite class; rather our access to it

[1]Supra p. 39

[2]Remember that the constituent occurring as meaning unites what would otherwise be the mere juxtaposition of entities. Russell's disagreement with Leibniz is over Leibniz's view that whatever united substances was, in reality, a fact about each individual substance. Russell wants to say that this unifying constituent is neither reducible to some fact about each individual in the collection, nor merely ideal. Complex meanings are not the work of the mind, and indeed the 'synthetic unity of apperception' requires a unified something in order to be 'apperceived'. See (Russell 1994c, page 316)

is conceptually mediated through a property, an intension. To use Russell's later way of putting it, this is how we come to know things about objects with which we have no acquaintance. The point to drawing a distinction between meaning and designation is to allow intensions that do not succeed in picking anything out:

> It is necessary, for the understanding of a proposition, to have acquaintance with the meaning of every constituent of the meaning, and of the whole. It is not necessary to have acquaintance with such constituents of the denotation as are not constituents of the meaning. It is thus that definitions proceed. They give a known meaning, and enable us to make propositions about what the meaning denotes, although we may have no acquaintance with this something. (Russell 1994e, page 307)

One of the conclusions of the last chapter was that, in some sense, Russell thought that the contradiction arose from treating every constituent in a complex as being fundamentally the same, as an individual. Because a complex was a collection of individuals with a structure, Russell began to suspect that the problem lay in not drawing out the differences between the individuals in a complex, and how they were combined to make up that complex. The distinction between meaning and denotation, which assumes such a prominent place in the manuscripts written after the *Principles*, is best seen as an attempt to come to grips with *structure*, by clarifying distinctions among the components of a complex.

A denoting concept which is about a class was to be formed by combining a propositional function and the notion *such that*. So it seemed reasonable to assume that determining the legitimacy of the class as one in any particular case would depend on specifying the right restrictions on propositional functions. The idea was that a distinction had to be drawn between functions occurring as meaning as opposed to an entity. When a propositional function occurred as entity, then it was said to denote a class. And so Russell was led very naturally to believe that the contradiction could be avoided through restrictions on the comprehension axiom and by making the distinction between functions as meanings and functions as entities in a principled way.

In the manuscripts written immediately after the discovery of the contradiction, Russell begins by restating fundamental principles of philosophical grammar similar to those found in the *Principles*: Proper names are used to refer without meaning. Verbs and adjectives have meaning but do not denote. Some expressions (including definite descriptions) have both meaning and denotation. An empty proper name, such as Apollo, is in fact a disguised description. Russell provides as example the sentence "the table is black", where *table* forms a constituent of the meaning, and also part of the denotation. (Russell 1994d, page 284) The denotation will remain invariant across the substitution of coextensive constituents in the meaning, though the meaning will change. Some phrases have meaning but no denotation. The meaning is a complex concept. Because truth or falsity has to do with what the sentence denotes and not with what the sentence means, "The present king of France is bald" is neither true nor false. It is axiomatic that the subject of a proposition is part of the proposition.

Although we have no presentation of an instance of a concept on many occasions, we do have a presentation of its meaning. Where we cannot give an instance of such a concept as *just this*, we may describe such a purported entity "by means of a collection of characteristics of the combination of which they are conceived to be the only instance" (Russell 1994d, page 285). Thus the complex aRb will have a meaning and (perhaps) a denotation. The meaning is to be understood as a separable entity. Take the phrase "the present Prime Minister of England". Each word will designate an object. However, the whole phrase designates something over and above the designations of each word in the phrase, for these will certainly not be constituents of Arthur Balfour (Russell 1994c, page 320). Rather, England etc, are part of the *meaning*.

The distinction between meaning and denotation is to allow Russell to avoid ontological commitments to certain entities, such as *the round square* or *the even prime other* *than two*. We achieve an understanding of such phrases without having to posit the existence (or subsistence) of the supposed denotation of such a phrase. Definite descriptions will all have a meaning but not all of them will have a denotation. So it seems natural for Russell to have looked to meanings, and a sharpened distinction between meaning and denotation, to find a solution to the paradoxes. Because Russell had assumed that the entities responsible for the complex's 'unity' were to be included among its individual constituents, it seemed reasonable to begin by taking these entities and somehow pushing them out of those complexes they were uniting. The heart of Russell's analysis of the paradoxes is that they they arise from treating every constituent in a complex as being of the same ontological level.

> For our purposes, *all* complexity is complexity of meaning, and whatever can be denoted may be regarded, when denoted, as simple, even when what is denoted is a complex meaning: that is, the complexity involved is to be of meaning, not of denotation (Russell 1994d, page 288).

Thus Russell's general strategy was to refine his analysis of complexes in terms of intension and extension. A complex was "determined by its constituents together with their mode of combination; it is not determined by the constituents alone" (Russell 1994a, page 98). A complex's unity stems from meaning. In 'A is greater than B', the complex is made up of the constituents A, is greater than, and B. But what is essential is the way these constituents are combined. For the complex 'A is greater than B' is a different complex from 'B is greater than A'. This additional element is the way in which the constituents are brought together. Given this framework, it is not implausible to surmise that Russell found in this distinction the beginning of a solution to the paradoxes:

> The mode of combination of the constituents of a complex is not itself one of the constituents of the complex. For if it were, it would be combined with the other constituents to form the complex; hence we should need to specify

> the mode of combination of the constituents with their mode of combination; thus what we supposed to be the mode of combination of the constituents would be only a mode of combination of *some* of the constituents. In short, in a complex, the combination is a combination of *all* the constituents, and cannot therefore be itself one of the constituents (Russell 1994a, page 99).[3]

To include this mode among the primary constituents meant generating a vicious regress. Russell expresses this worry by insisting that including the mode of combination among the basic constituents added to the complex in a way which made no difference to it, i.e., added to it while keeping it unchanged. Take the complex aRb. The relation R occurs in it as entity. But it also functions there as relating. It is this relating function that cannot be counted as a fundamental constituent. This relating relation is the common element bringing the constituents together to form a complex. The point, then, seems to be that treating this common element as an entity in the complex would add nothing to it; that is, the complex is aRb, and not aRb + the *glue*. That, whatever it is, is a different complex. Treating the glue as some*thing* one could fasten on to R would add a new constituent to the original complex, creating a new complex. The difficulty however would stem from trying to specify the ontological status of this unifying constituent and trying to isolate the meaning of a denoting complex as a logical subject. Denoting complexes posed a problem for Russell in that he had to clarify how these constituents could be both entities and meanings, and similarly, how propositional functions were occurring as meanings and as entities. (Presumably, the entity occurrence of such a function would be predicative in the sense of defining a class as its extension).

[3]Russell, remarks that "A mode of combination, like everything else, is an entity; but it is not one of the entities occurring in a complex composed of entities combined in the mode in question" (Russell 1994a, page 98)

5.3 Russellian Complexes

Any constituent in a complex which can simply be varied is called a term. What remains constant in a complex as an entity is varied is the meaning. If this distinction was at the heart of Russell's strategy in dealing with the contradiction, the problem was finding a way of mentioning this unifying constituent. In any complex, that which does the unifying can only perform this function *as meaning.* As such it is not a term in the complex. Thus, what is needed is a way of denoting a *meaning.* This is done by using a proper name (which has no meaning) to denote a function (an object occurring as meaning in a complex). This constituent may be designated only by means of a proper name, in which the meaning is denoted. Once denoted, it becomes an entity, though not a term of the original complex. Russell's aim was to distinguish the different kinds of roles entities played in a complex.

Thus it remained the case that any entity in a complex could be replaced by any other entity, with this position marked by the occurrence of an entity variable, while what remained invariant across substitution of terms could not be marked by the presence of a variable for entities. For this invariant constituent was the mark of meaning, and some restrictions would have to be placed on it. Russell held the hope of finding some recourse that would keep entity variables as unrestricted, while finding precise conditions under which a complex's structure could be treated as a logical subject.

> Every complex has meaning and being. *Qua* meaning, it is not one entity, but a compound of several. A complex may occur in two ways, as meaning or as entity. Complexes may differ as meaning without differing as entity. What the complex *is* is what we have called its denotation. There is no entity which *is* the complex as meaning, because the complex as meaning is not one entity...Thus we cannot use a single letter such as C, to stand for "any complex", because a complex is only distinguished by *structure*; as far

> as its being is concerned, it is not different from an entity in general. Thus any general notation for a complex must have structure. (Russell 1994b, page 366)

The paper *On Fundamentals* attempts to solve this problem with the notion of an entity's position in a complex. There, $(C)(x)$ is to indicate a complex containing the entity x. The mode of combination is indicated by $C\ x$ (Russell 1994b, page 360). A distinction is drawn between meaning position and entity position in a complex. The idea must be that constituents endowed with a dual role, either as meaning or as entity, will display these roles depending where they are put in a complex. Philosophically, this has the advantage for Russell of making such a role depend on an external relation (i.e., its position in a complex) as opposed to some property internal to the entity involved. Presumably, by putting a denoting concept in a meaning position, the concept will function as such (i.e., as meaning). When the concept is put into entity position, the concept will function as entity (i.e., will denote). In effect, in entity position, any extensionally equivalent denoting concept may be substituted without affecting the truth of the whole. Thus when the denoting complex occurs in entity position, the propositional complex is not about the denoting complex, but rather, it is about what the denoting complex denotes. Again, this strategy is to resolve the paradoxes by drawing a sharper distinction between the meaning of a complex and its denotation. For the paradoxes arise by the improper substitutions of the one for the other, as when a meaning is treated as an entity.

The whole point of positing meaning and entity positions in a complex is to avoid requiring these roles to be performed by separate entities. What Russell has been entertaining (with denoting concepts) is one entity with two sides; meaning and being. When a complex occurs as meaning, its complexity is essential to it, but when it occurs as being, it is not.

> When complexes occur as *meaning*, their complexity is essential, and their constituents are constituents of any complex containing the said complexes;

> but when complexes occur as *entities*, their unity is what is essential, and they are not to be split into constituents. Hence generally: When a complex A occurs in a complex B, if A occurs as *meaning*, its constituents are constituents of B, but if it occurs as *entity*, its constituents are not constituents of B (Russell 1994b, page 373).

As meaning, the constituents of a complex are united. As being, they comprise a single thing. Meaning and being seem to be surface forms of an underlying reality. But is this underlying reality itself a constituent of the complex? And, if so, what work does this entity do here? The trouble with denoting concepts is that they do not allow a sharp distinction between meaning and denotation. Such an entity was to have two-sides, which would lead one to suspect that this entity could be specified independently of them. However, the problem appears in identifying this element as a possible logical subject. Then if there is no unifying content which binds the meaning and denotation as parts of a single underlying thing, then better to do without it. The conclusion Russell draws is that, if the unifying constituent in a complex is separate from what it unifies, there can be no object which performs both functions i.e., that of unifying and that of being the object of that unification. Thus Russell rejects denoting concepts because they are entities with 'two sides'.

While Russell was especially concerned to eliminate the ontological ambiguity which seems inherent in the 'two entity' view, this problem is not restricted to denoting concepts. Relations (as both relating and as entities) are affected by the same sort of problems. In the complex aRb, a and b are bound by the sense of the relation R. It is this additional ingredient acting on a and R and b which generates distinct complexes.[4] If the way the basic constituents of a complex are combined cannot be one of the basic constituents of

[4]Another way, perhaps, of understanding the role of meaning is by way of explaining how the same collection of atomic constituents can generate different complexes. Nelson Goodman expresses this by insisting that different aggregates be distinguished in terms of content: "The Nominalist denies that two different entities can be made up of the same entities" (Goodman 1956).

the complex (which Russell views as paradoxical), some way must be found of locating it outside the complex and accounting for its ontological status. Maintaining this unifying component as an entity 'out of' the constituents it united generated the problem of its treatment as logical subject.

Meaning had been used in the *Principles* to capture the idea of identity of content. For Russell this relation united the atomic constituents of a complex into a whole, and was thought of as an individual, on an ontological par with the other atoms. As the reader will recall Russell stressed the foundational importance of recognising aggregates with a being, or individual nature, which transcended its parts (Russell 1903, page 141). The resulting change in views was drastic, for

> We shall not now say that e.g. "x is a man" *is* the mode of combination involved, and only *denotes* its instances; we shall say, on the contrary, that "x is a man" *is* any one of its instances, though not a mentionable one. That is, we shall adopt the view of ambiguous denotation *without* meaning. We shall recognize a *similarity* between "Socrates is a man" and "Plato is a man", which similarity is of a special kind, namely the kind which constitutes them instances of one type, of which an ambiguous instance is "x is a man". But they are not instances of one type because there is a type to which they both belong; on the contrary, the type is constituted by the relation of sameness of type (Russell 1994b, page 401).

Treating this common element now in terms of similarity reflects deep changes in Russell's thinking and marks the beginning of his "retreat from Pythagoras" (Russell 1959, pages 154-158). Indeed, this 'common' element is no longer being expressed in terms of an identity to another term. What is less clear is that Russell now regarded any fact about a complex entity as reducible to a fact about the individuals making up the complex. Whatever was affecting the unity of a complex is not now deprived of an individual nature. Although Russell engaged in some (uncharacteristic) 'hand waving' over this, he clearly wanted to count these structures as independent of our conceptual schemes. Yet accounting for this individual nature would prove to be very difficult.

In the "dogmatic summary" Russell offers at the end of *On Fundamentals* he remarks that denoting phrases may still denote if their substitution for a name in a proposition yields the expression of a proposition whose truth value remains unchanged (Russell 1994b, page 408). Indeed in such a case the denoting phrase can be regarded as a name of *this* entity. It remained to specify the conditions under which a definite description behaves as a name and when substitutions are admissible. In extensional contexts the description can be substituted for the name.

Russell's requirement that symbols for entities be single letters, but that symbols for objects reflect the structure of their referents, was crucial and problematic. But the seed for what would become the theory of descriptions is planted firmly here. The idea that emerges is that what functions syntactically as a name need not have any semantic correlate. What the syntax can do is reflect the ontological composition of complex objects. Complex objects are built up from individual entities and these entities can be combined in any number of ways. To be sure, not all such combinations may be actual. But the important thing is that the resulting combinations cannot be designated by a proper name, for this would obscure the ontological dependence of complexes upon their constituents. Indeed, by the fall of 1905, Russell felt that he had found the general form for all known contradictions:

> The contradictions result from the fact that according to current logical assumptions there are what we may call *self-reproductive* processes and classes. That is, there are some properties such that, given any class of terms all having such a property, we can always define a new term also having the property in question. Hence we can never collect *all* the terms having the said property into a whole; because, whenever we hope we have them all, the collection which we have immediately proceeds to generate a new term also

having the said property (Russell 1906b, page 144).

Here we have finally come to the heart of the matter: Russell's objection is that self-reproduction generates *a new entity*. Given this diagnosis, a strategy slowly took shape in Russell's mind whose purpose was to allow reference to plural logical subjects without generating these *entitates occultae*. However, it remains obscure why reference to such abstract entities has this puzzling consequence and Russell is not particularly helpful here. Clearly, however, Russell now thinks that certain structural features of complexes cannot be autonomous entities.

What Russell needed was a way of getting symbols to reflect the complexity of the objects they picked out, or designated. Paradoxes and contradictory 'entities' do not populate the world, and if our language indicates that they do, then so much the worse for language. The means by which we named and described the world would have to be regimented in such a way as to preclude expressions indicating such things as round squares or odd even numbers. Russell needed a way of ruling out expressions which putatively designated these illegitimate beings, but in a way which didn't rely on conceptual resources laying beyond the domain of logic. The project of a systematic regimentation of language would develop slowly, and the final result would not appear until the publication of the *Principia Mathematica*. But the first crucial step would be Russell's theory of definite descriptions.

5.4 *On Denoting* and Definite Descriptions

In the *Principles* denoting expressions such as "the author of Waverely"indicated denoting concepts which provided Russell with a way of accounting for the informativeness of identity statements. Russell begins by distinguishing two different kinds of identity statement. In the first case we have "Scott is Scott", and this expression simply asserts the identity of a proper name with itself. Here the name functions as a symbol designating a single object, Walter Scott, and such a statement is analytic and for Russell, uninformative. In the second kind of case, an expression such "Scott is the author of Waverley" asserts the identity between a proper name and a denoting expression. Substituting the denoting expression for the proper name renders this expression non-analytic and confers on it cognitive value.

According to the principle of the substitutability of identicals, two names for the same object can be substituted for each other in any true proposition. Or, if two expressions can be substituted for one another without changing the truth-value of any sentence in which they appear, then the things that those expressions designate are one and the same. In such a case we say that the expressions are intersubstitutable salva veritate and Russell took this to be a fundamental principle of logic. Problems with this fundamental principle are apparent from counterexamples like as the following. Consider these three sentences:

1. George IV wished to know whether Scott was the author of *Waverley*

2. Scott was the author of *Waverley*.

3. George IV wished to know whether Scott was Scott.

Now we know that Scott is the author of Waverley. If King George IV wants to know if Scott is the author of Waverley it's doubtful whether he's asking a question about the law of identity. Just as for Frege, the problem here is to isolate the origin of this additional cognitive information. That is we need some explanation for the apparent failure of the inference from (1) and (2) to (3).

Another puzzle involves apparent violations of the law of the excluded middle. Consider the following two sentences:

4. The present king of France is bald.

5. The present king of France is not bald.

If we enumerate all the things in the world that are bald, the present king of France will not be among them. So it is false that the present king of France is bald. On the other hand, if we enumerate all the things in the world that fail to be bald, we will not find the king of France among them either. So it equally appears that it is false that the present king of France is not bald. That is, it seems to follow, in violation of the law of the excluded middle, that the present king of France is neither bald nor not bald. Now consider:

6. The round square does not exist.

(6) seems straightforwardly true; but just how can it be true? One's initial temptation will be to think of (6) as the denial of a sentence in subject predicate form. The grammatical structure of (6) appears to be similar to that of 'Socrates was not a fool.' This last sentence is true if and only if a certain object-viz., Socrates- fails to have a certain property-the property of being a fool. Just so, one might think, for (6). (6) is true just in case a certain object-viz., the round square-fails to have a certain property-viz., the property of existence. However, in order for an object to lack a certain property, there must first be that object. If the property putatively in question is the property of non-existence, then nothing can be said truly to have that property. For if an object lacks the supposed property of existence, there is no object to begin with. If not, then (6) does not, after all, truly predicate a property of an object. Yet evidently (6) does express a truth. The problem is to explain just what truth it expresses and to say what that truth is a truth about.

Russell's solution to these puzzles in the *Principles*relies on the theory of denoting concepts. He regards these propositions as informative when identity is treated as a relation involving two separate entities, one being the subject, the other being a denoting concept whose verbal expression begins with the definite article, and which denotes a unique individual. So the statement

7. Edward VII is Edward VII.

is true but uninformative. On the other hand

8. Edward the VII is the King.

asserts a true identity whose informativeness lies in the conceptual content found in "the king". (8) becomes informative because on the one side the term itself is present. on the other side the denoting concept is present.

This theory has affinities with Frege's. In *On Sense and Reference* Frege splits the content of an expression into two parts: its 'sense' (Sinn) and its 'Reference' (Bedeutung). According to Frege. though there is more to the full meaning of a sentence than just its sense, (there is also force and tone) these mental associations are subjective. and therefore they play no part in communication. In communication we can only convey objective things. things which are common to everyone we can communicate with. It is the objective part of meaning that Frege calls sense. Sense determines reference. Two expressions with the same sense have ipso facto the same reference, although this does not hold the other way around. An expression such as "The king of England" is a name referring to an entity. Its sense is what Frege calls the mode of presentation of that entity. It is the way the reference is presented. It is of great importance that it is possible to be quite familiar with the sense of a name without knowing what its reference is. Sense is the 'mode of presentation', but familiarity with the reference of any given expression is merely a possibility and may not be assumed. Sense is merely a criterion by means of which reference may be determined under various circumstances. How does Frege's theory solve the paradox of identity? Take the following two statements:

9. The morning star is the morning star.

10. The morning star is the evening star.

The senses of 'the morning star' and 'the evening star' consist in the two different ways in which the two expressions determine their reference. For the morning star this could be made explicit as 'the brightest heavenly body in the eastern skies at dawn', and for the evening star it could be 'the brightest heavenly body in the western skies at sunset'. Once again it is quite possible to be familiar with the senses of the two names without knowing to what heavenly bodies they refer. So this gives us a simple explanation of the cognitive difference between 'The morning star is the morning star' and 'the evening star is the morning star'. Sentence (10) is true just in case the reference of 'the morning star' is the same as the reference of 'the evening star'. And that this is so is quite clear independently of what the reference in fact is. So to know that (9) is true is an a priori matter. Sentence (8) is true just in case the reference of 'the morning star' is the same as the reference of 'the evening star'. Whether this is the case cannot be determined solely on the basis of the meanings of the two names. It is necessary to know exactly what the references of the two expressions happen to be. so the truth of (10) was only apparent once astronomers discovered that they do both refer to the same celestial object, namely, the planet Venus.

Frege was then able to address a number of problems that arise with intensional constructions, in particular, the problems surrounding failure of substitutivity in quotational and propositional attitude contexts. Quotation marks and the sentential operator 'that' associated with propositional attitude operators (e.g. Bertrand believes that'), according to Frege create an *oblique context*, in which expressions take on a different referent from their customary referent. Whereas the expression 'Hesperus' customarily refers to the planet Venus, when occurring within quotation marks, as in the sentence 'The expression 'Hesperus' is a sequence of eight letters', it instead refers to itself. Such is the case when someone's remarks are quoted in "direct discourse," that is, when reporting the very words used by the speaker, as in 'Bertrand said 'Hesperus appears in the evening'. Analogously, when occurring in a 'that'-clause in a propositional- attitude attribution, as in 'Bertrand believes that Hesperus appears in the evening', the name 'Hesperus' refers neither to its customary referent nor to itself, but to its customary sense. Similarly, the entire embedded sentence 'Hesperus appears in the evening', when occurring within the 'that'-operator, refers to its customary sense rather than to its customary referent. Such is the case when someone's remarks are quoted in "indirect discourse," that is, when reporting the content of his or her remarks rather than the very words used, as in 'Bertrand said that Hesperus appears in the evening'. It is not the truth value of the sentence 'Hesperus appears in the evening' that Bertrand is said to believe or have asserted, but its information or thought content. Frege proposed that expressions do not have their normal references in intensional constructions but instead refer to their senses. He says that in such cases expressions have an indirect reference (ungerade Bedeutung), which is then the same as what is normally their sense.

Russell's explanation of these different expressions of identity strongly resembles Frege's. According to Russell's version of the theory of definite descriptions, the definite article 'the', when joined to a class concept, forms a denoting concept which denotes a unique term through that class concept. A thing can be described through the use of a denoting concept expression whose class concept applies to a single individual. This use of the definite article presupposes that each thing can be identified through a concept with a single instance. That is, each term is the instance of some class concept and the descriptive expression univocally refers to that thing without requiring direct epistemic contact.

This account would change as a result of the paradoxes. The change in Russell's strategy involved the new technique of contextual definitions of certain expressions, such as definite descriptions. Not all expressions indicate single entities; those that fail to are said to be 'incomplete symbols', their purported ontological correlates treated as 'logical fictions'. He hoped thereby to retain the unrestricted variable while filtering out those objects whose postulation as entities lead to contradictions. The theory of incomplete symbols was a way of preserving the unrestricted variable, and avoiding solving the paradoxes through resorting to a typed hierarchy of entities as a preliminary given.

The reader should bear in mind that from Russell's way of viewing the problem, if the entities presupposed in all reasoning came in a typed hierarchy, an account of this could not draw on the same said entities. Solving the paradoxes by means of a typed hierarchy is, technically, a straightforward task. Russell's primary concern here however was in the order of explanation. A solution to the paradoxes was to flow from a better understanding of syntax for two reasons. The first reason is that expressions which purportedly designated contradictory 'entities' had to be ill formed because such entities could not be part of the natural order of things. The second reason being that accounts of syntactic well-formedness did not presuppose by way of explanation anything of an ontological nature.

The solution involved eliminating certain expressions as legitimate symbols with meaning in their own right. Russell abandoned the view that all expressions picked out determinate meaning relata, urging that many expressions were "incomplete symbols" and which only become meaningful when inserted into an expression. Such expressions did not indicate entities (i.e., meanings). However, they did contain entities that could, when combined, be genuine components of propositions. The point of the theory of descriptions is that certain expressions do not indicate separate individual entities but that can nevertheless contribute to a complex entity. It is therefore important that symbols reflect the structure of the complex in some way. Even though what results when a definite description is broken down into its constituent entities is not itself a single thing, in extensional contexts a single name may be used for the description. When an individual does possess the properties in question, then substitution can be made. But the point remains that there will not always be an individual corresponding to the complex of properties being entertained.

In the paper *On Denoting* Russell abandons the theory of denoting concepts, denying them meaning. The problem is no longer in knowing when these expressions have an individual nature, because none of them have. What changes in the new account of descriptions is that the term is no longer correlated with a single entity, but to several. That is, denoting concepts are broken down into their constituents. Rather, definite descriptions are what Russell calls "incomplete symbols". If α is an incomplete symbol, then α has no meaning in isolation, but every sentence in which it does occur does have a meaning. The point is not that incomplete symbols lack meaning in isolation, but have meaning in context. Nor is it that such expressions are devoid of meaning in the way that, say, nonsense is. The point is rather that contrary to appearances, incomplete symbols are not proper grammatical constituents of the sentences in which they occur. For example, our initial assessment might be that the expression 'the round square' refers to a bona fide term and as such occupies subject position in a proposition. But Russell holds that 'the round square' is not a referring term at all and is not really a proper grammatical subject. In particular, Russell held that a paraphrase of this sentence could be found expressing its logical form so that (a) no constituent of that paraphrase could be substituted for the expression 'the round square' and (b) no constituent of the paraphrase occupied the role of grammatical subject. Where the original contains what appears to be a syntactically complex constituent, 'the round square', which apparently occupies the subject position, the paraphrase will contain only a collection of predicates, quantifiers, and variables, none of which can be regarded, either jointly or severally, as the grammatical subject of the paraphrase.

The general strategy is to take an expression such as "The Present King of France is bald" and translate it into the following logical formula:

1. x is presently king of France.

2. x is unique.

3. x is bald.

The *Principia Mathematica* formalises the account of definite descriptions given in

On Denoting:

$$*14 \qquad [(\imath x)(\phi x)].\psi\{(\imath x)(\phi x)\}. =: (\exists b) : \phi x \equiv .x = b : \psi b$$

This definition is an example of "definition in use", viz., a logical translation of an expression with the form "the so and so". Formalizing the analysis found in *On Denoting*, a definite description is analysed as a propositional function satisfied by a single value of its variable. The symbol $(\imath x)(\phi x)$ is like an individual constant in univocally designating a determinate individual. But these symbols are not the same. Russell argues in the *Principia* that a definite description and a proper name cannot be substituted for the other salva veritate. The argument is that a definite description doesn't function in the same way as a proper name in a statement of identity, (however identity is construed).

The first part of the argument takes the form of a reductio:Suppose that a definite description, like a proper name, signifies an object. Suppose that any proposition of identity expressing the identity of reference(meaning) of the proper names justifies their substitutability salva veritate. Therefore in virtue of a), any proposition of identity between a proper name and a definite description should allow the substitution salva veritate between a proper name and a definite description. It is clear enough that this isn't the case. So we have to revise our initial premisses. Consider the logical definition of identity:

$$*13.01 \qquad x = y. =: (\phi) : \phi!x. \supset .\phi!y \text{ Df}$$

This definition establishes that x and y can be considered identical when each predicative function satisfied by x is satisfied by y. When applied to variables such as x and y, the assertion of identity authorizes their substitution salva veritate. This kind of identity is said to be trivial in that it simple shows that each object is symbolized in two different ways. So it makes no difference whether x is used in place of y or vice versa. If, in accordance with premise a) we could assimilate definite descriptions and proper names, the substitutability of proper names should also apply to descriptions. But we know from the principles that introducing a description into an identity proposition bestows an importance on that proposition which it doesn't have with just proper names. From this first argument we seem able to conclude that any assimilation of the symbol $(\imath x)(\phi x)$ with that of an individual constant, and of a definite description with that of a proper name, according to premise a), is mistaken.

Now one objection that can be raised here is that the above argument can be challenged by objecting to the second premise, concerning the interpretation to be given to propositions of identity. One can reject the interpretation given by definition *13.01 according to which x and y are different symbols for the same object and instead consider identity as manifesting not the identity of the referent of the proper names, but rather the identity of the names of that object. To complete the proof we need a second argument maintaining the first premise (which we are trying to refute) and which suggests a new interpretation of identity. Suppose that a definite description, like a proper name, signifies an object.Suppose that any proposition asserting an identity between proper names expresses the identity of the names of that object thereby authorizing their substitution salva veritate. Therefore, by A' any proposition of identity between a description and a proper name allows the substitutability of the proper name and the definite description.

Once again, the conclusion doesn't follow. Here, while the informative content of the identity statement is preserved its truth value is lost: "what would have been required for its truth would be that Scott should have been called the author of Waverley: if he had been so called, the proposition would be true, even if some one else had written Waverley; while if no one called him so, the proposition would be false, even if he had written Waverley." We are left with no choice but to reject the first premise of the above argument assimilating a definite description with a genuine symbol directly signifying an object. This is what happens in the PM where the conclusion is drawn that "'the author of Waverley' signifies nothing". The first argument is essentially the analysis given in the discussion in "On Denoting" of the enigma posed by the curiosity of George IV. Though the second argument is not found in "On Denoting" it retains an affinity with Frege's argument given in "On Sense and Reference".

Considered as a genuine symbol, a proper name immediately designates an object which is given deictically and directly knowable. A definite description, on the other hand, is essentially discursive, where reference to an object is mediated by concepts.[5] Rather than designating, it describes, qualifies or characterizes the presupposed object. Logical translation shows clearly what separates a proper name from a definite description. A proper name is symbolised by a constant, standing for a definite individual. The description is symbolised by the complex symbol $(\imath x)(\phi x)$ which turns out to be a quantified propositional function. The function symbol ϕ expresses some sort of descriptive qualification and accounts for the informativeness of any proposition asserting an identity containing a description. Just like a proper name, a definite description can identify an individual but this identification doesn't operate through a deictic relation between the sign and the object named. One can identify an individual through a description without directly knowing it or being able to designate it. A proper name is represented by a constant which deictically picks out an individual, while the definite description appears as an incomplete symbol, being reduced to the characterization of an indeterminate object represented by the function ϕx. This is an example of a definition in use. In the expression "The present King of France is bald", the constituent description "The present King of France" does not refer to any determinate entity, and so the expression has no meaning in isolation. Rather, the expression's meaning is cashed out by breaking it up into component constituents that are part of the proposition. It is with these constituents that we have direct and unmediated epistemic contact.

The theory also solved the same puzzles that Russell had addressed in the *Principles*. These puzzles arose when propositions are construed as being of subject-predicate form. One such puzzle concerns the conditions in which an identity statement can be infor-

[5]The description may not secure a referent, as in the case of round squares.

mative. Given the subject predicate framework assumed in traditional logic, identity can never be both true and informative: the said proposition merely unpacks the properties inhering in the subject, in which case the proposition is true but uninformative. Given that a subject is nothing beyond its properties, the proposition can't be held to assert a relation holding between a property and a subject: the latter is, by hypothesis, unknowable. The Russellian reductive analysis in *On Denoting* shows that if the denoting expression has no signification by itself, it still contributes to the informativeness of the whole proposition by using a concept to describe an individual. In this respect there is no difference between Russell's analysis of definite descriptions in the *Principles* and the account offered in *On Denoting*. The advantage of the latter account is that it explains the semantic phenomena without admitting potentially contradictory entities (such as the round square) to the realm of being. In any judgment of identity linking a proper name and a denoting expression beginning with the definite article "the", the proper name effectively names the individual of whom it is affirmed that it satisfies the concept(s) to which the denoting expression is reduced. "Scott" names that x which, according to the dismembered elements of the expression, is unique, exists, and possesses the property of having written *Waverley*. To be sure, the reductive analysis of denoting expressions denies them significance on their own. Nevertheless, the explanation of the informativeness of identity statements still rests on the synthesis of various concepts. If denoting expressions have no significance by themselves, their dismembered elements nevertheless contribute to the significance of the proposition as a whole, by providing a description of some entity.

5.5 The Substitutional Theory

The contextual definitions of definite descriptions cleared the way for contextual definitions of classes through the so-called substitutional theory. The theory was first sketched out as the "No Classes" theory in Russell's paper *On Some Difficulties in the Theory of Transfinite Numbers and Order Types* (Russell 1906b). Russell expanded the theory in his paper *On the Substitutional Theory of Classes and Relations* (Russell 1906c) which he read before the London Mathematical Society in May of 1906. The substitutional theory was designed to dispense with positing classes and functions as independent entities[6] (be they in extension or intension) through an operation called 'substitution'.

We assume as given a universe of simple entities ('individuals', i.e., things and concepts) and complex entities (propositions), with the former constituting possible constituents of the latter. Once given this universe of entities and propositions we make the further assumption of a primitive relation expressed by the following propositional function:

(1) q results from p by the substitution of x for a in p,

where it is understood that by 'substitution' we mean the replacement of all occurrences of the entity a in the proposition p.[7] Furthermore, this relation is defined for all entities; p and q are not necessarily propositions. The symbolic notation for this primitive relation is:

$$p/a;x!q$$

which is read as (1) and is the only primitive formula of the theory.

Given this schema the contextual definition of matrices (symbols standing for classes) takes place in two steps. In the first step, we use the technique elaborated in *On Denoting* to provide the description of the resulting proposition q;

The result of substituting x for a in p, or;

$$p/a;x$$

for some context $f(\ldots)$, accompanied with the usual notation for quantifiers and identity.

[6] Russell sometimes speaks of this in terms of "abstinence from a doubtful assumption". (Russell 1906b, page 154)

[7] The reader is reminded that this is not simply a theory about *symbols*, since propositions and concepts are among the entities in the universe.

In other words, we define:

$$f(p/a; x)$$

(that is: $p/a; x$ has the property f by the following contextual definition:

$$f(p/a; x) = \exists q\{\forall r(p/a; x!r \equiv r = q \wedge f(q)\}$$

Now we can define the matrix p/a, which can only be introduced in very specific contexts. The first is

$$x \in p/a$$

which is defined as:

$$x \in p/a = \exists p \exists a(p/a; x \text{ is true})$$

This definition corresponds to the definition of $x \in z(\varphi z)$ in the *Principia*, that is, the definition of class membership for an element determined by the function φx. In place of the quantified variable function we have the variables p and a, which are propositional variables, or general variables for individuals (the goal of the substitutional theory being precisely to avoid having to quantify over functions and classes[8]. So $p/a; x$ is an example of an incomplete symbol defined in use. Because matrices are defined contextually they must always appear in an argument position. In other words, these symbols can only appear meaningfully when accompanied by a predicate. So the truth predicate occurring in the definiens of the above definitions is basic and cannot be eliminated.

Next we are given the definition spelling out the identity conditions for two matrices. Two matrices have the same value (designate the same class) if and only if the propositions obtained by the defining substitutions are materially equivalent (Russell 1906b, page 176). This definition guarantees the extensional character of matrices and allows that the symbol p/a be used in place of symbols for classes. The second context to be

[8]Russell says in effect: "Here the values of x for which $p(x/a)$ is true replace the class u; but we do not assume that these values collectively form a single entity which is the class composed of them." (Russell 1906b, page 155)

defined for matrices is thus

$$p/a = p'/a'$$

which is defined:

$$p/a = p'/a' = \forall x(p/a; x \text{ is true} \equiv p'/a'; x \text{ is true})$$

Russell proceeds in a step by step fashion, introducing a hierarchy of matrices. This procedure is fundamental from his point of view because the resulting hierarchy is generated without requiring any explicit rules of an ontological nature concerning types.

Thus classes of classes are as much logical fictions as the classes already introduced. Classes of classes will be matrices for which it is meaningful to say that matrices of the first type (having the form p/a) are members of it. Syntactically, the definitions require that matrices simulating classes of classes have the necessary number of variables, and if this condition is not respected the definiens will be meaningless. This means that it isn't necessary to stipulate that matrices have to be of some higher type than its arguments.

Russell next defines class membership:

$$r/c \in q/(p, a)$$

The point here is that we have to make sure that it is the class r/c that is being said to belong to $q/(p, a)$, and not the entities r and c. This requirement is met by the following definition:

$$r/c \in q/(p,a) = \exists q \exists p \exists a[(q/(p,a);(r,c) \text{ is true } \wedge \forall r' \forall c'(r'/c' = r/c \Rightarrow q/(p,a);(r',c') \text{ is true}.$$

The syntactic constraints on matrices by these definitions dispense with the need for grammatical rules stipulating which expressions are meaningful and which are not. This is important from Russell's perspective as it yields what he calls "homogeneity of type" (Russell 1906c, page 178) between symbols on each side of the membership sign without need for any metatheoretical stipulations to that effect. Homogeneity of type is given in the syntax because guaranteed in the definitions.

When a formula contains matrices, the test of whether it is significant or not is very simple: it is significant if it can be stated wholly in terms of entities. Matrices are nothing but verbal or symbolic abbreviations; hence any statement in which they occur must, if it is to be a significant statement and not a mere jumble, be capable of being stated without matrices. Thus for example '$p/a = q/b$' means:' Whatever x may be, if x is substituted for a in p and for b in q, the results are equivalent.' Here nothing but entities occur. But if we try to interpret (say) '$p/a = q/(b, c)$', we find, on supplying x, that we have a proposition on the left and a matrix on the right. Thus on the right another argument has to be supplied, but on the left there is no longer room for an argument. Hence the proposed formula is meaningless. (It is not *false*: its denial is just as meaningless as its affirmation.) Thus where matrices occur, significance demands homogeneity of type: this does not need to be stated as a principle, but results from the necessity of getting rid of matrices in order to find out what the proposition really means (Russell 1906c).

This sheds light on the true nature of contextual definitions: they are rules for translating from one logically obscure language, into an unambiguous, logically perfect language. They can be viewed as rules testing whether an expression is well formed. Any expression in the language containing matrices must be translatable into the base language, which does not contain matrices (class symbols) but rather consists in the minimal lexicon of the quantification theory with identity (quantifiers, connectives, variables for entities propositions and individuals, and the identity sign) supplemented with the primitive notion that q results from p by the substitution of x for a (the description $p/a; x$ is itself defined). These contextual definitions provide translation rules; if the translation is impossible in the end, it is because the expressions containing matrices are not well formed. It becomes impossible to write nonsense. This was important for Russell because it dispensed with the need for explicit rules for well formedness.

The fundamental point here is that such a string of symbols can be given content by inserting it within a meaningful set of symbols. Such a complex symbol will then form a name picking out some determinate entity:

> It is to be observed that a substitution or a matrix is not an entity, but a mere operation like d/dx; that is to say, the symbols x/a and p/a are wholly devoid of meaning by themselves and only become significant as parts of appropriate propositions... Thus when we say a matrix is not an entity, we mean that a matrix is a set of symbols, or a phrase, which by itself has no meaning at all, but by the addition of other symbols or words becomes part of a symbol or phrase which has meaning, i.e. is the name of something. (Russell 1906c. page 170)

The influence of Frege's context principle is evident here. That a matrix is not an entity but a *mere operation* suggests that matrices stand for some highly rarified kind of abstraction indeed. But what exactly is the nature of this supplement conferring entityhood when attached to other entities? Presumably, this contribution will be mind-independent. Furthermore, these contributions, or whatever it is that matrices bring with them, presuppose independent entities. So Russell does not view the contextual definitions given above as abandoning the account of logic and mathematics as synthetic a priori. Synthetic a priori propositions are informative precisely because properties are not contained in a subject and the mere operations Russell talks about are clearly intended to be structural features of a complex, and not constituents of the complex. These constituents, in turn, are independent of each other. They are autonomous.[9]

[9]This account of the informative nature of the synthetic comes at the expense of explaining how we make the connections between these independent constituents. That is, how do you account for the gaps, or jumps in reasoning? This is a problem with a long history. Hume, for example, explained a priori relations among ideas as a function of a property of homogeneity among ideas. It is because some ideas are so related that the understanding can move with ease from one idea to another. Again in §90 of the Grundlagen, Frege talks about self evident transitions from one proposition/judgement to another. Frege's chief concern was the rigorisation of arithmetic and the transitions from one proposition to another proposition in a proof. Proofs rely upon transitions involving a missing piece. A proof attempts to make continuous what is essentially discontinuous. The transition has to be recognised as a legitimate one and Frege remarks that the only way to do this is by establishing beforehand the legitimate rules for making such steps. But these rules for transitions must themselves be recognised as legitimate and correct. That is, an inference had to be in conformity with accepted laws of logical inference. But

So Russell felt that the essentials of the Logicist project were still intact. Mathematics remained derivable from a base of independently existent abstract entities knowable a priori. If paradoxes resulted from positing certain objects as independently real a solution was at hand where the offending objects were eliminated as entities. More specifically, the paradoxes sprang from treating structural features of complexes as constituents. Given proper attention to logical syntax, Russell's idea was that the contribution made by such structural features could be specified, without, as it were, compromising the integrity of the complexes to which they belonged.

doesn't this found the transition from judgement to judgement in intuition? Russell's point is that our intuitions have to be tested against and disciplined by the existence of independently existent abstract entities.

"The object is not to banish 'intuition', but to test and systematise its employment, to eliminate the errors to which its ungoverned use gives rise, and to discover general laws from which, by deduction, we can obtain true results never contradicted, and in crucial instances confirmed, by intuition" (Russell 1906a, page 194)

Chapter 6

The Ramified Theory of Types

6.1 Introduction

Chapter 4 of this essay surveyed Russell's strategy of eliminating ontological commitments to entities via the use of contextual definitions (the theory of descriptions and the substitutional theory). Russell was after the minimum number of abstract entities necessary for the reduction of mathematics to logic. Entities associated with paradoxes were eliminated. The substitutional theory, in an effort to avoid paradoxes related to classes, abstained from ontological commitments to functions and classes. However, Russell withdrew this paper (Russell 1906c) from publication on account of his (re)discovery of paradoxes involving propositions. The substitutional theory elaborated in (Russell 1906c) would prove to be defenceless against them.

While Russell discusses the *Epimenides* paradox in *Les Paradoxes de la Logique* (Russell 1906a), he had discovered that propositions are associated with a version of Cantor's paradox in appendix B of *Principles*: If m is a class of propositions, the proposition 'every m is true' may or may not be itself an m. But there is a one-one relation of this proposition to m: if n be different from m, 'every n is true' is not the same proposition as 'every m is true'. Consider the power set of the class of all propositions, P and the following classes of propositions; p_1, p_2, p_3 etc. as its members. Then the power set of the class of all propositions can be given a one-to-one mapping into the set of all propositions by the map which associates each set p_1, p_2, p_3 etc. with the proposition 'every p_1 is true', 'every p_2 is true', etc., respectively. We have a one-to-one mapping of a proper subset of the class of all propositions onto the power set of the class of all propositions.

Now consider the class m of all propositions of the form 'every p_1 is true', and the class formed by all those propositions which have this form while not having the property of being members of their respective p_1's. We call this class w. Let r be the proposition 'every q in w is true'. Is r a member of w? Since r is of the form 'every p_1 is true', and its image under the one-to-one mapping given above is the class w, it follows that it is a member of w only if it is not a member of w. Hence it is not a member of w. Then r does not belong to the class whose image it is under our mapping and therefore does belong to w. Then r does not belong to the class whose image it is under our mapping and therefore does belong to w. It belongs to w if and only if it does not belong to w. In other words, put;

$$w = \{p : \exists m((p = \forall q(q \in m \supset q)) \wedge (p \notin m))\}$$
$$r = \forall q(q \in w \supset q)$$

One proceeds by taking cases. First assume $r \in w$; then

(1) $\exists m((r = \forall q(q \in m \supset q) \wedge (p \notin m))$

(2) $(r = \forall q(q \in m_0 \supset q) \wedge (p \notin m_0)$

(3) $\forall q(q \in w \supset q) = \forall q(q \in m_0 \supset q)$

(4) $\forall q((q \in w \supset q) = \forall q(q \in m_0 \supset q))$

(5) $(r \in w \supset r) = (r \in m_0 \supset r)$

(6) $r \in w = r \in m_0$

(7) $\sim (r \in m_0)$

(8) $\sim (r \in w)$

Now assume $r \notin w$:

(1) $\sim \exists m((r = \forall q(q \in m \supset q)) \wedge (r \in m))$

(2) $\forall m((r = \forall q(q \in m \supset q)) \supset (r \in m)$

(3) $(r = \forall q(q \in w \supset q)) \supset (r \in w)$

(4) $r \in w$

The paradox is generated by very fine-grained identity conditions for propositions. Identical propositions have identical constituents. This assumption makes itpossible to identify the propositional constituents w and m_0, as these are constituents of identical propositions. Russell provides an informal presentation of his criteria for propositional identity in the *Principles* (Russell 1903, page 527-528) but it has been left to Alonzo Church (Church 1984) to provide a formally exact characterization of the assumptions involved in the derivation of Russell's propositional contradiction.

The *Epimenides* is produced by a similar diagonalization procedure. That is, this paradox is generated when a speaker asserts the following proposition:

(1) There is a proposition p which I affirm and which is false.

When (1) is asserted, what is asserted is a proposition p about all propositions, p itself among them, namely that they are false if they are asserted. The proposition p thus refers to itself in the sense that it-or more exactly, the sentence that expresses it-quantifies over (i.e., refers generally to all or some of the elements of) a collection of entities among which p itself is included. So let E be the set of all propositions asserted by Epimenides, and suppose that E has only one member: the proposition that all members of E are false. Then assume:

$$E(p). \equiv .p = (q)(E(q) \supset {\sim}q) .$$

Let us abbreviate $(q)(E(q) \supset {\sim}q)$ by ε. One then proceeds by taking cases. First assume

ε; then

(2) $(q)(E(q) \supset {\sim}q)$

(3) $E(p). \equiv .p = \varepsilon$

(4) $E(\varepsilon)$

(5) $E(\varepsilon) \supset {\sim}\varepsilon$

(6) ${\sim}\varepsilon$

Now assume ${\sim}\varepsilon$; then $(\exists q)(E(q) \wedge q)$ by the definition of ε. Let q_0 be such a q; then

(2) $E(q_0)$

(3) q_0

(4) $E(q_0). \equiv .q_0 = \varepsilon$

(5) $q_0 = \varepsilon$

(6) ε

What makes Russell's antinomies about propositions so perplexing is that despite the semantic or linguistic nature of these antinomies, they are generated by a diagonalization procedure. Because they shared this feature with those paradoxes that are more obviously "Mathematical", or "Logical", Russell's suspicion that the antinomies had a common source and a common solution seems very plausible. Russell suspected that the problem lay in the construal of propositions as entities[1]. As the simple theory of types in appendix B of *Principles* is unable to block these paradoxes, Russell returned to address a cluster of problems surrounding his understanding of individuals, or *terms*.

In his analysis of the liar paradox, Russell observes that the proposition 'I'm lying' contains an *apparent variable* (a variable bound by a quantifier), and that the paradox arises when this proposition is admitted as being a possible value of this bound variable. So suspicion fell on the ontology assumed by the simple substitutional theory of classes, with its admission of general propositions as entities. And after reading work by Poincaré (Poincaré 1905) suggesting that the paradoxes were the consequence of 'vi-

[1]The reader is refered to the discussion in (Pelham and Urquhart 1994)

ciously circular' definitions, Russell concluded that paradoxes were the outcome of a process of self-application, flowing from attempts to quantify over illegitimate totalities:

> I recognize... that whatever in any way concerns *all* or *any* or *some* (undetermined) of the members of a class must not be itself one of the members of a class. In M. Peano's language, the principle I wish to advocate may be stated: 'Whatever involves an apparent variable must not be among the values of that variable' ... let us call this the 'vicious circle principle'. (Russell 1906a, page 198)

Poincare objected to impredicative definitions, where an entity is picked out using quantifiers ranging over a domain that includes that entity. The solution Russell settled on was to organise abstract entities into a hierarchy and to exclude any 'entity' containing an apparent variable from the domain of being (i.e., classes, descriptions, functions, general propositions). The vicious circle principle is a thesis about the nature of abstract entities to the effect that these are endowed with a structure and that they are ontologically dependent on the objects they are abstracted from, whether or not the latter are concrete or abstract. The shift in emphasis is from employing contextual definitions to eliminate entities to using these definitions to reveal the underlying structure of these entities.

Thus any expression containing a bound variable would not be considered as an individual entity, which ruled out general propositions, descriptions, functions, and so on. The strategy was simply to extend the theory of incomplete symbols to cover all those abstract entities of a derived nature, such as general propositions. So an expression containing an apparent variable was to be treated as a logical fiction.[2] Furthermore, these

[2]The reader should bear in mind that Russell's initial reaction to the paradoxes was to suspect that the fault lay in the reification of abstract entities. As I suggested in chapter 5, the substitutional theory and the theory of descriptions were designed to dispense with having to postulate suspect entities. But with the vicious circle principle, Russell thought he finally had his finger on the kind of self application underlying the paradoxes. The result was a complex theory about the ontological presuppositions governing the realm of abstract entities. But the issue is the regimentation of the things found there, not

contextual definitions preserved the principle of the unrestricted variable. Another way of putting the point is that classes are affected by some sort of internal complexity indicated by the presence of the variable. This complexity could be thought of along the lines of the analysis of 'denoted' objects in the *Principles*: as 'combinations' of terms. Furthermore, that which contains an apparent variable presupposes, or involves, a totality of elements. The crucial issue here was whether entities such as classes, functions, general propositions possessed the sort of internal complexity Russell attributed to them.

Russell reasoned along the following lines: An aggregate is made up of a variety of constituents. In the simplest case a complex will contain a universal and a particular. This sort of case is relatively unproblematic in that universals are somehow ontologically dependent on the particulars they qualify (or relate in the case of relations), and the unity of the complex flows from this ontological asymmetry.[3] Other complexes, however, may contain variables. In this case, the fundamental idea is that such a variable will not have the very complex it is a constituent of in its range (this complex cannot be taken as a value of the variable). The complex, and the entities substituted for the variable(s) in that complex, must be entities of different types. Aggregates specified or defined impredicatively will be 'illegitimate' totalities.

Individuals are known by acquaintance and given proper names. These simple constituents in turn may be combined in various ways to form complex objects. A hierarchy of objects is established through a process of generalization and abstraction over well-defined totalities. All entities from such a totality form a type, and entities of higher type are constructed from entities found in lower types. So do 'incomplete symbols' indicate

their elimination. Russell's change in thinking about the nature of the entities needed to derive mathematics from logic would affect the account of these disciplines as synthetic apriori. Why not account for the presupposition relations among abstract entities in terms of analytic containment? Russell's rejection of formalism ruled that option out, but at the same time he could not appeal to Cartesian intuitions of clear and distinct ideas.

[3]Again, Russell, contra Frege, resolves the thorny problem of unity through this sort of presupposition relation. Russell's worries over the 'one and the many', and Frege's theory of unsaturated entities are facets of a broader worry over 'gaps' in chains of reasoning.

complexity? Well, an entity of a certain type must be picked out by a symbol appropriate to that type. Symbols for objects (or collections of entities) must reflect the complexity of the objects they purport to pick out. Symbols for such objects are 'incomplete' in the sense that they must be cashed out in a context displaying the way their simple constituents are part of a complex structure.

6.2 The Theory of Types

In *Mathematical Logic as Based on the Theory of Types* (Russell 1908)substitution is restricted to a range of significance. However, the vicious circle principle does not block the construction of the function "...is not self-predicable" (Russell's paradox in intensional form), as no quantifier is involved in its construction. So, the first difficulty is the lack of any direct link between the vicious circle principle and Russell's paradox in its intensional form. Gödel remarked that the simple theory of types is motivated by reasons having nothing to do with the vicious circle principle:

> The reason adduced (in addition to its "common sense") is very similar to Frege's ...The reason is that (owing to the variable it contains) a propositional function is something ambiguous (or, as Frege says, something unsaturated, wanting supplementation) and therefore can occur in a meaningful proposition only in such a way that this ambiguity is eliminated. (Gödel 1944, page 147)

Frege's fundamental idea in regard to concepts is that of 'unsaturatedness'. But Russell's notion is that of 'ambiguity', which is entirely different. A function is ambiguous, for Russell, because it has instances, via the objects satisfying the function. Given that ϕx has instances, it makes sense to say 'ϕx is true in all cases'. This is how Russell introduces a function as an argument to a higher-level function. Bringing the vicious circle principle to bear on this sort of construction requires an additional principle to the effect that a function presupposes the totality of its values, and thus, of its arguments. Thus, the theory of types is derived from the vicious circle principle being applied to functions, with the addition of a principle to the effect that a function presupposes the totality of the propositions which are its values (and so of the arguments that are the constituents of these propositions).

This principle is independent of the vicious circle principle. The vicious circle principle prevents a function expression that contains a quantifier from being an entity within that quantifier's range. The quantifier's range will be restricted to a certain order. What is needed is another principle to the effect that a function presupposes the totality of its arguments. It is by adding this principle that one can derive a second version of the vicious circle principle pertaining to functions, which prevents any function from taking as argument any entity which presupposes that function. As Gödel points out, without this amendment, the concept 'impredicability', which does not refer to any totality can be applied to itself as argument. Russell's justifications for the principle leans on an account of the nature of functions where a propositional function $\varphi\hat{x}$ denotes a totalityof propositions, which presupposes that this totality is well defined. So this function can't be a constitutive element of this totality.

Similarly, this is how Russell understands the 'ambiguity' of concepts and universals. They are ambiguous because they have particular instances, not because they are unsaturated, or somehow essentially predicative. We have, after all, acquaintance with all sorts of abstract entities. However, this amendment to the vicious circle principle prohibits identifying functions and concepts. Clearly, if a function presupposes its values, the constituents of a proposition cannot contain any functions. On the other hand, the concept *mortality* will be presupposed in any grasp of the proposition "Socrates is mortal". So while it was a fundamental principle of Russell's that understanding a proposition required acquaintance with its constituents, we don't have to be acquainted with any function in order to grasp this proposition. In other words functions, as opposed to concepts, are not constituents of propositions.[4]

Accordingly, in the introduction to the *Principia*, Russell reiterates that the paradoxes result from a vicious circle and justifies the ensuing logical hierarchy with the principle that a function presupposes the totality of its values alone and thus of its arguments (which Russell says amounts to the same thing). The stipulation that a function can have neither itself nor anything presupposing itself as argument is sufficient to rule out the construction of the function responsible for Russell's paradox, 'is not predicable of itself'. Not only does this block Russell's paradox, but we also obtain one facet of the theory of types: the notion of a function's range of significance. The variable x is restricted to the totality of possible arguments to the function.

It is also important to notice that the limitation of a variable to a function's range of significance does not have to be stated using some explicit meta-rule. Simply, a variable will have no meaning when it strays from that range. In practical terms the vicious circle principle places restrictions upon the possible range of significance of a propositional or functional variable. The range of significance of a function is all the arguments for which the function is true and all the arguments for which the function is false. In other words, the principle can be thought of as placing restrictions on the substitutions for such a variable. The basic premise is that 'a function is not well-defined unless all its values are already well-defined'. The meaning of a function presupposes the determinability

[4] On such an understanding of functions there is a sense in which their being is relative to the way in which they are abstracted from propositions. Reading Russell as a constructivist, one would interpret him here as saying that their existence is to a certain extent relative to our means of grasping them. How is functional abstraction and quantification over functions to be understood given that functions are "ambiguities"? There is an interpretation of the notion of functional ambiguity that treats them as full-fledged entities with an ontological status comparable to that enjoyed by concepts. In his 1944 paper on Russell's logic, Gödel suggests that by the vicious circle principle properties and classes are mind dependent entities. Gödel based this conclusion after distinguishing 3 different versions of the principle Gödel detects in Russell's writings. In one version the principle stipulates that whatever involves all of a collection must not be one of the collection and that whatever involves an apparent variable must not be among the possible values of that variable. Another version has it that if, provided a collection had a total, it would have members only definable in terms of that total, the said collection has no total. Yet again, given any set of objects such that, if we suppose the set to have a total, it will contain members presupposing this total, then such a set cannot have a total. Philosophers of a realist bent have found the consequences of all these versions of the principle to be unacceptable. See (Gödel 1944).

of its range of significance. Values that can't be determined prior to the determination of the function are to be excluded. The principle allows one to separate those values giving meaning to the function from those for which the function is meaningless. So no function can have as a value something presupposing it, i.e., whose values depend on the determination of that function.

6.3 The Ramified Hierarchy of the *Principia Mathematica*

In §5 of the Introduction to the *PM*, Russell sketches the construction of the hierarchy of types. A variable is introduced into a function given a well-determined totality for it to range over. So a function can only be introduced once such totalities have been established. Given a totality of individuals and the variables x, y etc ranging over them, one has expressions representing indeterminate elementary propositions such as φx, $\psi(x, y)$ and so on. The functions themselves seem to be created in the act of functional abstraction. This process is then iterated. The object of the theory of types is to guarantee values for those variables. That is, regardless of a logical formula's complexity, the hierarchy of types guarantees that the formula will be constructed from a base composed of referents of basic symbols.

The theory holds that we can only be sure of the legitimacy of properties if these can be constructed from the bottom up. These basic properties are those with which we are acquainted. At the lowest level of the hierarchy – at type 0– there is no problem in determining what an expression means. The lowest type is made up of individuals. That is, an individual corresponds to every proper name. An individual is known directly through empirical contact. Determining meaning at this level raises no logical issues that Russell had not dealt with earlier in the *Principles*. What this level provides is an ontological grounding of thought and talk. It is in determining the status of entities residing at levels higher than the first that questions arise about what the *Principles* called "logical meaning".[5]

The rules governing the construction of any function of one variable, such as ϕz, can be given as stipulations governing the use of the function symbol and the variable symbol respectively: 1)- "ϕ" is an elementary function symbol. 2)-"x" symbolizes an individual variable. These two stipulations are sufficient to eliminate any circle at this level. For example, the formula $\phi(\phi x)$ would be excluded because it violates the second rule relative to the domain of the variable. This rule limits the domain of the variable to individuals and guarantees that the function ϕ will not take itself as argument. So predicative functions with individual variables can be constructed without difficulty. However, because the logicist program of deriving mathematics requires symbols which are functions of propositional functions, additional rules are needed to ensure formulas such as "$f(\phi x)$" are properly predicative and represent functions whose arguments are themselves a function of individuals. The difficulty here lies, not in the status of ϕx, but in the new function f, which is not an elementary function. Should this new function f admit individuals as well as functions of individuals for arguments? That is, should we construct both the formulas $f(a)$ and $f(\phi a)$? To do so would violate the vicious circle principle. If f admitted a as argument, then the function could be determined when a was determined. But if ϕa is also admitted as a possible argument, f could be determined once ϕa was determined. Now ϕa is only determined once the individual a is determined. So the function f and

[5]In the *Principles* universals were construed as individuals where the word 'individual' was taken as synonymous with 'term' and 'entity', or having the capacity of being a logical subject. These properties and relations in intension did not exist merely in their instances but rather were thought to subsist separately in a timeless platonic realm. Determining how Russell understands such 'things' given the ontological hierarchy in the *PM*, however, is no easy matter. First there are the basic concepts picked out by general names. As one moves up the hierarchy, propositional functions which are applied as arguments to other propositional functions are governed by certain restrictions on well-formedness: viz. they must be decomposable into syntactic elements that refer to items of acquaintance. It is only by conforming to these restrictions on admissible substitution for variables that we will be assured of the possibility of denotation. In this way we can avoid generating properties that will yield paradoxes. Given that the existence of a property depends on the possibility of introducing a coherent predicate, the ramified theory of types guarantees that paradoxical predicates cannot be coherently introduced.

its argument ϕa both presuppose the same range of significance, which conflicts with the requirement that the argument be determined prior to that of its function. The rules for constructing functions of propositional functions will rule out these functions from taking individuals as arguments: their only arguments will be functions of individuals. Adhering to this stipulation generates predicative functions of functions of individuals, written $f!\phi x$. Extending this procedure yields a hierarchy of functions. A function can only have as its type, i.e., range of significance, arguments of a type immediately below it. Thus the domain is stratified into a regimented hierarchy.

The hierarchy of functions produces a hierarchy of types, or ranges of significance, of these functions. Classes constitute partitions on these ranges of significance and yield a hierarchy of classes. When a value is a member of a range of significance then it can belong to the class determined by that function. Individuals, for example, constitute the lowest type in the hierarchy, type 0. Type 1 consists of classes of individuals, type 2 of classes of classes of individuals, type 3 classes of classes of classes individuals, and so on. Unlike the homogeneous universe, members of a given class in the type hierarchy must all be drawn from a single logical type n, and the class itself must reside in the next higher type $n+1$. The definition of type 0 as a set of individual values suffices to guarantee the significance of all functions of x.

The result is that a class of some given type can only have as members elements from a type immediately below it. This theory of types successfully treats the mathematical paradoxes by reducing classes to those propositional functions determining them. The resulting hierarchy resolves the mathematical paradoxes, such as Russell's, by imposing a type distinction between a class and its members. A function determines a unique class of its type composed of arguments of a type immediately below it. This excludes a class determined by a function from being a value of that function. It is no longer possible to construct the class w: Thus if α is a class, the statement "α is not a member of α" is always meaningless, as is the phrase "the class of those classes which are not members of themselves". Hence the contradiction which results from supposing that there is such a class disappears.

The simple type hierarchy sketched in the above is not designed to address the sorts of paradoxes which were the explicit targets of the vicious circle principle. To deal with such paradoxes Russell was led to the more complicated system known as the ramified type theory. The ramified theory of types, like the simple theory, restricts the formation of certain constructions by establishing a 'vertical' hierarchy requiring that an expression of some type accept only expressions of some lower type as arguments. Parallel to the simple theory, then, the type of a function must exceed the type of its possible arguments. What complicates the hierarchy in the ramified theory is the presence of functions containing quantifiers. And rules are required in order to determine the proper range of significance for functions containing quantifiers. This was to circumvent the generation of collections which Russell took to be illegitimate under all circumstances: there can be no such property as that of having every property, for example, for such a property would both involve, and also be among, a certain collection. That is, it is incorrect to talk of "all properties" because this forms an illegitimate totality.

The key notion here is that, in addition to its arguments the ranges of the quantified variables involved in the function contribute to determining the function's position in the hierarchy. A function's position in the vertical hierarchy will be determined by a 'horizontal' hierarchy given by the kinds of quantifiers contained in the function. Technically this can be done in a number of ways, but perhaps the clearest expositions are those given by Alonzo Church (Church 1976), where a function's position in the hierarchy is called its order, and where the quantifier(s) in a function in turn determine the function's level. What 'ramifies' the ensuing hierarchy of functions is that a function's position (its order) in the hierarchy will be pegged by its level (the sorts of entities it quantifies over), its arguments, as well as all the entities presupposed in the determination of these.

For example, let's assign an individual 0 as its order. Now, the order of a function must exceed the order not only of its possible arguments, as in simple type theory, but also the orders of the things it quantifies over. So functions such as "x is a philosopher" and "x is as wise as all other philosophers" are first order properties, since they are true of, and, in the second instance, quantify over, individuals only. Properties whose order exceeds the order of their possible arguments by 1 are called predicative, and they are of the lowest possible order relative to their range of significance. Compare this with the property "x has all the properties of a great general", which can be represented by the complex formula$(\phi)f(\phi z, x)$, which in turn is obtained by universally generalizing over the function variable ϕ in the function of a propositional function of two variables $f(\phi z, x)$. This function is a function of x and represents a property of individuals. Its truth value depends solely on the real variable x. However, its significance presupposes objects other than objects of type 0. The presence of the apparent variable ϕz presupposes the determinability of all functions of individuals, i.e., objects of type 1. Furthermore, this type –the set of all functions of x– must be redefined to avoid any vicious circles. Restrictions are necessary to prevent the function $(\phi)f(\phi z, x)$ from being counted as belonging to that type. Since $(\phi)f(\phi z, x)$ quantifies over first order properties, according to the vicious circle principle it cannot be counted among them. Accordingly, in the ramified hierarchy, this is a second order property of individuals, and hence non-predicative: "It follows that the totality of values of ϕz concerned in $(\phi).f(\phi z, x)$ is not the totality of all functions in which x can occur as argument, and that there is no such totality as that of all functions in which x can occur as argument."(Russell 1910, 49). Similarly, the property "x is a (first order) property of all great philosophers" is also second-order, since its range of significance consists of objects of order 1 (and it quantifies only over objects of order 0); but since it is a property of first order properties, it is predicative.

What is required is the construction of a new hierarchy within the first, splitting each type into different functions with mutually exclusive orders. This distinction between orders would take into account not only the argument of the function, but also its internal syntactic complexity, reflecting the presence of quantifiers. If the truth value of propositions generated from the function was a function of the real variable, the function's significance would also have to take into account the organization of the apparent variables introduced by the generalizations. The aim here is to avoid complex functions as functional values presupposed by their apparent variables. This second hierarchy requires that any function containing at least one apparent variable be of an order superior to the highest order found amongst its variables, be they real or apparent. Church has provided the following formal characterization of Russell's ramified logical types (Church 1976):

(1) there is an r-type i to which all and only individuals belong, and whose order is stipulated to be 0.

(2) If $m \in \omega$, $n \in \omega - \{0\}$, and $\beta_1, \ldots \beta_m$ are any given r-types, then there is an r-type $(\beta_1, \ldots, \beta_m)/n$ to which belong all and only m-ary propositional functions of level n and with arguments of r-types $\beta_1, \ldots, \beta_m$, respectively; and the order of such a function is $N + n$, where N is the greatest of the orders corresponding to the types $\beta_1, \ldots, \beta_m$ (and $N = 0$ if $m = 0$).

The notion of the level of a propositional function of r-type $(\beta_1, \ldots, \beta_m)/n$ is required as a counterpart to Russell's use of the notion of an apparent variable occurring in that propositional function. That is, if N is the greatest of the orders corresponding to $\beta_1, \ldots, \beta_m$, and k is the greatest of the orders of the apparent variables occurring in that function, then $n = 1$ if $k \leq N$, and $n = k + 1$ if $N < k$. $(\alpha_1, \ldots, \alpha_m)/k$ is *directly lower* than $(\beta_1, \ldots, \beta_m)/n$ if $\alpha_1 = \beta_1, \ldots, \alpha_m = \beta_m$ and $k < n$. The force of the division of types is felt in the restriction on well formed formulas: $f(x_1, x_2, \ldots, x_m)$ is a *wff* iff f is a variable or constant of type $(\beta_1, \ldots, \beta_m)/n$, x_1 is a variable or constant of type β_1 or directly lower, ..., and x_m is a variable or constant of type β_m or directly lower.

So in the *PM* and in *Mathematical Logic as Based on the Theory of Types* this hierarchy of orders of functions is generated in a genetic fashion. First we take a propositional function containing no variables other than real variables. This is called a matrix. Complex functions of different types (but of the same order as that of the matrix) are created by existential or universal generalisation on some of these variables, generating the following hierarchy:

1. First Order Matrices: ϕx, $f(x, y)$ etc.

A variety of first order functions can be generated by generalizing on functions which only admit individuals for arguments:

First Order Functions:

$$F(x, y), \ldots, (y).f(x, y), (\exists y).f(x, y) \text{ functions of } f.$$

At this basic level, all 1^{st} order functions are also of type 1: They only presuppose the set of individuals.

2^{nd} Order Matrices: $f(\phi z), f(\phi z, x)$

2^{nd} Order functions:

$$f(\phi z) \ldots (x)/(\exists x).f(\phi! z) \text{ functions of } \phi! z \ldots (\phi)/(\exists \phi).f(\phi! z) \text{ functions of } x$$

Second order functions can be either of type 1 or type 2. So a second order function can admit as arguments either individuals or functions of individuals. Second order functions presuppose both individual entities of type 0 and objects of type 1, i.e., functions of individuals (whatever the type corresponding to a second order function turns out to be). For example, "x is courageous", "x is a shrewd strategist", etc, are first order functions involving just the domain of individuals. On the other hand "x possesses all the properties of a great general" is a second order function because it draws on 1^{st} order qualities characterizing great generals.

This hierarchy of orders blocks the generation of circular function formulas. Separating functions of a given type into different orders breaks up the totality of "all functions of x". All the functions of x are now divided into first order functions (whose only apparent variables are individuals), 2^{nd} order functions (whose only apparent variables are 1^{st} order functions), and functions of order $n+1$ (whose only apparent variables are functions of order n, of order $n-1$), etc. The expression $(\phi).f(\phi z, x)$ cannot thus apply to itself. This expression is a 2^{nd} order function of x. So it only presupposes (admits as arguments) individuals and 1^{st} order functions.

In this ramified form, the theory of types is strong enough to ground the significance of all functional formulas and provide a solution to paradoxes like the liar. In the *PM*, this is accomplished through a hierarchy of propositions. That is, the hierarchy of orders provides a solution to antinomies involving "illegitimate totalities" of propositions. The method involves splitting the illegitimate totality of "all propositions" into smaller, mutually exclusive totalities. This is obtained by deriving a hierarchy of propositions from the hierarchy of functions. When a proposition is the value of a function it is always of the order of the function generating it. A proposition with a determinate order is constructed by quantifying over all the real variables of a matrix of a given order.

Matrices: Propositions:

1^{st} Order:

$$\phi x \ldots (x/\exists x).\phi!x$$

2^{nd} Order:

$$f(\phi!x)(x/\exists x)(\phi/\exists\phi).f(\phi!x)$$

As Russell puts it:

> These 2^{nd} order propositions presuppose the totality of individuals and 1^{st} order propositions. Generally speaking, we obtain propositions of order n from matrices of order n: we may define a proposition of the nth order as one which involves an apparent variable of the $(n-1)$th order in the functional hierarchy. (Whitehead and Russell 1910-1913, page 55)

Paradoxical assumptions are incapable of being formulated in the ramified theory of types.

With this apparatus in place, Russell's approach to defining numbers can be better appreciated. The general strategy is to reduce the mathematical concept of set to the logical concept of class, and then complete the reduction by reducing the latter notion to the primitive idea of the propositional function of one variable. A class can now be defined as the totality of those possible values of the function for which the function is true. This is not completely satisfactory, because the relation between a function and a class is not univocal. Strictly speaking, a function does not "define" a class since the same class may be determined by any number of other functions. The class determined by "x is a man" can just as well be determined by the function "x is a featherless biped". With the help of the equivalence operator, a class is defined as that element shared by any number of propositional functions. We define formal equivalence as: $(x).\phi x \equiv \psi x$. Two functions are formally equivalent when they are both true for each possible argument., i.e. when every argument satisfying one satisfies the other, and vice versa. So "z is a man" is formally equivalent to "z is a featherless biped". When two functions are formally equivalent they have the same extension. The definition is found by calling this extension a "class". "Since extensional functions are many and important, it is natural to regard the extension as an object, called a class, which is supposed to be the subject of all the equivalent statements about various formally equivalent functions" (Whitehead and Russell 1910-1913, page 74).

Now this presupposes that the functions in question are extensional and that their truth value depends solely on the variation of the values of their argument(s). Not all functions have this feature. Certain functions are intensional in that their truth value depends on something other than the extensions of their arguments (if the argument is a function). If I'm unfamiliar with the concept of a featherless biped I can believe *that x is a man* without believing *that x is a featherless biped*. So Russell provides a procedure for deriving extensional functions of functions. That is, an extensional function is generated for $f(\psi z)$ by providing f with an argument $\phi!z$ formally equivalent to ψz and satisfying the function f.

The derived extensional function is then defined as that predicative function formally equivalent to ψz and satisfying f. So the derived function of a function is necessarily extensional. Furthermore, this derived function is now a function of the extension determined by the argument function and this is guaranteed by the formal equivalence between $\phi!z$ and ψz. The abstraction operator expresses this extension as $z(\psi z)$ and the function of the function is written as $f\{z(\psi z)\}$. This extension was then called a "class" and the construction of an extensional function of a function yielded a definition for the class determined by the argument function. In symbols:

$$*20.1 \qquad f\{\hat{z}(\psi z)\}. =: (\exists \phi) : \phi!x. \equiv_x .\psi x : f\{\phi!z\} \text{ Df}.$$

Here $f\{\hat{z}(\psi z)\}$ is a function of ψz, defined when $f\{\phi!z\}$ is significant for the predicative function $\phi!z$. The abstractor $\hat{z}(\psi z)$ symbolizes a class, with the notation expressing the extensionality of the function ψz.

This definition allows a class abstract to be eliminated from any sentence in which it occurs as grammatical subject. The sentence that the class of ψz has the property f is replaceable by a sentence saying that some predicative property with the same extension as ψz has the property f. Technically, this definition of class is the same as that of definite descriptions. Comparing the definitions *14.01 and *20.01, shows that in both cases, the symbols to be defined $-(\imath x)(\phi x)$ and $\hat{z}(\psi z)$ – are introduced in their respective definientia as arguments of a function and so are defined contextually. Logically, the class is not apprehended directly but instead is defined as an object determined by an extensional function.

6.4 The Axiom of Reducibility

Unfortunately, the order distinctions made among propositional functions and propositions rule out talk of "all of x's properties. Take the relation of identity, which is logically defined by the identity of indiscernibles and its converse, the indiscernability of identicals. So we have

$$(1) \qquad x = y . \equiv : (\phi) : \phi x . \supset . \phi y \text{ Df}$$

This definition violates order distinctions by quantifying universally over individual properties: two individuals are identical if they have all their properties in common. Again, it does not allow the definition of the identity of a and b in terms of their possessing all the same properties, in view of the illegitimate unrestricted quantification over properties. So this definition would be systematically ambiguous and would have to assign meanings according to their order. A first definition would satisfy all 1^{st} order functions, a second all 2^{nd} order functions etc. This solution is unsatisfactory because generality must be given as such. Indeed we can imagine two objects having all their 1^{st}, 2^{nd}, and nth order properties in common without being identical because they differ in their properties of order $n+1$. Two individuals are identical if they possess all their properties in common, whatever their order. To affirm that the definition of identity holds for any order whatsoever is precisely what is prohibited by the ramified theory of types.

The seriousness of the problem surfaces in the logicist derivation of number where an inductive number is defined as possessing every hereditary property of 0. This is illegitimate in the ramified theory in view of its unrestricted quantification over properties. Stating the principle of induction in logical terms requires universal quantification over the properties attributed to 0, n and $n+1$. The ramified theory's rejection of quantification over all properties requires distinct formulations of the principle of induction according to the order assigned to the quantified properties. But again, this systematic ambiguity doesn't satisfy the requirement of generality in mathematics. Consider two whole numbers m and n. The property of "being a finite number" is a second order property because in virtue of the principle of induction it ranges over the set of first order properties. So we can't prove, without violating order restrictions, that the sum of the whole numbers $m + n$ is itself a whole number. Take some arbitrary first order property P. Given the principle of induction, from $(m + 0)$ possessing P and given that, if $(m + n)$ possesses P then $[(m + n) + 1]$ possesses P, it's easy to deduce that if m possesses P, $m + 0$ possesses P and, more generally, $m + x$ possesses P. But this sort of inductive reasoning would not be applicable to any order higher than 1 and cannot be used to establish that, if m and n possesses the second order property of being a finite number, $m + n$ is a finite number. As Russell remarks this state of affairs renders most of elementary mathematics impossible.[6]

Russell introduces an axiom whose sole function is to ensure the reduction of the order of propositional functions, the axiom of reducibility. This axiom guarantees that every propositional function, of whatever order, is equivalent for all its values to some predicative propositional function. This axiom states that for every property there corresponds a coextensive predicative property, where a property is predicative iff it has the lowest order consistent with its type. In the simplest case, the function of individuals or ϕx, there exists an equivalent predicative function $\phi!x$ of an order immediately above its variable, in the case here of an individual variable of order 0, the function is of the 1^{st} order.

To see how the axiom works take the following example from *PM*, the proposition "Napoleon had all the qualities that make a great general". If we note by $\phi!x$ the first order functions which are qualities such as courage, being a good strategist etc, "having all the qualities of a great general" is a second order function of individuals in that it involves the set of first order functions. If we write this as $f(\phi z)$, our initial proposition

[6]Other examples are the least upper bound theorem and Cantor's proof that every class is less numerous than its power class.

is written as: $(\phi) : f(\phi!z) \supset \phi!(\text{Napoleon})$. The axiom of reducibility guarantees a particular predicative function equivalent to the second order function in question. This means that all great generals have in common some specific quality. This quality could be the disjunction of all their particular qualities. As for the definition of identity, we obtain a satisfactory definition by applying the original formula to predicative functions:

$$*13.01 \qquad x = y. =: (\phi) : \phi!x. \supset .\phi!y \text{ Df}$$

That is, x and y are identical if they share all their predicative functions. Because the axiom of reducibility guarantees that every function of x is reducible to a predicative function, the definition holds for every function of x. Clearly the same reasoning holds for the functions of y and restores the generality in the definition of identity ruled out by the distinction among orders.

Similarly, the axiom of reducibility provides a satisfactory formulation of the principle of induction. Stating it for any predicative individual function suffices. In the PM this comes down to giving a general definition for the ancestral by using predicative functions. This gives $xR * y : \phi!z.zRw. \supset .\phi!w : \phi!x :\supset .\phi!y$. If y possesses all the hereditary predicative properties belonging to x, in virtue of the axiom of reducibility, it possesses all the hereditary properties of whatever order belonging to x. Because the principle of induction applies without restrictions on orders, it is now possible to prove that, if m and n are finite numbers, their sum $n + m$ is also a finite number. The second order function "m is a finite number" can be replaced by the first order predicative function, call it F, which is equivalent. Applying the reformulated principle of induction to this function, we can deduce that $m + x$ is F. Given that this function is equivalent to the initial second order function, we have $m + x$ is a finite number, and therefore that $m + n$ is a finite number.

But does the axiom mitigate the effects of ramification to the point of reintroducing the paradoxes? Order distinctions were introduced to proscribe intensional paradoxes. However, while the axiom reduces orders, it still requires that the substitutable functions be formally equivalent. In the case of the Epimenides, the formal equivalence of the functions does not allow their substitutivity. As Russell puts it "an expression such as "Epimenides asserts ψx" is not equivalent to "Epimenides asserts $\phi!x$" even when ψx and $\phi!x$ are equivalent" (Russell 1908, 82-83).

So, for any function of whatever order, the axiom of reducibility postulates the existence of some coextensive predicative function.[7] It was impossible to simulate classes without the axiom. For if classes exist, then for any property φx there is a class α of just those things having φx, and so there is a predicative property coextensive with φx; viz.: x belongs to α. Such a property is predicative in that it is defined without the use of an apparent variable. Being a member of this class would be extensionally equivalent to φx and so would effect the desired reduction in orders. But this requires assuming the existence of the class and the axiom supplies these predicative functions without assuming classes exist:

> There is no advantage in assuming that there really are such things as classes, and the contradiction about the classes which are not members of themselves shows that, if there are classes, they must be something radically different from individuals. It would seem that the sole purpose which classes serve, and one main reason which makes them linguistically convenient, is that they provide a method of reducing the order of a propositional function. We shall, therefore, not assume anything of what may seem to be involved in the common-sense admission of classes, except this, that every propositional function is equivalent, for all its values, to some predicative function of the same argument or arguments. (Russell 1908, page 81)

[7]Where the number of individuals is finite it is always possible to construct a predicative function as the disjunction of each of those properties particular to each individual. The axiom is plausible when entities of finite complexity are considered. Because the totalities found at any order are the result of a proper construction, one can be sure of finding at the bottom of this hierarchy predicative specifications whose objects are given in acquaintance. But the axiom does its real work in mathematics where infinity comes into play by providing predicative functions that can be neither constructed nor named. But how is presupposition supposed to work here?

In this way we can always simulate a class which brings together only those values which satisfy the function in question. So, for functions with individual variables, the axiom of reducibility is called the axiom of classes and is written:

$$*12.1 \qquad (\exists f) : \phi x . \equiv . f!x$$

Similarly for functions of two individual variables called the axiom of relations

$$*12.11 \qquad (\exists f) : \varphi(x, y) . \equiv . f!(x, y) \ .$$

Russell's strategy in 1910 consisted in establishing the extensionality of all the required functions of functions by constructing an extensional function derived from an initial function of a function. To do this, one had to be assured that some predicative function $\phi!x$, equivalent to the function ψz also satisfies the original function of a function $f(\psi z)$. This guaranteed the extensionality of functions of functions and defined a class, such as $z(\psi z)$, as the extension of the function ψz . The motive for the extensionalisation of functions of functions should be clear: it is nothing other than the introduction of the concept of a class by means of the axiom of reducibility. The preceding definition relies crucially on the existence of a predicative function formally equivalent to a given function ψ. And this existence is only guaranteed by the axiom of reducibility. It is also easier to understand Russell's claim that admitting the axiom of reducibility is technically, though not ontologically, equivalent to admitting classes. In the *PM*, the choice was between assuming the existence of classes and then deducing the extensionality of functions of functions, or reducing classes and deriving all the needed extensional functions with the axiom of reducibility. Russell and Whitehead chose to dispense with classes and admit the axiom of reducibility for reasons that had nothing to do with ontological economy, but everything to do with intensionally grounding mathematics and logic in a way that avoided the paradoxes.

6.5 Conclusion

By allowing the expression "all the predicative functions of x" to be substitutable for the ambiguous formula "all the functions of x", the axiom of reducibility restored the sort of expressions of generality essential in mathematics. But not all mathematicians have taken impredicative definitions as illegitimate. In some sense the tallest man in the regiment 'involves' all the men of the regiment, but this does not prove that he is not a member of the regiment. Likewise, the tallest man in the regiment in some sense 'presupposes' the regiment, but this does not mean that the regiment 'has no total' or that we cannot quantify over all its members. As Gödel points out, if classes exist regardless of our thought and talk about them, as soldiers in the regiment do, there is no reason why it can not be granted that these classes can be specified impredicatively.

A realist about abstract entities will assert that how we specify them has no bearing on their nature; these entities exist independently of us and in principle each could be specified by a name. The constructivist about properties, on the other hand, will argue that the identity of a property will be a function of how we characterize it. After all, properties are taken on this view to be objects whose existence is wholly dependent on human thought and language. But Russell is not making this sort of claim. What he has in mind is this: allusions are made in thought and talk to entities which, given the metaphysical state of the universe, cannot be named. Such entities can only be described and any expression used in that description can't use a name assuming the existence of that entity without running the risk of circularity. It is something else to insist, as Gödel does, that one is thereby committed to holding that our way of specifying the entity forms part of its essence. And Russell would no doubt argue that he is not construing entities in this way. From *On Denoting* to *Principia* Russell urges that care be taken when inferring entities. That care is expressed in strict rules for constructing predicative expressions. Furthermore, the basis for inferring such entities must be grounded in acquaintance. But objects given in acquaintance exist independently of the minds acquainted with them. However, what is more troubling for Russell is the effect this new theory of abstract entities has on logic as synthetic a priori. There is a great tension in treating the relations involving abstract entities as synthetic while holding that these relations entail ontological presuppositions.

What is going wrong? The crucial point is that there is an underlying ambiguity in the axiom of reducibility. The axiom of reducibility implies that. however many ways individuals can be grouped together. these will be coextensive to a grouping determined via simple concepts. Yet, eliminating higher order intensional entities from our ontology leave descriptions in these upperregions *empty of content.* This would effectively kill the project of establishing mathematics and logic as synthetic a-priori. So Russell invokes additional content over and above an extensional listing of basic entities in order to supply an explanation for the resolution of, for example, the Epimenides paradox. Which begs the question: where does he get this? Presumably, the axiom of infinity provides all the properties and attributes which would be required. But this, to paraphrase Russell, has all the advantages of theft over honest toil. Indeed, after having gone through so much bother in specifying such a fine-grained hierarchy of attributes, it is not at all clear where this intensional content comes from. Furthermore, if our access to it is secured via acquaintance (which is, after all, an extensional relation) there is no reason to suppose that this content is acquired without our knowing *truths* about it.

The ensuing controversies over the significance of the axiom are hardly surprising, given the uncertainty over what, exactly, Russell had in mind. The objections to the axiom offered by Quine, Ramsey, and a legion of others, has been to insist that the axiom undoes the effect of ramification. But this is a mistake, largely the result of not seeing Russell's logicism as intensional. Gödel also suspected something amiss with the axiom, but settles on a critique which amounts to scolding Russell for not taking realism about properties more seriously. But it should be clear enough to the reader by now why Russell cannot be regarded as an anti-realist about properties. The difficulty for Russell is precisely in locating the kind of content that disallows substitution of coextensive properties, given the ramified hierarchy of types.

Russell must have had doubts as to the source of this additional content. Russell clearly felt that the axiom of reducibility did not undo the effects of ramification, but this was accomplished only due to the understanding's grasp of some kind of intensional content. But it is very difficult to see how this can be accomplished through an extensional relation; viz acquaintance. How such content could be supplied by simple entities is left unsaid. Russell's account is faced with the difficulty of explaining the informativeness of our talk about the most general features of the world.[8] At a fundamental level, Russell could not see how the source for this informativeness could spring from some internal feature of human beings.

[8]Indeed, to the point of regarding the identity of indiscernibles as analytic. Russell would also remark that "The importance of "tautology" for a definition of mathematics was pointed out to me by my former pupil Ludwig Wittgenstein" (Russell 1919, page 205)

Chapter 7

Propositions and Logical Forms

7.1 The Theory of Belief

The theory of incomplete symbols had given Russell reason to hope that the contradictions could be avoided by dispensing with the assumption of entities such as classes and functions. By 1906 general propositions had lost their status as unified entities. Sentences expressing such propositions were recast in the language of quantifiers and propositional functions. Like the pseudo-objects apparently denoted by definite descriptions, general propositions are complexes to be broken down into constituents through analysis. During the period where Russell regarded truth and falsity as intrinsic properties of propositions, false propositions had been endowed with the same ontological status as true ones. While elementary propositions remained part of Russell's ontology during the writing of "On Denoting" and the subsequent development of the theory of incomplete symbols, the elimination of elementary propositions called for a revised account of truth for propositions.

This issue emerged as a problem over the nature of the object(s), if any, of the 'propositional attitudes'. In *On the Nature of Truth* (?), Russell considers two theories about belief. According to the first theory, when a belief is true there is a corresponding fact. When the belief is false there is no such fact. According to the second theory, a belief always has an object and this object is a true proposition, or *fact*, when the belief is correct and a false proposition, or *fiction*, if the belief is mistaken. The first option is consonant with common sense and ontologically parsimonious. The difficulty with it is to explain what it is we believe in when our belief is mistaken. If our belief is in nothing it is not a belief. This objection makes the second alternative attractive, but Russell would try to amend the first option to make it more acceptable. He rejects what he takes to be one of its presuppositions. The presupposition requires that a belief must have a single unitary thing as its object. That is, a complex mental event would have as its object a unitary entity, in this particular case a proposition. This presupposition requires accepting 'non-facts' into our ontology as well as facts, objective falsehoods as well as objective truths, false propositions as well as true ones. Russell rejects this and insists that aRb may be the object of a belief without positing a corresponding complex as a fact. One has the idea of a, the idea of b, and the idea of R. Now, each of these singular ideas has as its object: a, b, R; but there is no need to postulate some additional thing aRb, a fact corresponding to theobject of the belief. Whether aRb turns out to be a fact or not in no way effects the principle that to every belief there corresponds an object, and every belief without a corresponding fact remains a belief. Thus, the objects of a belief are entities separate from the entities which must exist, or subsist, to make the belief true.

In 1910, in the chapter of the *Philosophical Essays*, entitled "On the Nature of Truth and Falsehood", Russell returns to the theory of belief sketched out in 1907. Here he is no longer concerned with beliefs apart from the propositional attitude of believing. He is no longer merely saying that there are no false propositions, but rather that there are no propositions. There are some correct beliefs and so there are facts. What is different is that the latter no longer serve as objects of the former; they are not *propositions*. The theory Russell ends up with is that beliefs are not correlated with unitary objects and that there are no propositions. As propositions are nothing other than logical fictions, a belief is no longer a two-termed relation between a subject and an object. Rather, it is a relation involving a subject with many objects. These objects constitute a unity, a *fact*, if the belief is correct, and nothing at all if the belief is mistaken.

What is required accordingly is a new theory of truth that makes truth an extrinsic property of judgements rather than of propositions. Judgement is a relation between a subject and an extra-linguistic reality, a world made up of *facts*[1]. A judgement is true when a correspondence holds between a judgement's content and this complex object. Facts are mind-independent. While a proposition is the result of a synthesis effected by a subject, a fact is a datum. Unlike other data, however, a fact is not simple. A true proposition's discursiveness is a representation of the complexity of the corresponding fact. Facts are independent of one another and are reducible to basic simples related to each other by external relations. While elementary propositions disappear, they make room for elementary facts.

One of Russell's problems had been to explain how relations could figure both as propositional constituents and as source of propositional unity. In 1910 the problem is overcome by assigning the job of providing to the *act* of judging. According to the theory of types, being able to determine an atomic proposition was a precondition for being able to determine a propositional function. Yet propositions are not terms. Rather, the idea of a proposition is explained through the new concept of a 'judgement of perception', which becomes the new ground for meaning and truth. In 1910, an elementary atomic proposition is no longer the object of direct knowledge. Given the resulting fragmenting of propositions into isolated constituents, the job of bringing these together to form

[1]Russell is trying to ensure that the synthetic unity of apperception is mind independent. Now that abstract entities are ordered through presupposition relations Russell wants to ensure that the world of facts is not so constrained. It cannot be overemphasised that Russell was not particularly worried about the ontological status of abstract entities, but he certainly was worried about the presupposition relations holding among them. *This* marks the fundamental shift caused by the paradoxes. And Gödel's point in his 1944 paper is then that such entities just *are* independently of relations they may have to each other.

suitable objects of 'propositional attitudes' was to be affected in judgement by the mind.

7.2 The Theory of Judgement

The picture in *Theory of Knowledge* is that of a mind constructing propositions using constituents that are somehow grounded through direct epistemic contact. The acquainted mind is a passive receptacle of its objects. This account makes sense in the light of Russell's professed realism. The items of acquaintance exist independently of our identification and characterization of them. The trouble then can be put very simply: the ability to distinguish the entities corresponding to the symbols in the language seems to transcend what could be delivered through the epistemic relation of acquaintance.

It is a commonplace about Russell that he took the search for certainty to be one of philosophy's central tasks. This search found its expression in the attempt to specify the extent to which our descriptions of the whole of our universe pertain to something mind-independent. By elaborating the basic intuition that things and their relations are quite independent of any propositional attitude we may have toward them, Russell developed an account of how the same propositional content could be believed by more than one person (even when the proposition was false) as well as a ground for the logical notion of valid inference. Throughout his work, he was guided by the conviction that all informative content was derived from a source located outside the private confines of the individual subject. If epistemology asks what we can know and what are our grounds for saying that we know, Russell's answer was framed in terms of an account of what lies outside of us and the kinds of inference required to pass from this content to the rest of what one wants to count as knowledge. However, Russell would attempt to ground the sort of certainty he sought in an epistemological doctrine that collapsed under the weight of the phenomena it was called to account for.

The broad metaphysical picture Russell painted in *Theory of Knowledge*, and in other writings, is one of subjects having epistemic contact with the two fundamental constituents of the external world: concrete things (sense data) and subsistent abstract entities (such as universals[2]). A sense-datum was a particular object qualified by a property and/or standing in a relation to other sense data. A sense-datum was a spatio-temporal particular referred to directly without being approached via any descriptive content, the sort of object named by means of such demonstratives as 'this' and 'that'. When a sense-datum was brought under a concept (i.e., a universal), the result was a complex object. These complex objects were then called sensible facts. An example of a complex object of this type is the fact described by the sentence, "This is to the left of that". Russell concluded that phenomenal qualities (sense-data) and universals were objective constituents of the world, having a mind independent life, and that they served as suitable denotata for linguistic expressions. It was out of his account of the nature of a subject's contact with these constituents that a fundamental distinction was drawn between two different kinds of knowledge,knowledge by acquaintance and knowledge by description:

> If I am acquainted with a thing that exists my acquaintance gives me the knowledge that it exists. But it is not true that, conversely, whenever I can know that a thing of a certain sort exists, I or some one else must be acquainted with the thing. What happens, in cases where I have true judgment

[2]It is not easy to get clear on Russell's understanding of the distinction between universals and particulars notwithstanding his insistence on its importance. Individuals are defined in the *PM* as non-functional constituents of elementary propositions, which are propositions containing no apparent variables. The question has been whether or not a concept or universal should be construed as an individual, or rather identified with the functions extracted from the proposition. The latter alternative is attractive in that it would go quite a way towards marking a clear ontological distinction between the two, because if universals are functions they are thus intrinsically structured. But if functions are genuine components of propositions (or of the facts to which such expressions correspond), there is a conflict with the vicious circle principle. The other alternative is to regard universals as non-functional constituents of propositions and so as individuals. The problem here, as Wittgenstein noted, is then that the multiple relations theory allows the assertion of nonsense:

"If I analyse the proposition Socrates is mortal into Socrates, mortality and $(\exists x, y) \in_1 (x, y)$ I want a theory of types to tell me that 'mortality is Socrates' is nonsensical. Because if I treat mortality as a proper name ... there is nothing to prevent me to make the substitution the wrong way round". (Wittgenstein 1961, page 122)

> without acquaintance, is that the thing is known to me by *description*, and that, in virtue of some general principle, the existence of a thing answering to this description can be inferred from the existence of something with which I am acquainted. (Russell 1912, page 45)

Since, according to a well-known principle he sets down, a proposition can be understood only when one has acquaintance with all of its constituents, we can understand propositions about things with which we are not acquainted only if these are tied to elements with which we do have acquaintance. Knowledge by acquaintance is a dyadic relation between a knowing subject and the constituents of a fact given in direct awareness. I say that I am *acquainted* with an object when I have a direct cognitive relation to that object, i.e., when I am directly aware of the object itself. By 'directly aware' Russell meant without the intermediary of any process of inference or any knowledge of truths.

> The fundamental characteristic which distinguishes propositions (whatever they may be) from objects of acquaintance is their truth or falsehood. An object of acquaintance is not true or false, but is simply what it is: there is no dualism of true and false objects of acquaintance. And although there are entities with which we are not acquainted, yet it seems evident that nothing of the same logical nature as objects of acquaintance can possibly be either true or false. (Russell 1984, page 108)

Truth, then, required that we go beyond the immediate deliverances of the senses, and no judgment could consist of mere acquaintance with a simple sense datum. Most of what we know, in fact, is not knowledge by acquaintance, but knowledge by description. This type of descriptive knowledge would have to be inferred from items with which we were directly aware. Once we had secured these primitive elements a comprehensive body of descriptive knowledge could be given by the logical analysis of the structure of molecular or complex propositions, all of which are truth-functionally compounded out of atomic propositions by the logical constants. Russell's (preliminary) view then was that if we believe a proposition, there must be a subject, a relation of belief, and a proposition, a group of objects brought together by the act of judging

Unfortunately, this intuitively plausible account began to unravel when Russell tried to provide an account of the truth conditions for belief statements. Since Russell had abandoned propositions as entities because of the paradoxes, he had to account for what is known as the propositional attitudes. Russell's solution was that such attitudes are complexes with a structure, and that truth is a relation between beliefs and facts. Falsity is merely the lack of any such correspondence. However, a difficulty can be located in the tension between Russell's views of truth as correspondence with fact, on the one hand, and his realist assumptions concerning acquaintance on the other. Simply put, if true belief is to be accounted for through it's correspondence to some actually existing thing (i.e., a fact), what are we to do with those false beliefs that (our robust sense of reality assures us) have no objects to serve as denotata? Furthermore, with mental acts being construed as directed towards an object, false propositions could not be genuine objects of belief. Although a belief is true by virtue of its corresponding with reality, if the falsity of a belief is due to its lack of correspondence with reality, what does this lack of correspondence amount to?

To see why Russell's reticence to admit false propositions[3] posed a difficulty, let us look at the following case of false belief:

(1) Othello believes that Desdemona loves Cassio

Russell did not doubt the existence of the objects that the belief was about, e.g., Cassio, Desdemona, and the dyadic relation "Loves". Rather, because Desdemona does not love Cassio, even though Othello believes this is the case, a strategy had to be found

[3] See also Russell's argument to the effect that a false proposition is not an entity in (Russell 1984, page 109).

to account for the false proposition that

(2) Desdemona loves Cassio.

Proposition (2) must not occur as a fully-fledged constituent in the final analysis of (1). The strategy Russell adopted was to provide an account of beliefs that does not make (2) or more generally, what is believed) propositional in its own right. Russell's multiple relation theory of judgment tried to eliminate this problem by analyzing (1) as a single fact consisting of a relation of belief that relates the subject Othello with the individuals Cassio, Desdemona and the universal Loves. Since belief would be the only "relating relation" in (1), (2) would not be a propositional component of (1). Thus belief consisted in a complex, many-termed relation between a subject (in our example, Othello) and these objects (Desdemona, Loving and Cassio). In this way, even though a false proposition could appear as the object of Othello's belief, there was no need to construe the 'proposition' as a constituent of the belief and as either existing or subsisting.

In the new theory (known as the multiple theory of judgement), words continued to stand for particulars in the world, but now, when these words were brought together to form sentences, the things that these words stand for come together to form the constituents of propositions. Russell always associated a unique semantic object with each syntactic unit. When such objects joined with others to constitute a proposition, he naturally called them the 'meanings' of the corresponding expressions in the sentence. The main function of Russell's meanings was to act as the building blocks with which the propositional complex was constructed. The subject would thereby be related by an act of acquaintance with the particulars constituting the proposition. Subjects would come to grasp the semantic units that lay at the basis of such a construal of the propositional complex by being acquainted with the constituent entities (of a proposition) that the terms in a sequence, expressing the proposition, designate. We understand sentences because we understand their constituent phrases, and we understand these because we stand in an immediate and direct relation to the meaning of those terms. The propositional complex was constituted by a mental act in which the subject would bring together the objects with which this act of judgment was concerned. Judgement, then, was a multiple termed relation between a judging mind and the diverse constituents of what was maintained to be the proposition.

In Russell's discussions of the multiple theory of judgment, there was a tendency for him to focus on the perception of a complex object, say 'a-is-similar-to-b'. If attention showed that the object was complex, the perceiver could then analyse the complex by decomposing it into its simple constituents. Analysis consisted in isolating the components of a complex fact with which one was acquainted by transferring one's attention from the whole of that complex to its constituent parts. Assuming that a subject could simultaneously attend to the complex and to its parts, analysis of a perceived complex would require perceptions of the constituents as parts of the complex. A complete analysis would provide the grounds for judging whether a and b stood in the relation *is-similar-to*. The judgment was true if this complex sense-datum existed, false otherwise. Acquaintance with sense-data was thus seen to be the basis and ground of our judgments of sensation. Simple sense data were the constituent objects of our judgments, without which the judgments could not even exist. Complex sense data were the facts that make the judgments true or false.

To make this account of the act of judging more intuitive, let's look at a variant of Russell's Othello example in *The Problems of Philosophy*,

(3) Othello believes that a is similar to b.

Othello, as the subject, would unite the terms a, and b, and the relation similarity, into a propositional complex. This complex could be schematically represented something along the following lines;

(4) Believes {Othello, a, similar, b}

Each of these terms would be the 'bricks' of the proposition, brought together (or 'synthesised', to use Russell's expression) to form a unity by the subject's act of judging. Judgment is a mental act which functions as the cement which holds together the bricks- similarity, a, and b. If Othello believes truly, then there is a complex with *a-is-similar-to-b* contained therein; if Othello believes falsely, then there is no such complex. For a belief to exist there must be a subject to have it, but not its truth. The belief is true when it corresponds to a certain associated complex. If the belief is true then the complex already exists- its association does not rely upon the belief to be true- but the complex must already exist or else belief could make it true.

As attractive as this account is, it soon became apparent that there is a difficulty concerning the status of the embedded relation 'similarity'[4]. When Othello combines these three constituents in thought how does he know that the combination is meant for the possibility that a is similar to b and not for the possibility that each of these constituents exists separately, or, for that matter, that some nonsensical concatenation of the terms such as 'similarity b a' should be excluded? The embedded relation 'similarity' seems to be doing more work here than the other constituents of the complex, more work, that is, than we would expect of a term we were merely acquainted with. It became apparent that understanding cannot be analyzed into a set of acquaintances. A complex such as a-is-similar-to-b cannot be analyzed by the understanding as a set of acquaintances $\{a, b, \text{similarity}\}$ precisely because such a discreet list leaves their order in the complex undetermined. In Russell's account this relation is treated as just another term in the complex. Russell demanded that acquaintance be extensional, yet mere acquaintance with an embedded relation seems to enable the understanding to grasp the role that this embedded relation contributes to ordering the particulars in the proposition.

[4]This account of Russell would be subjected to a number of criticisms. Peter Geach, for example, held that Russell's treatment of universals as constituents of the judgment relation in his 1910-1912 versions of the Multiple Relation Theory of Judgment is inadequate. Since a universal like "larger than", as a relative term, "is incomplete; it carries with it, so to say, two blanks that need filling up" (Geach 1956, page 56). Geach, of course, did not have access to the *Theory of Knowledge* manuscripts, and from the manuscript we can see that Russell is clearly aware of the problem.

And this is to treat the embedded relation as more than a brick[5].

Russell wants the subject's contact with these constituents to be unmediated and direct. The intensional hierarchy of his ramified theory of types presupposes some given base of entities and predicative properties. However, sometimes a relation relates and at other times it doesn't. If the understanding is only passively acquainted with this relation, how can the understanding then make the distinction between the relation as such, and the relation as relating? Once it became plain that mere acquaintance with the types of the terms in a complex was insufficient for the understanding to provide an ordering of the terms in that complex, a reconstruction of the proposition became pressing. And it was also clear that a solution to the problem would require that the understanding relation have some means of distinguishing and arranging the constituents of the object-complex of understanding when this object complex is not present in perception.

In *Theory of Knowledge* Russell would attempt to overcome the problem by allowing the understanding to see the need for the terms and their relation to the universal through acquaintance with a new sort of logical object. In fact, he would claim that we have acquaintance with the *logical form* of the complex in order to synthesize the constituents into an atomic propositional thought.

> In order to understand "A and B are similar", we must be acquainted with A and B and similarity, and with the general form of symmetrical dual complexes ... But these separate acquaintances, even if they all coexist in one momentary experience, do not constitute understanding of the one proposition "A and B are similar", which obviously brings the three constituents and the form into relation with each other, so that all become parts of one complex. It is this comprehensive relation which is the essential thing about

[5]Russell became aware of the difficulty, and by the time he amended these doctrines in the light of (presumably Wittgenstein's) criticism's, he would say of the view "I held formerly that the objects alone sufficed, and that the 'sense' of the relation of understanding would put them in the right order...no longer seem to me to be the case." (Russell 1984, page 116)

> the understanding of a proposition. Our problem is, therefore, to discover the nature of this comprehensive relation. (Russell 1984, page 112)

The logical form of a relation complex involving a universal now enters as a new constituent of the analyzed form of an understanding of that relation complex. The new argument holds that while our thought must in some way "unite" the terms and the relation it cannot unite the given terms in the given relation out in the world, but can be understood as relating them to the general logical form. Without the logical form of a complex, the understanding would not know what to do with the terms and the relation in cases where the complex is not present to acquaintance. Russell thought that an acquaintance with forms was required to explain the fact that we can understand propositions we have never seen before.

Once we are acquainted with the logical types of the constituents in a complex, we need acquaintance with the logical form of the proposition in order to synthesise the constituents into an atomic propositional thought. Thus the understanding of the proposition involves acquaintance with a, and with b, and the relation *similarity*, but also involves understanding how these terms are put together by the understanding, and this understanding comes from being acquainted with the propositions form.

7.3 The definition of Propositions in *TK*

Like every object in the intensional hierarchy, propositions were constructs of a sort. They did not enjoy a status as independent entities yet they were not merely mental. But Russell did not think of them as inhabitants of some metaphysical limbo:

> We arrived at the proposition, in the first place, as something which a number of mental events have in common. If two men judge that A and B are similar, or if one man makes this judgement on two occasions, it is obvious that the difference between the two events is only on the subjective side, and that on

> the objective side there is a similarity consisting not only in the fact that the same objects are concerned, but also in the fact that the different judgements bring the objects into the same relation to each other ... It is this common element that we call the "proposition", and wish, if possible, to isolate from its subjective context. (Russell 1984, pages 114-116)

Thus we must also be acquainted with an additional entity if we are to understand the proposition expressed by 'aRb'. This additional entity is the logical form of the proposition. Russell accounts for logical form as what remains when the non-logical constants of a proposition have all been replaced by variables. This characterization fits in with his view of logic as the most general of the sciences. If a proposition mentioned anything specific, whether particular or universal, it was not logical. A "touchstone by which logical propositions may be distinguished from all other" is that they result "from a process of generalization which has been carried to its utmost limit" (Russell 1984, page 97).

So, in the complex 'a is similar to b', the terms and relation were to be replaced by a process of existential generalisation to yield xZy, and so on for all complexes endowed with a similar structure. The logical form of this particular complex was seen as expressing the generality that "something bears some relation to something". The truth of this proposition was taken to be self-evident, and indeed to be a simple object, known immediately by direct and unmediated acquaintance:

> The importance of the understanding of pure form lies in its relation to the self-evidence of logical truth. For since understanding is here a direct relation of the subject to a single object the possibility of untruth does not exist, as it does when understanding is a multiple relation. (Russell 1984)

Acquaintance with logical forms is thus necessary for the propositional understanding of complexes. The form tells us how the terms are to be put together, and the role of thought is to bring the terms in the complex together with their form. The form connects the constituents into a complex without itself being a member of this proposition.

Many relations appear to come with direction, because not all dual relations are symmetrical. Understanding a non-symmetrical relation such as 'loves' seems to require that a subject be acquainted with, and grasp, something which is called the 'sense', or "direction" of that relation. For example, let "R" be the temporal relation of precedence; in a representation of the fact that "S understands that a precedes b", it is not enough to mention the terms 'a', 'b', 'sequence', and the form '$Z(x, y)$'. In 'a precedes b', the relation 'precedes' comes with a 'sense'. All of this was evident to Russell:

> When someone tells us that Socrates precedes Plato...what we understand is that Socrates and Plato and 'precedes' are united in a complex of the form 'xRy', where Socrates has the x-place and Plato has the y-place. It is difficult to see how we could possibly understand how Socrates and Plato and 'precedes' are to be combined unless we had acquaintance with the form of the complex (Russell 1984).

Asymmetrical relations called for a more refined treatment of form, and understanding Russell's account requires understanding a particular property of relations; some relations display a property known as 'homogeneity' which is just to say that their flanking terms can be permuted while retaining a significant expression. Other relations have the property of heterogeneity, which is displayed by an asymmetrical relation when its terms cannot be permuted without rendering the new expression nonsensical.

Russell's reconstruction of the propositional complex is then accomplished in two steps. The first step was merely to deny that the sense of an asymmetrical relation such as *precedes* was to be found in the relation itself. Some words that putatively refer to relations are, rather, incomplete symbols. Lying disguised beneath the surface grammatical form of an expression referring to an asymmetrical relation was a 'pure' relation not requiring flanking terms in order to deliver it as an object of awareness. His second move was to regard logical form as being characterized by the position of the terms in a 'sequence-complex'. Once the embedded asymmetrical relation was analysed in terms of a sequence complex, the relation of constituent to complex then becomes 'heterogeneous'; only one sense made sense. The object terms bound by an asymmetrical relation in a complex receive a supplementary analysis where the complex 'a-precedes-b' is broken up into a further complex composed of the object term bearing an asymmetrical relation to the complex 'a-precedes-b' itself, something like 'there is a complex α, in which a is earlier and b is latter'. In this way, a's position in the complex is marked by a heterogeneous asymmetrical relation; i.e..in this example, the relation 'is earlier in'. For example, it makes no sense to say that the complex "(a-precedes-b)-is earlier-in-a".

What is important to keep in mind here is that a relation's sense is disambiguated when it is flanked by terms of different logical type. Once the reduction of a homogeneous asymmetrical relation to a heterogeneous asymmetrical relation is effected, the position of each constituent in the complex is marked out and the sense of the asymmetrical relation is made clear. The positional relation of constituent to complex is such that only one direction can make sense. The reduction of homogeneous asymmetrical relations to heterogeneous relations is designed to explain how meaningful combinations are derived, by saying that in such a sequence two different types of objects are involved-an object term, and the complex itself. In these cases the problem of sense does not arise. The heterogeneous relation "is-earlier-than" does not relate two objects of the same type but rather of two different logical types.

Let us consider how this revised account applies to 'a is similar to b'. This proposition contains two object terms related by a symmetrical relation, viz., 'similarity'. The subject must be acquainted with the 'form' of a dual complex to ensure against uniting the 'bricks', 'a', 'b', and 'similarity' into a nonsensical concatenation of terms, such as 'similarity a b". Symbolized in Russell's fashion, the subject's understanding of this proposition is represented as:

$$U\{S, a, b, similarity, Z(x, y)\}$$

The relation of acquaintance provides direct contact between the subject and such constituents. Russell's realism requires that this hierarchy of entities be mind independent. Indeed, forms are objects in a non-spatiotemporal world (Russell 1984, page 114):

> It will be remembered that, according to our theory of the understanding of propositions, the pure form is always a constituent of the understanding-complex, and is one of the objects with which we must be acquainted in order to understand the proposition. If this be true, then the understanding of the pure form ought to be simpler than that of any proposition which is an example of the form. Since we desired to give the name "form" to genuine objects rather than symbolic fictions, we gave the name to the "fact" "something is somehow related to something". If there is such a thing as acquaintance with forms, as there is good reason to believe that there is, then a form must be a genuine object; on the other hand, such absolutely general "facts" as "something is somehow related to something" have no constituents, are unanalyzable, and must accordingly be called simple. They have therefore all the essential characteristics required of pure forms. (Russell 1984, page 129)

Russell appeals to forms to account for propositional complexity, yet they are supposed to be simple. The difficulty is in understanding how general facts can be simple entities.

The nature of Russell's problem is clear enough; there is a need for a distinction between acquaintance with a complex (which is extensional) and knowledge about a complex (which is intensional). It is obvious that a mere list of universals and items of sense data which are objects of acquaintance for a subject is not enough to enable that subject to know how these items are to come together in a proposition. Understanding the proposition "A tomato is red, round, and smooth to the touch", does not consist of merely listing off these properties. Someone who went out and saw a round ball, a red traffic light, and touched a role of silk and then claimed to have found a tomato would not be said to have understood the proposition. Russell's answer to this problem in *TK* is that the logical form of a dual complex is presupposed both in our perception of the complex and in our characterization of a complex when we are not confronted with it in experience. Without bringing in the form to which there is an abstract cognitive relation we could not succeed in unifying the constituents of the complex.

The reasonable objection, here then, is that in order to mark the move from understanding to truth we need something which marks a transition from mere acquaintance with a complex to a propositional understanding of it, and that it is difficult to see how an extensional relation such as acquaintance could provide us with any such thing. But by Russell's own account in *TK*, understanding and acquaintance are very different kinds of relation. Acquaintance is not supposed to involve knowledge of truths. Yet *that something is related to something* is certainly a truth, and Russell clearly thinks of the acquaintance as delivering truth in the case of logical form. Russell has to show that in the case of general propositions (the logical forms), understanding and acquaintance are the same. The argument is that entirely general propositions are self-evident and that their truth is grasped the moment they are understood. The point is that acquaintance with a form is a kind of understanding. The form is a proposition yet we are acquainted with it.

Acquaintance is held to be a direct relation between a subject and a simple object. It is precisely because general propositions contain no constituents that Russell feels confident enough to characterize the subject's *understanding* of them as an act of acquaintance. However, if (as Russell insists), the question of truth and falsity does not arise with acquaintance with an object, then his attempt to establish the self-evidence of logical forms by arguing that they are simple objects containing no constituents is in need of some refinement. Either acquaintance with a simple object is an extensional relation, in which case it makes no sense to talk of it as either true or false. Or the acquaintance relation is intensional, in which case Russell loses the use of those terms whose sense in a given context is given by picking out non-descriptively the objects they are being used to signify. The dilemma here is that the doctrine of acquaintance requires that we know the form as a form ordering terms of different types. This in turn requires that our acquaintance with the form be intensional or context-dependent. But acquaintance cannot explain how we locate this extra ingredient. Russell doesn't intend this element to be a mental projection onto the external world. But then acquaintance cannot explain how a form can be the bearer of information. So intensionality here has the effect of canceling out whatever independence was granted to entities with the relation of acquaintance.

Another way of making the same point is to say that a subject's acquaintance with the objects a and b and the relation R as well as the form simply cannot account for how the subject knows these terms to be of the appropriate types to be synthesised into the complex aRb by this form. Russell is struggling here to account for representational complexity out of simple items of awareness. Plaguing Russell's account is precisely his insistence that logical form be treated as a *simple* object. This is to saddle logical form with responsibilities it is incapable of bearing, for in the end, the extensionality of the acquaintance relation fails to keep the perceptual criteria we use to govern the application of an expression distinct from the logical function that expression may have. The upshot is that it is difficult to grant that acquaintance can bear this dual role. There is a very serious problem in reconciling the connection between the doctrine of acquaintance, on the one hand, with the doctrine that propositional understanding presupposes a subject's grasp of the self evidence of the appropriate logical form.

7.4 Conclusion

The vigorous realism of the *Principles* had relented under pressure from the paradoxes. There certainly remains a commitment to abstract entities of a general kind presupposed in Mathematics. These are the logical Forms of *TK*. And these forms are presupposed in mathematics and in all forms of reasoning about generality. The problem Russell faced, which he inherited from his predecesors, was how to account for necessity and the a priori, in a way that did not make these functions of human cognitive powers. This was the real intent behind Russell's objections to psychologism. I have argued that Russell's account of Forms cannot be made to work with the definition he provides for propositions. Russell pictures these forms as mind independent entities. ordered into a hierarchy of presupposition relations. Since forms are mind independent. the relation between them and subjects is one of acquaintance. Yet forms are also intensional entities. We have an a priori grasp of them that stems from their generality. But in grasping this we are grasping a truth and it is unclear how this sort of epistemic phenomenon is anything like acquaintance. Forms represent synthesis because they bring together discreet entities to form a complex. According to the Russellian hierarchy these forms cannot be constituents of the propositional complex. The reasons for this are twofold. On the one hand, if forms were constituents of the complexes they unified, their relation to the complex would be that of analytic containment. The second reason was that such containment would make forms the same kind of entities as the other constituents of the complex. And Russell was convinced that this sort of thing generated paradoxes. It is clear why these two concerns meshed together in Russell's mind. But the whole account is very unstable.

Russell abandons the *TK* because he could not get his account of the structural aspects of the understanding relation to work. Consider the general proposition "something is related to something" and associate with it a series of particular propositions such as "*a* is related to *b*", "*c* is related to *d*" constituting an exhaustive list of all such relations.

It is clear that Russell did not regard this list as equivalent to 'something is related to something". That is, it does not provide an analysis since presumably someone could know this disjunction and not know "something is related to something". Yet Russell claims that truth involves correspondance with facts (A propositions truth depends in some manner upon its correspondence with a fact). Then understanding generality presupposes general facts. Suppose "Something is related to something" is self-evidently true. What kinds of facts are there which correspond to its truth? The problem is to give a catalogue of all the facts which acknowledging the truth of "Something is related to something" would require. If the conditions under which "Something is related to something" is true are the very conditions under which "aRb", "bRc" etc. then the facts required by "Something is related to something", if true, are the very facts required by "...". But then, "Something is related to something" cannot express anything over and above what is expressed by "aRb", "bRc" etc. Indeed, if our epistemic relation with these facts is one of acquaintance, it is difficult to see what more it could be. And if it is we who project this intentionality onto the world, what function then does the form serve? The dilemma here is inescapable.

We saw in Chapter 2 how Kant and Locke drew a distinction between a priori propositions whose predicate is not contained in the subject, and those whose predicates are so contained. To say that mathematics and logic are analytical a priori is to say that the propositions of logic and mathematics are derivable from the law of non-contradiction alone. So one can deduce a contradiction from the denial of such a proposition, that is, an analytic judgement cannot be denied without self-contradiction. A synthetic a priori proposition is necessary but not logically demonstrable [6]. Russell can be seen as

[6]The kind of criticism Russell levels at Leibniz, to the effect that Leibniz's analytic propositions presuppose certain synthetic propositions (see *Supra* chapter 2), could be extended to Frege. If one considers Frege's definition of analytic truth, Frege says that in the course of proving a proposition "we come only on general logical laws and on definitions, then the truth is an analytic one, bearing in mind that we must take account also of all propositions upon which the admissibility of any of the definitions depends Foundations of Arithmetic, transl. Austin p.4e). On pain of circularity, these definitions could not be analytic.

attempting to de-psychologize necessary truth without resorting to a formal account of 'carving out the content' in an analytic fashion. This presupposes that we have a direct grasp of universals and the necessary connections between them. From the *Principles* through *TK* Russell rejected the suggestion that the formal analysis of concepts yielded informative content. The full measure of the changes which followed the abandonment of *TK* are reflected in Russell's account of the nature of logic and mathematics given in the *Introduction to Mathematical Philosophy*:

> It is clear that the definitions of "logic" or "mathematics" must be sought by trying to give a new definition of the old notion of "analytic" propositions. Although we can no longer be satisfied to define logical propositions as those that follow from the law of contradiction, we can and must still admit that they are a wholly different class of propositions from those that we come to know empirically. They all have the characteristic which, a moment ago, we agreed to call "tautology". This, combined with the fact that they can be expressed wholly in terms of variables and logical constants... will give the definition of logic and mathematics (Russell 1919, pages 204-205).

When Russell spoke of entities as logical constructions out of percepts, he often made misleading references to such constructions as 'fictions', as somehow unreal. This was the case when Russell spoke of classes as having no ontological status. But what Russell actually meant was that expressions standing for classes were to be replaced by symbols standing for properties. That is, class symbols were contextually defined in terms of propositional functions. So statements about classes of individuals were to be given as statements about properties of individuals, statements about classes of classes of individuals were to be translated into statements about properties of properties of individuals, and so on. In this sense, and in this sense only, is the relation of class inclusion to be eliminated in terms of the relation of formal implication between propositional functions.

But there would now emerge a strong tendency to deny an independent reality to logical constructions (the elimination of designative expressions referring to unobservables). Once Russell began substituting constructions for inferred entities, it was simply a matter of time before things became defined as constructions from percepts. Russell's new idea is that no complex of symbols can evoke a concept without the understanding having had some antecedent acquaintance with an instance of it. This marks the beginning of Russell's empiricist turn, which would become increasingly pronounced from this point on. It is curious that the Anglo-American philosophical tradition, in particular logical positivism, would draw so much inspiration from Russell given his views on the synthetic a-priori. One can only suspect that had philosophers actually bothered to read the Russell of the *Principles*, the evolution of the Analytic school would have been very different.

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