Frequency Domain Equalization for High Data Rate Multipath Channels

by

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Abstract

High data rate transmission over multipath channels requires equalizers of long impulse response. In such cases, frequency domain implementation of the block least mean square (FBLMS) algorithm offers low complexity growth relative to the time domain techniques. The work presented herein is devoted to a study of the FBLMS algorithm in comparison to the time domain equalization approach. We examine the bit error rate (BER) performance for high data rate, binary and quadrature phase shift keying (BPSK, QPSK) transmission over multipath channels. Some useful concepts that can be attained with FBLMS are as follows. Fractionally-spaced equalizers are capable of performing the function of receive filters. We show that for low data rate transmission, where the length of the equalizer is comparable to that of the receive filter, the receive filter can be eliminated. By virtue of its low complexity growth, a receiver consisting solely of a fractionally-spaced FBLMS equalizer offers similar computational complexity as compared to that of a combination of receiver filter and a symbol-spaced equalizer. For high data rate transmission it is shown that the computational complexity of the FBLMS equalizer is significantly lower than that of its time domain counterpart. Finally, normalizing the step size of the FBLMS algorithm according to the power distribution of the input process results in a significant improvement in the equalizer convergence relative to the time domain methods. All these aspects are important in high data rate cases where the equalizer can have as many as 64 tap gains.

To my loving family

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List of Abbreviations

AWGN	Additive White Gaussian Noise
BER	Bit Error Rate
BLMS	Block Least Mean Square
BPSK	Binary Phase Shift Keying
DCT	Discrete Cosine Transform
DFT	Discrete Fourier Transform
FBLMS	Fast Block Least Mean Square
FFT	Fast Fourier Transform
ISI	Intersymbol Interference
KLT	Karhunen-Loève Transform
LMS	Least Mean Square
MMSE	Minimum Mean Squared Error
MSE	Mean Squared Error
NFBLMS	Normalized Fast Block Least Mean Square
QPSK	Quadrature Phase Shift Keying
RC	Raised Cosine
RLS	Recursive Least Square
RMS	Root Mean Square
RRC	Root Raised Cosine
SIR	Signal-to-Interference Ratio

SNR	Signal-to-Noise Ratio
SNIR	Signal-to-Noise-and-Interference Ratio
TDAF	Transform Domain Adaptive Filtering
TDMA	Time Division Multiple Access

List of Symbols

$\mathbf{C}$	circulant matrix
$d(n)$	desired equalizer output
$\mathbf{D}$	diagonal matrix of input signal power
$\hat{\mathbf{D}}$	estimate of $\mathbf{D}$
$c(n)$	scalar error at the equalizer output
$\mathbf{e}(n)$	error vector at the equalizer output
$\mathbf{e}_{\mathcal{F}}(n)$	Fourier-transformed error vector
E_B	energy per data bit
$\mathcal{F}$	DFT matrix
$f(t)$	combined impulse response of the transmitter, channel and the receive filter
$h(t)$	combined impulse response of the transmit filter and the channel
$h_{t\mathbf{x}}(t)$	transmit filter impulse response
$h_{r\mathbf{x}}(t)$	receive filter impulse response
$h_c(t)$	channel impulse response
$h_{RC}(t)$	raised cosine impulse response
$h_{RRC}(t)$	root raised cosine impulse response
g_k	samples of the overall equalized channel response
I_k	transmitted symbols
$\tilde{I}_k$	detector output

$J(n)$	mean square error
$J_B(n)$	block mean square error
J_{excess}	excess mean square error
$J_{\min}$	minimum mean square error
$J_{\mathcal{T}}(n)$	transformed mean square error
$J_{\mathcal{T}}^{\mathcal{N}}(n)$	normalized transformed mean square error
$\mathbf{K}$	error block constraint matrix
L	data block size
$\mathcal{M}$	adaptation misadjustment
N	number of equalizer taps
N_0	complex noise PSD at the channel output
$\mathbf{p}$	cross-correlation of input bits with the desired response
$\mathbf{P}$	gradient constraint matrix
$\mathbf{q}_i$	eigenvector of the input data auto-correlation matrix
$Q(\cdot)$	complementary error function
$\mathbf{R}$	input data auto-correlation matrix
$\mathbf{R}_{\mathcal{F}}$	Fourier-transformed input data auto-correlation matrix
$\mathbf{R}_{\mathcal{T}}$	transformed input data auto-correlation matrix
$\mathbf{R}_{\mathcal{T}}^{\mathcal{N}}$	normalized transformed input data auto-correlation matrix
T	symbol duration
$T_{\mathcal{F}}$	Fourier series period for series computation of BER
$\mathcal{T}$	unitary transformation matrix
$\mathbf{v}(n)$	difference between equalizer taps and the optimum equalizer solution
$\mathbf{v}_{\mathcal{T}}(n)$	transformed vector $\mathbf{v}(n)$
$\mathbf{v}_{\mathcal{T}}^{\mathcal{N}}(n)$	normalized transformed vector $\mathbf{v}(n)$

$\mathbf{v}_{KLT}(n)$	KLT-transformed vector $\mathbf{v}(n)$
$\mathbf{w}(n)$	equalizer tap weight vector
$\mathbf{w}_{\mathcal{F}}(n)$	Fourier-transformed equalizer tap vector
$\mathbf{w}_{\mathcal{T}}(n)$	transformed equalizer tap vector
$\mathbf{w}_{\mathcal{T}}^{\mathcal{N}}(n)$	normalized transformed equalizer tap vector
$\mathbf{w}_0$	optimum equalizer tap vector
$\mathbf{x}(n)$	equalizer input vector
$\mathbf{x}_{\mathcal{F}}(n)$	Fourier-transformed equalizer input vector
$\mathbf{x}_{\mathcal{T}}(n)$	transformed equalizer input vector
$\mathbf{x}_{\mathcal{T}}^{\mathcal{N}}(n)$	normalized transformed equalizer input vector
$\mathbf{X}_{\mathcal{F}}(k)$	diagonal matrix of the Fourier-transformed equalizer input samples
$y(n)$	scalar equalizer output
$\mathbf{y}(n)$	equalizer output vector
$\mathbf{y}_{\mathcal{F}}(n)$	Fourier-transformed equalizer output vector
β	excess bandwidth factor of the raised cosine pulse
γ	smoothing factor for iterative signal power estimation
$\delta(\cdot)$	Dirac delta function
Δ	BER series error
λ_i	eigenvalue of the input auto-correlation matrix
Λ	diagonal eigenvalue matrix for the input auto-correlation matrix
μ	adaptation step size
μ_B	block adaptation step size
ρ	auto-correlation of input data sample
σ_v^2	noise variance at the equalizer input
σ_z^2	noise variance at the equalizer output

$\sigma_{\tau\tau}^2$	power estimate of the i^{th} transformed input sample
∇	gradient operator
$\hat{\nabla}$	estimate of the gradient operator

Chapter 1

Introduction

1.1 Motivation

For time division multiple access (TDMA), the fundamental challenge in high data rate wireless transmission is overcoming intersymbol interference (ISI) caused by multipath propagation. Along with cell sectorization, beamforming and various diversity techniques, time domain equalization has been the most common method of mitigating ISI. As the signaling rate increases, the number of dispersion symbols grows linearly with it. As a result, the number of equalizer coefficients, N , needed to eliminate ISI increases accordingly. Since the complexity per block of data of two common types of equalizers - linear and decision feedback, is on the order of N^2 , computational complexity presents a potential barrier, even with today's state of digital signal processing technology.

Channels of interest in this thesis include those for high data rate applications, where the intersymbol interference length is large. Such cases require equalizers with a large number of taps. Using the common time domain techniques to compensate for distortion introduced by such channels may be very computationally intense. Frequency domain equalization is a technique that offers low complexity growth with an

increase of equalizer length in comparison to the time domain approach. By efficiently implementing block data processing with the use of the fast Fourier transform (FFT), the complexity can be reduced to the order of $N\log N$ per block of N data symbols. For a large number of equalizer taps this translates to substantial computational savings.

This thesis presents an analysis of the frequency domain equalization as applied to a number of multipath channels. Performance measures such as the bit error rate, convergence properties and computational complexity are examined and compared to the time domain equalization approach.

1.2 Literature Review

Adaptive equalizers were pioneered by Lucky in 1965 [1][2] who is credited for the development of the zero-forcing algorithm. This algorithm, based on the peak distortion criterion, led to the development of high speed modems. Concurrently to Lucky's work, Widrow [3][4] developed the least mean square (LMS) algorithm, which has been a popular choice in many applications.

The first frequency domain implementation of the LMS algorithm was presented in 1973 by Waltzman and Schwartz [5], who introduced an adaptation algorithm based on an isolated training sequence. The algorithm presented there included a gradient constraint implemented using Rosen's projection method. In 1978, Dentino *et al.* [6] proposed a frequency domain LMS algorithm which updated the equalizer taps once per block of data. However, since proper data sectioning was not implemented, the algorithm performed circular convolution thus yielding a set of tap values different from the Weiner solution. It did offer significant decrease in computational complexity as no gradient constraint was included. The above shortcomings were finally overcome

in 1980 by Ferrara [7] who recognized the effect of circular convolution and correctly applied overlap-save sectioning to correct it. Ferrara also implemented a gradient constraint, and the resulting frequency domain block LMS algorithm converged to the proper set of tap values. The development of the non-block frequency domain LMS algorithm (referred to as sliding-DFT) is credited to Narayan [8].

Since the initial development, a generalization of the frequency domain algorithm to a number of different transforms has been developed and their advantages discussed [9][10][11]. The convergence properties have been studied extensively and its dependence on the disparity of the input correlation matrix has been recognized [12]. An improvement in the algorithm's learning curve has been obtained by normalizing the step size by the power distribution in the input spectrum. An excellent tutorial paper summarizing frequency domain adaptive filtering has been presented by Shynk [13].

Recently, Clark [14] has applied the FBLMS algorithm to a high data rate wireless environment. Promising performance over fading channels has been obtained using the frequency domain approach with diversity combining. Clark's algorithm however used a cyclic prefix method of achieving linear convolution, which decreased the transmission efficiency. Clark [14] states that for high data rate cases, resulting in large intersymbol interference and needing long equalizers, only the frequency domain method is practical. This thesis examines this claim for 1-4 Mbps transmission over such channels.

1.3 Thesis Contributions

We present a performance analysis of the normalized fast block least mean square (NFBLS) algorithm for high data rate channels. Unlike Clark's implementation,

which uses a cyclic prefix, the algorithm uses overlap-save data sectioning in order to ensure proper adaptation results.

The thesis offers a detailed derivation of the NFBLMS algorithm based on the concepts of block processing and transform domain equalization. It is shown that, if properly implemented, this algorithm offers the advantages of reduced computational load and improved convergence properties relative to the time domain equalization techniques.

An NFBLMS equalizer is applied to channels with short impulse response. For such cases, we show that a fractionally-spaced equalizer is capable of performing the function of a receive filter. For cases where the length of the receive filter is comparable to that of the equalizer, the former is removed and a fractionally-spaced equalizer is used. Due to the equalizer's low complexity growth with the increase of the number of taps, such arrangement offers comparable computational complexity while enabling for a single structure receiver. Results are presented to show that such a receiver offers essentially equal bit error rate performance to a conventional cascade of a receive filter and a T -spaced equalizer.

We present a study of bit error rate (BER) performance for two high data rate channels and show that satisfactory results can be obtained up to 2.0 Mbps BPSK and 4.0 Mbps QPSK within a signal-to-noise ratio (SNR) of 20 dB. We confirm the findings by including an analysis of the equalized signal-to-noise-and-interference ratios (SNIR) which predict the BER results.

We show that for both types of channels considered, by normalizing the step size in the adaptation process according to the distribution of power across the input spectral bins, the rate of convergence of the frequency domain algorithm can be greatly increased in comparison to the LMS algorithm.

For high data rate channels, it is demonstrated that when coupled with a receive filter, the FBLMS algorithm offers a considerable savings in computational complexity over the time domain LMS algorithm by virtue of an efficient implementation of block processing.

1.4 Thesis Outline

The thesis is organized in the following manner. Chapter 2 discusses the theoretical background by first presenting frequency domain adaptive filtering as a specific case of transform domain adaptive filtering. The equivalence of the frequency domain algorithm to a commonly known block adaptive algorithm is also demonstrated. Chapter 3 presents an analysis of performance measures for channels with relatively short impulse responses, including bit error rate, convergence properties and computational complexity analysis. Chapter 4 examines the same performance criteria as applied to channels containing a large number of ISI terms that occur in high bit rate applications. Finally, Chapter 5 contains a summary of the findings of this work along with presenting a number of suggestions for further work in the area.

Chapter 2

Theoretical Background

2.1 Introduction

Adaptive equalization is one of many applications of the signal processing tool known as adaptive filtering. Other important uses include interference and echo cancellation, linear prediction and spectral estimation. While this thesis deals strictly with adaptive channel equalization, the following theoretical treatment, based mainly on [13],[15],[16] and [17] is applicable to the general problem of adaptive filtering.

The primary purpose of this chapter is to derive the *fast block least mean square* (FBLMS) equalization algorithm, which operates in the frequency domain. In doing so, we first introduce the most common steepest descent search method, the *least mean square* (LMS) algorithm. The treatment covers an analysis of its computational complexity as well as detailed discussion of the convergence properties. The latter includes eigenanalysis, which gives a good insight into the effects of the eigenvalue spread of the input data correlation matrix on the adaptation learning curve. As will be shown, the LMS algorithm suffers from slow convergence in the case of highly disparate eigenvalues of the correlation matrix.

The concept of *transform domain adaptive filtering* (TDAF) is introduced as a

solution to the poor convergence properties in case of high eigenvalue spread. We show how by applying an orthogonal transformation to the correlation matrix the latter is diagonalized and subsequently normalized to reduce the eigenvalue disparity.

We present the concept of *block adaptive filtering* as an intermediate step in the derivation of the FBLMS algorithm. Unlike the LMS and TDAF algorithms, which use an *instantaneous* value of the gradient in the adaptation process, the block LMS (BLMS) algorithm uses an average gradient estimate over a block of data, which allows for less noisy convergence. Block processing of data in the BLMS algorithm lends itself to a very efficient implementation via Fourier transform techniques. We show how the operations of correlation and convolution can be efficiently performed via the fast Fourier transform (FFT). A distinction between linear and circular convolution is made and it is demonstrated how the undesirable wrap around effects inherent to convolution via the FFT can be avoided provided proper data sectioning is performed. We describe two such methods: the *overlap-save* and the *overlap-add* techniques.

Finally we present the FBLMS algorithm which utilizes the overlap-save method to efficiently carry out the operations of convolution and correlation. Apart from the savings in computational complexity, the FBLMS inherits the fast convergence properties of transform domain equalization, provided that step size in each adaptation is normalized according to the input power distribution. The algorithm which uses the variable step size is referred to as the *normalized* FBLMS (NFBLMS).

2.2 Wiener Filters

A general filtering problem involves linear estimation of a desired signal based on a filter input sequence. The goal is to select the filter parameters such that the error of this estimation is minimized. The linear filter most often used for channel equalization

Figure 2.1: A Transversal Filter

is a transversal filter depicted in Figure 2.1. The equalizer input and the tap weights are defined, respectively, as column vectors [15][16]

$$\mathbf{x}(n) = [x(n) \ x(n-1) \ \dots \ x(n-N+1)]^T \quad (2.1)$$

and

$$\mathbf{w}(n) = [w_0^*(n) \ w_1^*(n) \ \dots \ w_{N-1}^*(n)]^T, \quad (2.2)$$

where n is the time index and the superscripts $*$ and T denote complex conjugation and a transpose. The output of the equalizer, denoted as $y(n)$, is the sum of the input samples weighted by the tap coefficients, that is

$$\begin{aligned} y(n) &= \sum_{i=0}^{N-1} w_i(n)x(n-i) \\ &= \mathbf{w}^H(n)\mathbf{x}(n), \end{aligned} \quad (2.3)$$

where the superscript H denotes a Hermitian transpose. The tap values are to be chosen such that the output $y(n)$ closely approximates the desired signal which, in the case of channel equalization, is the transmitted symbol $d(n)$. The error of this estimation is defined as

$$e(n) = d(n) - y(n). \quad (2.4)$$

The process of selecting the optimum tap values is usually done by optimizing some kind of performance function. The criterion most commonly used is the *mean square error* (MSE), defined as

$$J(n) = E[\|e(n)\|^2], \quad (2.5)$$

where $E[\cdot]$ and $\|\cdot\|$ denote statistical expectation and the norm operations respectively. If we define the cross-correlation vector $\mathbf{p}$ and the auto-correlation matrix $\mathbf{R}$ respectively as

$$\mathbf{p} = E[\mathbf{x}(n)d^*(n)] \quad (2.6)$$

and

$$\mathbf{R} = E[\mathbf{x}(n)\mathbf{x}^H(n)], \quad (2.7)$$

the MSE can then be expressed as

$$J(n) = E[d^2(n)] - 2\mathbf{w}^H(n)\mathbf{p} + \mathbf{w}^H(n)\mathbf{R}\mathbf{w}(n). \quad (2.8)$$

It can be shown [15][16] that the MSE as a function of $\mathbf{w}(n)$ forms a surface in $N + 1$ dimensional Euclidean space, referred to as the performance surface. Furthermore, since the auto-correlation matrix $\mathbf{R}$ is positive definite, this surface is convex [15][16]. Its global minimum corresponds to the optimum tap values, that is those minimizing

the MSE. This point, defined as $\mathbf{w}(n) = \mathbf{w}_o$, can be obtained by setting the partial derivatives of J with respect to the tap weights to zero, that is

$$\nabla J(n) = 0, \quad (2.9)$$

where ∇ is the gradient defined as

$$\nabla = \left[\frac{\partial}{\partial w_0} \quad \frac{\partial}{\partial w_1} \quad \cdots \quad \frac{\partial}{\partial w_{N-1}} \right]^T. \quad (2.10)$$

The solution to (2.9) is the well known Wiener-Hopf equation [15][18]

$$\mathbf{R}\mathbf{w}_o = \mathbf{p}, \quad (2.11)$$

where $\mathbf{p}$ and $\mathbf{R}$ are defined by (2.6) and (2.7).

2.3 The LMS Algorithm

Solving the Wiener-Hopf equation requires inverting the auto-correlation matrix $\mathbf{R}$. An alternative way of minimizing the MSE avoids this operation by solving for the optimum tap values iteratively. The most common of such techniques is the method of *steepest descent* [18], where the solution is obtained by starting at an arbitrary point on the performance surface, $\mathbf{w}(0)$, and progressively moving towards the global minimum. Since the function is convex, moving in the direction of the steepest slope guarantees reaching the optimum solution. At each iteration, a gradient of the function is computed as

$$\begin{aligned} \nabla J(n) &= \frac{\partial J(n)}{\partial \mathbf{w}(n)} \\ &= 2\mathbf{R}\mathbf{w}(n) - 2\mathbf{p} \\ &= -2E[\mathbf{x}(n)\epsilon^*(n)] \end{aligned} \quad (2.12)$$

and a step in the direction opposite to it is taken. Thus the method of steepest descent can be described by the following tap update equation

$$\mathbf{w}(n+1) = \mathbf{w}(n) - \mu \nabla J(n), \quad (2.13)$$

where μ is a small constant that controls the step size and $J(n)$ is defined by (2.12).

In practice, the main difficulty with this approach, as with solving the Wiener-Hopf equation through matrix inversion, is that it requires the knowledge of the statistics of the underlying input signal. The gradient vector $J(n)$ depends on both the correlation matrix $\mathbf{R}$ and the covariance vector $\mathbf{p}$, which are generally unknown. For an ergodic input sequence this problem can be eliminated by estimating the gradient by a time rather than ensemble average [15]. The most famous iterative method of computing optimum tap values uses the simplest such approximation - the instantaneous value. Originally proposed by Widrow in 1966 [4], the *least mean square* (LMS) algorithm uses an estimate of $\nabla J(n)$, defined as

$$\hat{\nabla} J(n) = -2\mathbf{x}(n)e^*(n). \quad (2.14)$$

Since $E[\hat{\nabla} J(n)] = \nabla J(n)$, $\hat{\nabla} J(n)$ is an unbiased estimate of the true gradient vector.

Thus the tap update equation for the LMS algorithm is

$$\mathbf{w}(n+1) = \mathbf{w}(n) + 2\mu\mathbf{x}(n)e^*(n). \quad (2.15)$$

Equation (2.15) contains a dependence on the error $e(n)$, which in turn by virtue of (2.4), requires the knowledge of the transmitted symbol $d(n)$. As a result, when implementing the LMS algorithm, the equalizer is initially provided with a training sequence of bits known to be transmitted. This initial adaptation stage is referred to as the *training mode* [18] of equalization. Once the equalizer taps have converged, the training sequence is replaced by the decisions at the receiver output. This method,

referred to as *automatic* or *decision-directed* [18] equalization, also allows the equalizer to track slow changes in the channel response.

2.3.1 Computational Complexity of the LMS Algorithm

In describing the computational complexity of the algorithm, we examine the number of real multiplications necessary to process a data block of length N . Although other factors, such as the number of additions and storage requirements, also contribute to the complexity, the analysis presented here provides a reasonably accurate estimate.

For a real input signal, the adaptive equalizer of length N implemented using the LMS algorithm requires N real multiplications to compute the output and another N real multiplications to compute the gradient per sample. Thus, processing a block of N real input samples requires a total of $2N^2$ real multiplications.

In the case where the input signal is complex, the equalizer taps are also complex valued. Since each complex multiplication is equivalent to 4 real multiplications, the complexity increases by a factor of 4, resulting in $8N^2$ multiplications per block of data.

2.3.2 Convergence Properties of the LMS Algorithm

It is well known [12] that the eigenvalue disparity of the input signal correlation matrix $\mathbf{R}$ determines the rate of convergence of the gradient descent method. In order to examine this property in more detail, it is helpful to express the equalizer tap coefficients $\mathbf{w}(n)$, in terms of the eigenvalues of $\mathbf{R}$, λ_i , and the corresponding eigenvectors $\mathbf{q}_i$. This relationship can be expressed as [15]

$$\mathbf{w}(n) = \mathbf{w}(0) + \sum_{i=0}^{N-1} \alpha_i (1 - 2\mu\lambda_i)^n \mathbf{q}_i. \quad (2.16)$$

where α_i is a constant. A similar relationship can be obtained for the MSE, namely

$$J(n) = J_{\min} + \sum_{i=0}^{N-1} \alpha_i^2 \lambda_i (1 - 2\mu\lambda_i)^{2n}. \quad (2.17)$$

The above expressions indicate that the convergence of the equalizer taps for the gradient descent method, and thus for the LMS algorithm, is governed by a sum of N exponential terms each being a function of a separate eigenvalue λ_i . Every term independently controls one mode of convergence in the direction of its eigenvector $\mathbf{q}_i$. Thus, the convergence is rapid when $\|1 - 2\mu\lambda_i\|$ is small. In addition, from (2.16) and (2.17) we note that in order to ensure the stability of the LMS algorithm, the step size μ must satisfy $0 < \mu < 1/\lambda_{\max}$. These two conditions are impossible to satisfy if the eigenvalue spread $\lambda_{\max}/\lambda_{\min}$ is large [18].

The dependence of the convergence rate on the spread of the eigenvalue $\lambda_{\max}/\lambda_{\min}$ translates to a dependency on the power spectral density of the filter input. More specifically, the eigenvalues of the correlation matrix are bounded by the minimum and maximum values of the power spectral density of the input process [15]. When the filter input is white, that is the spectrum is flat across the entire range of frequencies, the LMS algorithm converges rapidly. However, in the case where the spectrum contains nulls, some of the modes converge slowly, thus greatly reducing the overall rate of convergence.

A geometrical interpretation of the above properties can be obtained by examining the shape of the performance surface as a function of the eigenvalues and eigenvectors of $\mathbf{R}$. Using (2.11), the MSE can be rewritten as

$$J(n) = J_{\min} + (\mathbf{w}(n) - \mathbf{w}_o)^H \mathbf{R} (\mathbf{w}(n) - \mathbf{w}_o), \quad (2.18)$$

where the minimum MSE, $J_{\min}$, is given by

$$J_{\min} = E[d^2(n)] - \mathbf{w}_o^H \mathbf{R} \mathbf{w}_o. \quad (2.19)$$

The positive definiteness of the matrix $\mathbf{R}$ implies that the second term in (2.18) is always non-negative, and equal to zero only at the point $\mathbf{w}(n) = \mathbf{w}_o$, that is, corresponding to $J(n) = J_{\min}$. Thus, $J(n)$ is a convex function of tap weights. By defining a vector

$$\mathbf{v}(n) = \mathbf{w}(n) - \mathbf{w}_o, \quad (2.20)$$

(2.18) can be rewritten as

$$J(n) = J_{\min} + \mathbf{v}(n)^H \mathbf{R} \mathbf{v}(n), \quad (2.21)$$

which is equivalent to shifting the origin of the Euclidean space to the point $\mathbf{w}_o$. Furthermore, $\mathbf{R}$ can be decomposed using the unitary similarity transformation [15]

$$\mathbf{R} = \mathbf{Q} \mathbf{\Lambda} \mathbf{Q}^H, \quad (2.22)$$

where the matrix $\mathbf{\Lambda}$ is diagonal with the eigenvalues of $\mathbf{R}$, λ_i , as its entries, while the columns of the matrix $\mathbf{Q}$ consist of the corresponding orthonormal eigenvectors $\mathbf{q}_i$. If we define

$$\mathbf{v}_{KLT}(n) = \mathbf{Q}^H \mathbf{v}(n), \quad (2.23)$$

in which the subscript KLT refers to what is known as the Karhunen-Loève Transform, (2.21) can be expressed as

$$\begin{aligned} J(n) &= J_{\min} + \mathbf{v}_{KLT}(n)^H \mathbf{\Lambda} \mathbf{v}_{KLT}(n) \\ &= J_{\min} + \sum_{i=0}^{N-1} \lambda_i v_{KLT,i}^2(n). \end{aligned} \quad (2.24)$$

Equation (2.23) is equivalent to rotating the axes such that they point in the direction of the eigenvectors of $\mathbf{R}$. Thus the performance surface is in the shape of a

hyperparabola in $N + 1$ dimensional space whose axes are the N tap-weight variables and the MSE. The contours of this surface, corresponding to a specific value of $J(n)$, consist of hyperellipses whose principal axes lie in the directions of the eigenvectors of $\mathbf{R}$. It can be shown [15] that the size of each principal axis is inversely proportional to the square root of the corresponding eigenvalue. Thus the larger the eigenvalue spread, the greater the eccentricity of the hyperparabolas.

Misadjustment of the LMS algorithm

Another property that should be taken into consideration when examining the learning curve of an adaptive algorithm is the fluctuation of the tap values around the optimum solution at steady state. The steady state error of the algorithm is referred to as the *excess MSE* and denoted by J_{excess} . The excess MSE is often normalized to J_{min} , the result given the term misadjustment

$$\mathcal{M} = \frac{J_{excess}}{J_{min}}. \quad (2.25)$$

For the LMS algorithm, one can show [15] that for misadjustment values of 10% or less

$$\mathcal{M}_{LMS} = \mu \text{tr}[\mathbf{R}], \quad (2.26)$$

where $\text{tr}[\cdot]$ refers to the trace operation.

2.4 Transform Domain Equalization

As pointed out previously, the convergence of the LMS algorithm is controlled by N distinct modes, each characterized by an eigenvalue of the correlation matrix $\mathbf{R}$. Since the eigenvalue spread is directly linked to the power spectral density of the input process, the convergence behaviour is also dependent on the spectral characteristics

of the input process. The LMS algorithm will converge rapidly in the frequency bands where the adaptive filter is well excited, and slowly in the poorly excited bands. An intuitive solution to this problem would be to employ a set of bandpass filters to separate the input into subbands and subsequently use a normalization process to equalize the energy content in each band. The signal normalized in such manner can then be used for the adaptation process of the equalizer taps.

Transform domain equalization, which was first proposed by Narayan and Peterson in 1981 [8], implements the above procedure. It uses an orthogonal transform to partition the input data into separate frequency bins, following which the step size in the recursion is normalized according to their power content.

A diagram of a transform domain adaptive filter is shown in Figure 2.2. The input vector $\mathbf{x}(n)$ is transformed into a new set of samples by multiplying it by a unitary matrix $\mathcal{T}$. Defining the transformed vector as

$$\begin{aligned}\mathbf{x}_{\mathcal{T}}(n) &= \mathcal{T}\mathbf{x}(n) \\ &= \begin{bmatrix} x_{\mathcal{T},0}(n) & x_{\mathcal{T},1}(n) & \dots & x_{\mathcal{T},N-1}(n) \end{bmatrix}^T\end{aligned}\quad (2.27)$$

and the similarly transformed equalizer taps as

$$\begin{aligned}\mathbf{w}_{\mathcal{T}}(n) &= \mathcal{T}\mathbf{w}(n) \\ &= \begin{bmatrix} w_{\mathcal{T},0}^*(n) & w_{\mathcal{T},1}^*(n) & \dots & w_{\mathcal{T},N-1}^*(n) \end{bmatrix}^T,\end{aligned}\quad (2.28)$$

the output of the filter at time n is given by

$$\begin{aligned}y(n) &= \sum_{i=0}^{N-1} w_{\mathcal{T},i}(n)x_{\mathcal{T},n-i}(n) \\ &= \mathbf{w}_{\mathcal{T}}^H(n)\mathbf{x}_{\mathcal{T}}(n).\end{aligned}\quad (2.29)$$

Once the input data has been partitioned by the transform $\mathcal{T}$, the step size for each spectral component is set inversely proportional to the power content at that

Figure 2.2: Transform Domain Adaptive Filter

frequency. This is equivalent to replacing the step size μ with $\mu \mathbf{D}^{-1}$ where $\mathbf{D}$ is the diagonal matrix with power estimates as its entries, that is

$$\mathbf{D} = \begin{bmatrix} E[x_{\mathcal{T},0}(n)x_{\mathcal{T},0}^*(n)] & 0 & \cdots & 0 \\ 0 & E[x_{\mathcal{T},1}(n)x_{\mathcal{T},1}^*(n)] & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & E[x_{\mathcal{T},N-1}(n)x_{\mathcal{T},N-1}^*(n)] \end{bmatrix}. \quad (2.30)$$

In practice, the matrix $\mathbf{D}$ is replaced by $\hat{\mathbf{D}}$ with estimates of $E[x_{\mathcal{T},i}(n)x_{\mathcal{T},i}^*(n)]$ on its main diagonal. Each entry in $\hat{\mathbf{D}}$ is computed by a recursion formula [15]

$$\hat{\sigma}_{x_{\mathcal{T},i}}^2(n) = \gamma \hat{\sigma}_{x_{\mathcal{T},i}}^2(n-1) + (1-\gamma) \hat{\sigma}_{x_{\mathcal{T},i}}^2(n), \quad i = 0, 1, \dots, N-1, \quad (2.31)$$

where γ is often referred to as the *smoothing factor*. The resulting recursion formula for the TDLMS algorithm is thus given by

$$\mathbf{w}_{\mathcal{T}}(n+1) = \mathbf{w}_{\mathcal{T}}(n) + 2\mu \hat{\mathbf{D}}^{-1} \mathbf{x}_{\mathcal{T}}(n) \epsilon^*(n). \quad (2.32)$$

To show how this algorithm normalizes the power in each input bin to unity, (2.32) can be decomposed into separate equations for each equalizer tap,

$$w_{\mathcal{T},i}(n+1) = w_{\mathcal{T},i}(n) + \frac{\mu}{\hat{\sigma}_{x_{\mathcal{T},i}}^2(n)} x_{\mathcal{T},i}(n) \epsilon^*(n), \quad i = 0, 1, \dots, N-1. \quad (2.33)$$

We can see from the above expression that the step size for each tap is inversely proportional to the corresponding power estimate. As a result the weakly excited subbands will have a larger step size, thus speeding up their convergence.

It can be shown that the transform domain LMS algorithm is equivalent to the conventional LMS algorithm with the normalized tap input vector, defined as $\mathbf{x}_{\mathcal{T}}^N(n)$, where

$$\mathbf{x}_{\mathcal{T}}^N(n) = \hat{\mathbf{D}}^{-1/2} \mathbf{x}_{\mathcal{T}}(n). \quad (2.34)$$

Furthermore, if we define

$$\mathbf{w}_{\mathcal{T}}^{\vee}(n) = \hat{\mathbf{D}}^{1/2} \mathbf{w}_{\mathcal{T}}(n) \quad (2.35)$$

and multiply (2.32) by $\hat{\mathbf{D}}^{1/2}$, we obtain

$$\mathbf{w}_{\mathcal{T}}^{\vee}(n+1) = \mathbf{w}_{\mathcal{T}}^{\vee}(n) - 2\mu \mathbf{x}_{\mathcal{T}}^{\vee}(n) e^*(n). \quad (2.36)$$

Noting that

$$\begin{aligned} e(n) &= d(n) - y(n) \\ &= d(n) - \mathbf{w}_{\mathcal{T}}^H(n) \mathbf{x}_{\mathcal{T}}(n) \\ &= d(n) - \mathbf{w}_{\mathcal{T}}^{\vee H}(n) \mathbf{x}_{\mathcal{T}}^{\vee}(n) \end{aligned} \quad (2.37)$$

shows that (2.32) is equivalent to the LMS algorithm with the normalized input vector $\mathbf{x}_{\mathcal{T}}^{\vee}(n)$.

2.4.1 Computational Complexity of the TDLMS Algorithm

While the TDLMS algorithm was developed mainly to improve the rate of convergence, it exhibits a tradeoff in the form of increased computational complexity. The computation of the transform requires on the order of N^3 multiplications, with the exact number depending on the type of input data and the transformation used.

2.4.2 Convergence Properties of the TDLMS Algorithm

As discussed in the Section 2.3.2, the eigenvalue spread affects the shape of the performance surface, which in turn dictates the speed of convergence. To examine the effect of applying the transformation to the input data on the shape of the surface, we recall that the performance surface without any transformation can be expressed

as

$$J(\mathbf{v}) = J_{\min} + \mathbf{v}(n)^H \mathbf{R} \mathbf{v}(n), \quad (2.38)$$

where $\mathbf{v}(n)$ is defined by (2.20) and $\mathbf{R}$ is the input data correlation matrix. To compare this function to that for a transformed input sequence we define a transformed vector $\mathbf{v}_{\mathcal{T}}(n)$ as

$$\mathbf{v}_{\mathcal{T}}(n) = \mathcal{T} \mathbf{v}(n), \quad (2.39)$$

and the auto-correlation matrix of the transformed input sequence as

$$\mathbf{R}_{\mathcal{T}} = \mathcal{T} \mathbf{R} \mathcal{T}^H. \quad (2.40)$$

The performance surface for a transformed input sequence $\mathbf{x}_{\mathcal{T}}(n)$ is then

$$J_{\mathcal{T}}(\mathbf{v}_{\mathcal{T}}) = J_{\min} + \mathbf{v}_{\mathcal{T}}(n)^H \mathbf{R}_{\mathcal{T}} \mathbf{v}_{\mathcal{T}}(n). \quad (2.41)$$

Substituting (2.39) and (2.40) yields

$$\begin{aligned} J_{\mathcal{T}}(\mathbf{v}_{\mathcal{T}}) &= J_{\min} + \mathbf{v}^H(n) \mathcal{T}^H \mathcal{T} \mathbf{R} \mathcal{T}^H \mathcal{T} \mathbf{v}(n) \\ &= J_{\min} + \mathbf{v}^H(n) \mathbf{R} \mathbf{v}(n) \\ &= J(\mathbf{v}), \end{aligned} \quad (2.42)$$

where we have used the fact that $\mathcal{T}^H \mathcal{T} = \mathbf{I}$ for any unitary transformation matrix [15]. This shows that the performance surface in both cases is identical, regardless of the transform used. The only effect of the transform is rotating the coordinate axis according to (2.39), which does not in itself affect the convergence rate.

If, however, in addition to the transformation, power normalization is performed, (2.41) becomes

$$J_{\mathcal{T}}^{\vee}(\mathbf{v}_{\mathcal{T}}^{\vee}) = J_{\min} + \mathbf{v}_{\mathcal{T}}^{\vee H}(n) \mathbf{R}_{\mathcal{T}}^{\vee} \mathbf{v}_{\mathcal{T}}^{\vee}(n), \quad (2.43)$$

where

$$\mathbf{v}_{\mathcal{T}}^{\mathcal{N}}(n) = \mathbf{D}^{1/2} \mathbf{v}_{\mathcal{T}}(n) \quad (2.44)$$

and

$$\mathbf{R}_{\mathcal{T}}^{\mathcal{N}} = \mathbf{D}^{-1/2} \mathbf{R}_{\mathcal{T}} \mathbf{D}^{-1/2}. \quad (2.45)$$

Substituting the above expressions into (2.41), we obtain

$$\begin{aligned} J_{\mathcal{T}}^{\mathcal{N}}(\mathbf{v}_{\mathcal{T}}^{\mathcal{N}}) &= J_{\min} + \mathbf{v}_{\mathcal{T}}^H(n) \mathbf{D}^{1/2} \mathbf{D}^{-1/2} \mathbf{R}_{\mathcal{T}} \mathbf{D}^{-1/2} \mathbf{D}^{1/2} \mathbf{v}_{\mathcal{T}}(n) \\ &= J_{\min} + \mathbf{v}_{\mathcal{T}}^H(n) \mathbf{R}_{\mathcal{T}} \mathbf{v}_{\mathcal{T}}(n) \\ &= J_{\mathcal{T}}(\mathbf{v}_{\mathcal{T}}) \\ &= J_{\mathcal{T}}(\mathbf{D}^{-1/2} \mathbf{v}_{\mathcal{T}}^{\mathcal{N}}). \end{aligned} \quad (2.46)$$

The above relationship demonstrates that the performance surface for the normalized case is equivalent to that defined by (2.43), but with the coordinate axes scaled according to (2.44). This has the effect of reducing the eccentricity of the hyperparabola, thus increasing the rate of convergence.

The most important task in implementing the TDLMS algorithms is the selection of the transform $\mathcal{T}$, such that the elements of the transformed input vector have mutually exclusive spectral bands. Ideally, one would like to obtain a vector $\mathbf{x}_{\mathcal{T}}(n)$ with uncorrelated components [19], which in turn results in a diagonal correlation matrix. Proper normalization would then yield an identity matrix with eigenvalue spread of unity. A transform with such properties is the Karhunen-Loève Transform (KLT), defined as [15]

$$\mathcal{T}_{KLT} = \mathbf{Q}^H. \quad (2.47)$$

where $\mathbf{Q}^H$ is a matrix with columns equal to eigenvectors of the correlation matrix $\mathbf{R}$, as in the unitary similarity transformation. Recalling that $\mathbf{R}$ can be decomposed as

$$\mathbf{R} = \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^H, \quad (2.48)$$

we have

$$\begin{aligned} \mathbf{R}_{\mathcal{T}_{KLT}} &= E[\mathbf{x}_{\mathcal{T}_{KLT}}(n)\mathbf{x}_{\mathcal{T}_{KLT}}^H(n)] \\ &= E[\mathbf{Q}^H\mathbf{x}(n)\mathbf{x}^H(n)\mathbf{Q}] \\ &= \mathbf{Q}^H\mathbf{R}\mathbf{Q} \\ &= \mathbf{Q}^H\mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^H\mathbf{Q} \\ &= \mathbf{\Lambda}. \end{aligned} \quad (2.49)$$

The above shows that the auto-correlation matrix of the KLT-transformed sequence is diagonal. If $\hat{\mathbf{D}}$ is a good estimate of $\mathbf{D}$, we have that the normalized correlation matrix is

$$\begin{aligned} \mathbf{R}_{\mathcal{T}_{KLT}}^{-1/2} &= \hat{\mathbf{D}}^{-1/2}\mathbf{R}\hat{\mathbf{D}}^{-1/2} \\ &\simeq \mathbf{I}. \end{aligned} \quad (2.50)$$

While the KLT is an ideal choice for the transform $\mathcal{T}$, as it produces an uncorrelated input sequence, it requires the knowledge of $\mathbf{R}$, which is not always available. Therefore in practice other transforms, such as the Discrete Fourier Transform (DFT) or the Discrete Cosine Transform (DCT), are often used.

2.5 Block LMS Algorithm

The algorithms presented thus far use an instantaneous value of the gradient as an approximation to its statistical expectation. Such estimations are noisy and cause

random fluctuations of taps during the adaptation process. One method of improving the gradient estimate is to use a time average over L samples

$$\hat{\nabla} J_B(n) = -\frac{2}{L} \sum_{i=0}^{L-1} \mathbf{x}(n+i)e^*(n+i), \quad (2.51)$$

where the subscript B denotes that the gradient is estimated based on a *block* of L data samples. The above is referred to as the block LMS (BLMS) algorithm. Since the gradient is updated every L samples, the taps are held constant for the duration of the block. The output of the BLMS equalizer is given by

$$y(n+i) = \mathbf{w}^H(n)\mathbf{x}(n+i), \quad i = 0, \dots, L-1. \quad (2.52)$$

The resulting tap update equation is

$$\begin{aligned} \mathbf{w}(n+L) &= \mathbf{w}(n) - \mu_B \hat{\nabla} J_B(n) \\ &= \mathbf{w}(n) + 2\mu_B \sum_{i=0}^{L-1} \mathbf{x}(n+i)e^*(n+i), \end{aligned} \quad (2.53)$$

where $\mu_B = \mu L$ is the block step size. Since the BLMS algorithm operates at a lower sampling rate than that of the input sequence, it is convenient to define a new time index k , one increment of which corresponds to L increments in the original index n . Equation 2.53 can then be rewritten as

$$\mathbf{w}(k+1) = \mathbf{w}(k) + 2\mu_B \sum_{i=0}^{L-1} \mathbf{x}(kL+i)e^*(kL+i), \quad (2.54)$$

where the indices n and k are related by

$$n = kL + i, \quad i = 0, \dots, L-1. \quad (2.55)$$

2.5.1 Convergence Properties of the BLMS Algorithm

Since the BLMS algorithm computes a more accurate estimate of the gradient, the descent direction at each step is more aligned with the true gradient, thus resulting

in smoother convergence [13]. This improvement becomes greater with a larger block size. However, when choosing the block length L , it is important to realize that since the taps are updated every L samples, a delay is introduced at the output of the filter. If L is large, the tracking performance of the BLMS algorithm may degrade.

In this thesis, we are mainly interested in the case where $L = N$, since, as it will be shown, it allows for an efficient implementation of the BLMS algorithm. The rate of convergence does not change, since much like in the case of the LMS algorithm, it is governed by the eigenvalue spread of the input data correlation matrix, which is the same for both algorithms [15].

2.5.2 Computational Complexity of the BLMS Algorithm

The complexity of the BLMS algorithm is the same as that of the LMS algorithm. That is, it requires, respectively $2N^2$ and $8N^2$ real multiplications per real and complex data blocks of length N . However, there exists a very efficient implementation of the algorithm using DFT's, which offers a significant computational savings, and which is described in the following section.

2.6 Fast Block Least Mean Square Equalizers

While transform domain adaptive filtering improves the convergence properties by reducing the eigenvalue spread of the data correlation matrix, it does it at the expense of increasing the computational complexity. In applications requiring large number of equalizer taps such complexity can be prohibitive. In this section we show how a computationally efficient implementation of BLMS algorithm can be achieved in the frequency domain [13][15].

Equation (2.52) is essentially a linear convolution of the data vector with the

equalizer taps, while the block gradient in (2.51) is a linear correlation of the error signal with the input data. It is known from digital signal processing theory [17] that these can be efficiently performed via the discrete Fourier transform (DFT). However, since the DFT is a signal representation in terms of frequency samples, it corresponds to a periodic time signal. Inherently, this leads to cyclic (or circular) convolution/correlation which differs from the linear counterpart obtained by filtering sequences in the time domain. To correct this, there exist two common techniques of data sectioning that allow us to achieve linear convolution via the DFT. The following sections, based mainly on [17] and [20], discuss these algorithms and how they are incorporated into the FBLMS algorithm.

2.6.1 Linear and Circular Convolution

Linear convolution of two sequences $x_1(n)$ and $x_2(n)$, both of length N , is given by

$$\begin{aligned} x_3(n) &= x_1(n) * x_2(n) \\ &= \sum_{m=-\infty}^{\infty} x_1(m)x_2(n-m), \end{aligned} \quad (2.56)$$

where $x_3(n)$ is the resulting sequence of length $2N - 1$. Thus, $x_3(n)$ is obtained by multiplying one sequence by a time-reversed and *linearly* shifted version of the other and summing the values of the products for all m .

We now examine the effects of computing convolution via the discrete Fourier transform. The respective N -point DFTs of the sequences $x_1(n)$ and $x_2(n)$ are given by

$$x_{\mathcal{F},1}(k) = \sum_{n=0}^{N-1} x_1(n)e^{-j2\pi kn/N} \quad k = 0, 1, \dots, N-1 \quad (2.57)$$

and

$$x_{\mathcal{F},2}(k) = \sum_{n=0}^{N-1} x_2(n)e^{-j2\pi kn/N} \quad k = 0, 1, \dots, N-1. \quad (2.58)$$

Multiplying two DFTs yields a sequence $x_{\mathcal{F},3}(k)$,

$$x_{\mathcal{F},3}(k) = x_{\mathcal{F},1}(k)x_{\mathcal{F},2}(k) \quad k = 0, 1, \dots, N-1, \quad (2.59)$$

whose IDFT is given by

$$\begin{aligned} x_3(n) &= \frac{1}{N} \sum_{k=0}^{N-1} x_{\mathcal{F},3}(k) e^{j2\pi kn/N} \\ &= \frac{1}{N} \sum_{k=0}^{N-1} x_{\mathcal{F},1}(k) x_{\mathcal{F},2}(k) e^{j2\pi kn/N}, \end{aligned} \quad (2.60)$$

Substituting (2.57) and (2.58) into (2.60) we obtain

$$\begin{aligned} x_3(n) &= \frac{1}{N} \sum_{k=0}^{N-1} \left[\sum_{m=0}^{N-1} x_1(m) e^{-j2\pi km/N} \right] \left[\sum_{l=0}^{N-1} x_2(l) e^{-j2\pi kl/N} \right] e^{j2\pi kn/N} \\ &= \frac{1}{N} \sum_{m=0}^{N-1} x_1(m) \sum_{l=0}^{N-1} x_2(l) \left[\sum_{k=0}^{N-1} e^{j2\pi k(n-m-l)/N} \right]. \end{aligned} \quad (2.61)$$

Let the third summation of (2.61) be expressed as

$$\sum_{k=0}^{N-1} a^k = \begin{cases} N, & a = 1 \\ \frac{1-a^N}{1-a}, & a \neq 1 \end{cases}, \quad (2.62)$$

where a is defined as

$$a = e^{j2\pi(n-m-l)/N}. \quad (2.63)$$

We can see that $a = 1$ when $n - m - l$ is a multiple of N . Also, from (2.63) we have that $a^N = 1$ for any value of $a \neq 0$. Thus (2.62) can be expressed as

$$\sum_{k=0}^{N-1} a^k = \begin{cases} N, & l = n - m + pN = \langle n - m \rangle_N, \quad p \text{ an integer} \\ 0, & \text{otherwise} \end{cases}. \quad (2.64)$$

where $\langle \cdot \rangle_N$ denotes modulo- N arithmetic. Using (2.61) and (2.64) we obtain an expression for $x_3(n)$

$$x_3(n) = \sum_{m=0}^{N-1} x_1(m) x_2(\langle n - m \rangle_N). \quad (2.65)$$

In contrast to linear convolution, here $x_3(n)$ is obtained by multiplying one sequence by a time-reversed and *circularly* shifted version of the other and summing values of the product for all m . Thus multiplying DFT's of $x_1(k)$ and $x_2(k)$ and taking the inverse DFT of the product corresponds to performing N -point *circular* convolution on $x_1(n)$ and $x_2(n)$, denoted as

$$x_3(n) = x_1(n) \circledast x_2(n). \quad (2.66)$$

In order to show how to perform linear convolution via the DFT, we first note that frequency domain equivalent of the linear convolution, defined by (2.56), of sequences of lengths L and P is [17]

$$X_3(\omega) = X_1(\omega)X_2(\omega). \quad (2.67)$$

In order to represent the output sequence $x_3(n)$ in the frequency domain by the samples of its spectrum $X_3(\omega)$, the number of distinct samples must equal or exceed its length, that is $L + P - 1$. Therefore, a DFT of length $N \geq L + P - 1$ is needed to properly represent the sequence $x_3(n)$ in the frequency domain [17]. Thus a linear convolution of sequences of lengths L and P can be achieved by multiplying the respective DFT's provided that each DFT is of length $N \geq L + P - 1$. One way of ensuring this is to augment both sequences with an appropriate number of zeros. However, computing a convolution in such manner is impractical in situations when the length of one of the sequences is very large, since storing it would require a large buffer size. More importantly, since the filter output is not computed until all the samples have been collected, there is a large processing delay. For these reasons, most practical approaches involve breaking the input sequence into blocks and computing the output in segments which are then appropriately combined. This is referred to as *block convolution*. The most common methods of block filtering in the frequency

domain are the *overlap-save* and the *overlap-add* techniques presented below. In their discussion we assume that the long input data stream is broken down into blocks of length L . The filter is assumed to be of length P .

Overlap-Add Method for Implementing Linear Convolution

The overlap-add sectioning is depicted in Figure 2.3. In this method, the input blocks of length L are padded with $P - 1$ zeros. The filter response (of length P) is padded with $L - 1$ zeros, to produce a sequence of total length $N = L + P - 1$. N -point DFT's are computed and the IDFT of their product is taken. The individual output blocks are free from aliasing, since as described in the previous section, the DFT's are of length $N = L + P - 1$. Since the input blocks contain $P - 1$ zeros, the last $P - 1$ points of each individual output must be overlapped and added to the first $P - 1$ points of the next output block.

Overlap-Save Method for Implementing Linear Convolution

The overlap-save method is shown in Figure 2.4. Similar to overlap-add, this method also generates input data blocks of size $N = L + P - 1$. Here, however, each data block consists of the last $P - 1$ points from the previous block followed by L new data points. Like in the case of overlap-add, the filter response of length P is padded with $L - 1$ zeros. Again, the IDFT of the product of the N -point DFT's is computed. Since the first $P - 1$ points of each individual output is a result of aliasing, it is simply discarded, while the last L points are appended to the successive output blocks.

2.6.2 Fast Block LMS Algorithm

The most common implementation of the FBLMS algorithm uses the overlap-save method to process the input bits. The filtering is equivalent to that discussed in the

Figure 2.3: Overlap-Add Method of Linear Convolution

Figure 2.4: Overlap-Save Method of Linear Convolution

preceding section with $L = 2N$ and $P = N + 1$, where N is the length of the equalizer. Choosing the data block size to be equal to the number of equalizer taps is known to minimize the computational complexity of the FBLMS algorithm [15]. The block diagram of the algorithm is shown in Figure 2.5. The data is broken down into blocks of length $L = 2N$, consisting of N samples from the previous block followed by N new ones. The resulting samples can be expressed as a diagonal matrix

$$\mathbf{X}_{\mathcal{F}}(k) = \text{diag}\{\mathcal{F}\{x(kN - N), \dots, x(kN - 1), x(kN), \dots, x(kN + N - 1)\}\}. \quad (2.68)$$

The DFT of the equalizer taps is a column vector of length $2N$, as the tap values are augmented at the end with N zeros, that is

$$\mathbf{w}_{\mathcal{F}}(k) = [\mathcal{F}\{w_o(n), \dots, w_{N-1}(n), 0, \dots, 0\}]^T. \quad (2.69)$$

The output vector is the element-by-element product of the transformed sequences is given by

$$\mathbf{y}_{\mathcal{F}}(k) = \mathbf{X}_{\mathcal{F}}(k)\mathbf{w}_{\mathcal{F}}(k). \quad (2.70)$$

As discussed in the section on the overlap-save technique, the first N samples of the product are invalid as they are the result of circular convolution. The vector $\mathbf{y}_{\mathcal{F}}(k)$ must be transformed into the time domain, where the invalid samples are discarded and only the last N are used to compute the error

$$e(kN + m) = d(kN + m) - y(kN + m), \quad m = 0, \dots, N - 1. \quad (2.71)$$

The above constraint can be formulated using the following matrix notation. By defining a sectioning constraint matrix for the error block as

$$\mathbf{K} = [0_N \quad I_N]. \quad (2.72)$$

Figure 2.5: FBLMS Algorithm Using Overlap-Save Sectioning

where 0_N is a $N \times N$ zero matrix, I_N is an $N \times N$ identity matrix, the output vector can be expressed as

$$\mathbf{y}(n) = \mathbf{K}\mathcal{F}^{-1}\mathbf{X}_{\mathcal{F}}(k)\mathbf{w}_{\mathcal{F}}(k), \quad (2.73)$$

where $\mathcal{F}$ is a $2N \times 2N$ DFT matrix with elements $\mathcal{F}_{ij} = e^{-j2\pi ij/2N}$. The resulting error samples are augmented with N zeros

$$\mathbf{e}_{\mathcal{F}}(k) = [\mathcal{F}\{0, \dots, 0, e(kN), \dots, e(kN + N - 1)\}]^T. \quad (2.74)$$

The zeros are inserted due to the fact that linear correlation has to be computed between the error vector and the complex conjugate of the input vector $\mathbf{X}_{\mathcal{F}}^H(k)$. Using the constraint matrix, vector $\mathbf{e}_{\mathcal{F}}(k)$ can be rewritten as

$$\mathbf{e}_{\mathcal{F}}(k) = \mathcal{F}\mathbf{K}^T\mathbf{e}(k). \quad (2.75)$$

The next section in the flow chart is the gradient constraint. Since there are only N tap weights being updated in the time domain, it is necessary to ensure that only N elements of the gradient are non-zero. Thus, the gradient is transformed into the time domain and the last N elements are replaced with zeros before transforming it back to frequency domain and updating the tap values. This also means that initially the last N equalizer taps must be set to zero.

The gradient constraint can be expressed by using a constraint matrix similar to that used for the error block. By defining $\mathbf{P}$ as

$$\mathbf{P} = \begin{bmatrix} I_N & 0_N \\ 0_N & 0_N \end{bmatrix}, \quad (2.76)$$

the augmented gradient can be expressed as

$$[\hat{\nabla}\mathbf{J}(k), 0, \dots, 0]^T = \mathbf{P}\mathcal{F}^{-1}\mathbf{X}_{\mathcal{F}}^H(k)\mathbf{e}_{\mathcal{F}}(k). \quad (2.77)$$

The tap update therefore becomes

$$\mathbf{w}_{\mathcal{F}}(k+1) = \mathbf{w}_{\mathcal{F}}(k) + 2\mu(k)\mathcal{F}\mathbf{P}\mathcal{F}^{-1}\mathbf{X}_{\mathcal{F}}^H(k)\mathbf{e}_{\mathcal{F}}(k). \quad (2.78)$$

If the step size $\mu(k)$ set to a constant, the resulting algorithm is the non-normalized FBLMS. However, as described in Section 2.4, the convergence properties can be improved by making the step size $\mu(k)$ inversely proportional to the power content in each frequency bin, that is

$$\mu(k) = \mu\hat{\mathbf{D}}^{-1}. \quad (2.79)$$

where $\hat{\mathbf{D}}$ is computed similarly as in the LMS algorithm. The resulting algorithm is referred to as the normalized FBLMS or NFBLMS.

In the case of both FBLMS and NFBLMS algorithms, the gradient constraint introduces two additional DFT operations, which significantly increases the computational load. In some applications, this constraint is dropped in order to decrease the complexity. This is referred to as the unconstrained FBLMS algorithm. In such cases, the gradient vector no longer contains N non-zero entries, and wrap-around effects are introduced affecting the performance of the adaptive algorithm. This results in an increased steady state MSE and the final taps do not correspond to the ideal Wiener solution. The unconstrained algorithm can be obtained from Figure 2.5 by simply removing the dotted section labeled as 'Gradient Constraint'.

Misadjustment of the FBLMS and NFBLMS algorithms

The relationship between the steady state misadjustment value and the adaptation step size for the FBLMS and NFBLMS algorithms has been derived in [15]. In the case of the non-normalized FBLMS algorithm, the misadjustment of 10% or less can

be expressed as

$$\mathcal{M}_{FBLMS} = \mu N \rho_0, \quad (2.80)$$

where ρ_0 is the diagonal element of the correlation matrix $\mathbf{R}$. For the normalized counterpart the expression is given by [15]

$$\mathcal{M}_{NFBLMS} = \mu N / N', \quad (2.81)$$

where N' is the total size of the overlapped input vector, usually chosen to be $2N$.

2.6.3 Computational Complexity of the FBLMS algorithm

The real advantage of using the FBLMS algorithm is the savings in the computational complexity. Recall that the LMS and the BLMS algorithms require $2N^2$ real multiplications per input block, while the TDLMS algorithm, due to the need to compute the transform after each input sample requires $2N^3$ multiplications.

The Fast Block LMS algorithm with the gradient constraint requires a total of 5 FFT's of length $2N$. Assuming the algorithm uses a radix-2 decimation in time, the total number of real multiplications for a transform of block N of complex data is $2N \log N$. For real input, the computational load decreases two-fold due to symmetry [21], thus requiring only $N \log N$ multiplications. In addition, the FBLMS algorithm requires $2N$ complex multiplications for computing the gradient as well as for updating the equalizer taps. The total number of real multiplications required to process a block of N data is $10N \log 2N + 16N$ and $20N \log 2N + 16N$ for real and complex signals respectively.

Furthermore, the step-normalized FBLMS algorithm requires an additional $2N$ multiplications of real power estimate by a complex gradient. Thus the complexity

of the NFBLMS is given by $10N\log_2 N + 20N$ for real input and $20N\log_2 N + 20N$ for complex input.

Chapter 3

Performance over Low Rate Channels

3.1 Introduction

Performance analysis of the time and frequency domain equalization as applied to low data rate multipath channels is discussed here. First, an overview of the communication system model used is given, including description of modulation methods, pulse shaping techniques and channel impulse response representation. Details of the simulations performed are described, following which we present the results. The performance criteria considered are the bit error rate, convergence properties and the computational complexity of the algorithms. Also included is a discussion of the Fourier Series technique of computing the BER for channels corrupted with additive Gaussian noise.

3.2 Communication System Model

The communication system model used consists of a cascade of a data source, transmit filter, channel, receive filter, equalizer, a decision device and a data sink. This is depicted in Figure 3.1.

Figure 3.1: Communication System Model

The data source generates a stream of bits to be transmitted over the channel. It is assumed that the source translates the bits onto a sequence of symbols depending on the type of signaling used. These symbols, denoted as I_k , are applied to a transmit filter which generates a pulse to be transmitted over the channel. The impulse response of the transmit filter is denoted as $h_{tx}(t)$ where tx stands for transmitter. The channel, with an impulse response $h_c(t)$, introduces intersymbol interference (ISI) and corrupts the signal with additive white Gaussian noise (AWGN) before the input to the receive filter. The impulse response of the receive filter is denoted by $h_{rx}(t)$, where rx stands for receiver. The signal at the receive filter output is sampled prior to the equalizer. In Figure 3.1 t_0 is the sampling time offset for best symbol detection. The output of the equalizer is fed to the decision device which generates a sequence of detected symbols denoted by $\tilde{I}_k$. These symbols are then mapped back to a stream of bits and sent to the sink.

The system model described in Figure 3.1 represents a baseband equivalent of a passband system. In most real applications, in order to transmit the data over a channel, the transmitted signal spectrum is shifted onto a passband centered around a suitable carrier frequency. The process of modulation is accomplished at the trans-

mitter by multiplying the signal by a sinusoidal carrier. Demodulating such signal requires shifting the passband signal back onto baseband by multiplying it by a replica of the carrier. In this thesis it is assumed that ideal coherent demodulation is performed, that is the carrier used at the receiver is in phase with that at the transmitter. With this assumption, the baseband equivalent model of a passband system is justified.

3.2.1 Signaling Methods

The input to the transmit filter is an impulse train of symbols as generated by the source, given by

$$I(t) = \sum_{k=-\infty}^{\infty} I_k \delta(t - kT), \quad (3.1)$$

where $\delta(t - kT)$ is the unit impulse function and T is the symbol interval. To optimize transmission in a bit error rate sense, mapping of the bits onto the symbol alphabet is chosen such that the Euclidean distance between the symbols I_k is maximized for a given average power constraint [18]. The simplest method of accomplishing this is antipodal signaling, where the symbol alphabet is $\{-1, 1\}$. This is used in conjunction with binary phase shift keying modulation (BPSK), where the phase of the sinusoidal carrier is altered by 180 degrees depending on the polarity of the symbol. An alternative method achieves double the data rate for a given bit rate by mapping each pair of bits onto an alphabet consisting of $\{1+j, -1+j, -1-j, 1-j\}$. The modulation used with such symbol mapping is referred to as quaternary phase shift keying (QPSK), which is equivalent to transmitting two BPSK streams on orthogonal carriers.

3.2.2 Pulse Shaping

The shape of the received pulse is determined by the overall impulse response of the transmit filter $h_{tx}(t)$, the channel $h_c(t)$ and the receive filter $h_{rx}(t)$. In order to achieve zero ISI, the receive filter must be chosen such that the overall pulse shape

$$f(t) = h_{tx}(t) * h_c(t) * h_{rx}(t) \quad (3.2)$$

satisfies Nyquist's first criterion. To fulfill such a requirement, the receive filter is ideally chosen such that it is matched to the combined response of the transmitter and the channel, that is

$$h_{rx}(t) = h^*(-t), \quad (3.3)$$

where

$$h(t) = h_{tx}(t) * h_c(t). \quad (3.4)$$

In practice the channel characteristics may be unknown and/or changing with time. As a result, suboptimum receiver filtering is used, where the received filter impulse response is matched solely to the shape of the transmit filter. Such system would satisfy Nyquist's first criterion in the absence of channel induced ISI. A commonly used shape is the raised cosine (RC) pulse, given by [22]

$$h_{RC}(t) = \frac{\sin(\pi t/T) \cos(\beta \pi t/T)}{(\pi t/T)(1 - 4\beta^2 t^2/T^2)}, \quad (3.5)$$

where β is the *excess bandwidth factor*. Pulse shaping is evenly split between the receiver and the transmitter, so that the convolution of their pulses yields (3.5). Such pulse is the root raised cosine (RRC) shape given by [22]

$$h_{RRC}(t) = \frac{4\beta}{\pi\sqrt{T}} \frac{\cos((1 + \beta)\pi t/T) + \frac{t}{4\beta T} \sin((1 - \beta)\pi t/T)}{1 - 16\beta^2 t^2/T^2}. \quad (3.6)$$

In practice, in order to ensure causality of the filter, a delay term must be included such that the entire pulse exists only for positive time values. In the analysis here, this delay term is set to zero, with the assumption that in a real life implementation it is non-zero, thus resulting in a realizable filter.

3.2.3 Channel

The pulse generated by the transmit filter passes through the channel where ISI is introduced. The primary source of ISI is multipath propagation. The presence of reflecting objects and scatterers in wireless channels results in multiple versions of incoming signal arriving from different directions and with different propagation delays [23]. Such a channel can be modeled by a sequence of appropriately delayed impulses, each representing a separate multipath component. The temporal variation of the position of objects in a wireless channel results in fading or fluctuation of amplitude of multipath components. In this thesis, however, the study is limited to stationary channels, where the impulse response does not vary with time.

Apart from introducing intersymbol interference, the channel is also subjected to noise. White noise, defined to have a flat power spectral density over the entire spectrum, is modeled by a stochastic process. Since noise is a cumulative effect of a large number of processes, it is assumed to be Gaussian via the Central Limit Theorem [24]. In the model here presented, noise is introduced into the system just prior to the receive filter.

3.2.4 Equalizer

To combat the ISI introduced by the channel, an equalizer is included at the receiving end of the system. The equalizers considered in this model are discrete-time and are implemented using tapped delay line adaptive filters. They are placed after the

sampler and operate on the sampled values of the received signal. This thesis examines linear equalizers, both symbol-spaced and fractionally-spaced.

3.3 Simulation Methodology

In the simulations performed, the data source was implemented using a uniform random number generator which generated antipodal symbols for BPSK and symbols with phases separated by 90 degrees in the case of QPSK modulation.

The generator output was convolved with the transmit filter impulse response. The transmit filter used was the RRC shape with a roll-off value of $\beta = 0.35$ chosen to conform with the IS-54 digital standard [22]. The pulse was chosen to be of length $8T$, that is, it was truncated after 4 symbol intervals at each side of the peak value. The sampling resolution for the RRC filter was $T/8$.

The channels examined in this section are two short impulse response channels chosen such that the overall impulse response of the transmitter, channel and receiver is that described in [18]. The real valued channel, referred to as Channel A, was used with BPSK transmission and consisted of a main component and two intersymbol interference terms. Its response is given by

$$h_c(t) = -0.45\delta(t - 1.25T) + 0.7\delta(t) + 0.45\delta(t + 1.25T). \quad (3.7)$$

The second channel, referred to as Channel B, used with QPSK transmission, was similar to Channel A described above, but included a phase offset

$$h_c(t) = 0.5756 \exp(j\pi/4)\delta(t - T) + 0.815\delta(t) + 0.5756 \exp(j\pi/4)\delta(t + T). \quad (3.8)$$

Suboptimum receive filtering was used, with the receive filter transfer function matched to the transmit filter. In the study presented here, the simulations were

performed using digital receive filters. Implemented as a tapped delay line, their sampling rate was varied among $T/8$, $T/4$ and $T/2$ to determine the tradeoff between receiver complexity and performance.

To eliminate ISI, the receive filter of varying complexity were coupled with T -spaced equalizers. Also, to examine the ability of fractionally-spaced equalizers to provide receive filtering [25], simulations using $T/8$, $T/4$ and $T/2$ equalizers and no receive filters were also performed. The length of the equalizer was chosen to provide sufficient elimination of ISI. For channels with short impulse response described above, equalizers spanning 16 symbol intervals were sufficient.

Tap adaptation was implemented in time domain using the non-block LMS algorithm and in the frequency domain using FBLMS. The latter included constant step size adaptation as well as normalized algorithm in which the step size was varied inversely proportional to the power content in each spectral bin. Estimation of the power distribution was done by means of (2.31) with the smoothing factor $\gamma = 0.95$. Both frequency domain algorithm used the overlap-save technique to obtain proper data sectioning thus yielding linear convolution.

A training sequence of bits was initially transmitted. For both channels, 10 blocks were sufficient to obtain convergence of the algorithm. Since the channels considered were assumed to be stationary, subsequent training sequences were not necessary, that is, data did not have to be broken down into blocks containing an initial preamble.

3.4 Bit Error Rate Performance

3.4.1 BER Simulation Method

To evaluate the bit error rate performance, the simulations were performed for signal-to-noise ratios of 5 to 30 dB. The signal-to-noise ratio was defined at the input to the

receiver, given by

$$SNR = \frac{E_b}{N_0}, \quad (3.9)$$

where N_0 is the noise variance and E_b is the energy per data bit computed as

$$E_b = \sum_{k=-\infty}^{\infty} h_k h_k^*, \quad (3.10)$$

where h_k is the discrete combined impulse response of the transmit filter, the channel and the receive filter as defined by (3.4).

3.4.2 Series Evaluation of BER

In [26], Beaulieu developed an efficient method for calculating the bit error rate for channels with both cross-talk and in-phase ISI and corrupted by additive Gaussian noise. The method is based on a Fourier series expansion of the noise cumulative distribution function (cdf) and can be extended to other noise distributions provided there exists a series representation for their cdf. To compute the BER, the T -spaced samples of the equalized impulse response must be known, along with the noise variance at the output of the equalizer. The discrete equalized system response, $\{g_k\}$ is the discrete convolution of the total channel samples with the equalizer coefficients, that is

$$g_k = \sum_{m=-\infty}^{\infty} f_m w_{k-m}. \quad (3.11)$$

As derived in [27], the BER series for QPSK signalling is given by

$$P(e) = \frac{1}{2} - \frac{2}{\pi} \sum_{m=1,3,5,\dots}^M \left\{ \frac{\exp\left(-\frac{m^2 \omega^2}{2}\right)}{m} \cdot \sin\left(\frac{m \omega g_0^I}{\sigma_z^I}\right) \cdot \left[\prod_{\substack{n=-N_1 \\ n \neq 0}}^{N_2} \cos\left(\frac{m \omega g_n^I}{\sigma_z^I}\right) \right] \cdot \left[\prod_{k=-N_1}^{N_2} \cos\left(\frac{m \omega g_k^Q}{\sigma_z^I}\right) \right] \right\} + \Delta, \quad (3.12)$$

where ω is the Fourier series frequency, Δ is the error term and N_1 and N_2 are, respectively, the number of postcursor and precursor intersymbol interference terms in the overall equalized response. In (3.12), g_0^I is the in-phase component of the desired signal term of the overall response, while $g_n^I (n \neq 0)$ and g_k^Q , the quadrature component of the overall response, constitute the ISI. The noise contribution to the BER is σ_z^I , the standard deviation of the noise samples in one component at the equalizer output, computed using Equation (3.41) of [22]. The above formulation of the BER can also be used for BPSK signaling by noting that in such cases the second product series in (3.12) equals to 1.

The accuracy of the series is determined by three factors: truncation of the infinite series to M terms, truncation of the ISI terms and the Fourier series period $T_F = 2\pi/\omega$. The error term, Δ , in (3.12) is given by [27]

$$|\Delta| \leq Q \left(\frac{T_F}{2} - \left| \frac{g_0^I}{\sigma_z^I} \right| - \sum_{\substack{n=-N_1 \\ n \neq 0}}^{N_2} \left| \frac{g_n^I}{\sigma_z^I} \right| - \sum_{n=-N_1}^{N_2} \left| \frac{g_n^Q}{\sigma_z^I} \right| \right) + \frac{\sqrt{2\pi}T_F}{\pi^2 M} Q \left(\frac{2\pi M}{T_F} \right), \quad (3.13)$$

where $Q(\cdot)$ is the complementary error function

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty \exp\left(-\frac{u^2}{2}\right) du. \quad (3.14)$$

In this thesis, all of the BER calculations were performed using $M = 401$ and $T = 300$, which yields results accurate to 10^{-14} for SNR up to 30 dB [22].

3.4.3 BER Results

Figure 3.4.3 shows the bit error rate results for BPSK transmission over Channel A. Time domain LMS and both frequency domain block LMS algorithms converged to the same set of equalizer taps thus yielding the same BER curves. The results obtained here are in agreement with those presented in [18]. For comparison, a

Figure 3.2: Channel A BER Results

matched filter bound, the performance achievable assuming transmission of isolated symbols (or equivalently a zero-ISI condition) [24], is also included in Figure 3.4.3.

The $T/8$ -spaced receive filter coupled with a T -spaced equalizer came within 2 dB of the matched filter bound at BER of 10^{-8} . When the complexity of the receive filter was reduced to tap spacing of $T/4$, the performance dropped an additional 2.5 dB. Finally, another two-fold reduction in receiver spacing to $T/2$ with a T -spaced equalizer resulted in a further performance drop of 3 dB.

Removing the receive filter and increasing the complexity of the equalizer by a corresponding amount resulted in a slight performance increase over the former

configuration. A $T/8$ -spaced equalizer without the receive filter outperformed the $T/8$ -spaced receiver coupled with a T -spaced equalizer by a fraction of a dB. Similarly, $T/4$ and $T/2$ -spaced equalizers yielded better BER results in comparison to $T/4$ and $T/2$ -spaced receivers with T -spaced equalizers. The reason for this slight increase in performance is that the fractionally-spaced equalizer performs a function of a matched filter as a receive filter [25]. Recall that the receive filter is of a root raised cosine shape, that is, it is matched to the transmit filter and not to the combination of the transmitter and the channel.

Bit error rate performance for Channel B using QPSK signaling is shown in Figure 3.4.3. High complexity $T/8$ -spaced receiver followed by a T -spaced equalizer resulted in 3.5 dB loss at BER of 10^{-8} compared to the matched filter bound. A two-fold reduction in the sampling rate of the receiver resulted in an additional 2 dB loss. Finally, a receiver sampled at twice the signaling rate achieved BER of 10^{-8} at signal-to-noise ratio of 21 dB.

In the case of Channel B, removing the receive filter and transferring the corresponding complexity to the equalizer resulted in no performance loss.

From the results presented above, it can be seen that a fractionally-spaced equalizer is capable of effectively performing the function of a receive filter. This finding has been discussed in [25]. Later in this chapter we examine the complexity of integrating the receive filter into the equalizer.

3.5 Convergence Properties

This section presents a study of the convergence properties of time domain non-block LMS and the frequency domain block LMS algorithms as applied to low data rate channels. The learning curve of the FBLMS algorithm was examined without step

Figure 3.3: Channel B BER Results

normalization and subsequently with step size normalized according to the power distribution across spectral bins.

Equalizers with sampling rate equal to the symbol interval were used and were preceded by a $T/8$ -spaced receiver filter. The simulations were run over Channels A and B at signal-to-noise ratio of 25 dB. An initial training sequence of 10 data blocks was transmitted. Each simulation consisted of an ensemble of 100 independent runs. To compute the mean squared error, the squared error was obtained after each algorithm iteration and averaged over the ensemble of tests.

In order to fairly compare the convergence properties of each algorithm, the step size was selected such as to obtain a final misadjustment value of 10% according to (2.26), (2.80) and (2.81). The step size in the tap update equation for the NFBLMS algorithm was chosen to be inversely proportional to the power estimate in the corresponding frequency bin. This estimation was calculated iteratively based on (2.31) with the smoothing factor $\gamma = 0.95$. In order to increase the stability of the algorithm in the event of a weakly excited frequency, a constant value of $\alpha = 0.1$ was added, resulting in the power estimate equation given by

$$\hat{\sigma}_{\mathcal{F},i}^2(n) = \gamma \hat{\sigma}_{\mathcal{F},i}^2(n-1) + (1-\gamma) \hat{\sigma}_{\mathcal{F},i}^2(n) + \alpha, \quad i = 0, 1, \dots, N-1. \quad (3.15)$$

The above recursion formula for power estimate does not always yield an accurate result during the first few iterations. In order to avoid an initial spike in the learning curve due to an incorrect step size normalization, the initial estimate of the power content was computed as an average over the first 3 iterations. During these iterations the step size was held constant.

Figure 3.5 shows the learning curve of the LMS, FBLMS and NFBLMS adaptation schemes for BPSK signaling over Channel A. Iteration number shown on the horizontal axis refers to the number symbols processed by the equalizer, or equiva-

Figure 3.4: Channel A Equalizer Learning Curves

lently, the number of tap adaptations for the LMS algorithm. As a reference point, the minimum mean square error as calculated by (2.19) is included.

As the results indicate, the LMS and FBLMS algorithms converged at the same rate, which supports the theory presented in Chapter 2. Transforming the input data sequence to the frequency domain does not affect the shape of the performance surface, but only rotates it. Consequently the learning curves for both algorithms are nearly identical. Defining convergence point as the iteration number where the mean square error comes within approximately 5% of the steady state value, we see that both algorithms converge to near the minimum mean square error after approximately

Figure 3.5: Channel B Equalizer Learning Curves

500 iterations.

As predicted by the theory presented in Chapter 2, normalizing the step size according to the power estimate of the input signal at each frequency bin results in a significant improvement in the learning curve. When using the NFBLMS algorithm, the steady state value of the MSE is reached at roughly 250 iterations.

To support these findings, we now examine the eigenvalue spread of the input correlation matrix for the LMS, FBLMS and NFBLMS equalizers. For the case of the LMS and FBLMS algorithms, the eigenvalue spread of the correlation matrix for Channel A can be calculated to be

$$\frac{\lambda_{\max}}{\lambda_{\min}} = 5.346. \quad (3.16)$$

In contrast, transforming the input data block into the frequency domain and normalizing the correlation matrix, results in a eigenvalue spread

$$\frac{\lambda_{\max}}{\lambda_{\min}} = 1.270, \quad (3.17)$$

which, as shown in Figure 3.5, causes more uniform convergence in each mode.

Figure 3.5 shows the learning curves of the algorithms for QPSK transmission over Channel B. Once again, the LMS and FBLMS algorithms exhibit nearly identical convergence properties. In the case of the LMS and FBLMS algorithms, the eigenvalue spread of the input correlation matrix for Channel B was calculated to be

$$\frac{\lambda_{\max}}{\lambda_{\min}} = 9.434. \quad (3.18)$$

With these eigenvalues controlling the modes of converges, the LMS and FBLMS algorithms reach a near steady state MSE after approximately 800 iterations. Similarly to the findings for Channel A, normalizing the step size results in a significant

improvement in the learning characteristics. The normalized correlation matrix of the data transformed to the frequency domain has the eigenvalue spread of

$$\frac{\lambda_{\max}}{\lambda_{\min}} = 3.073. \quad (3.19)$$

As a result, the NFBLS algorithm converges after 300 iterations.

3.6 Computational Complexity

In this section, the computational complexity of the receiving end of the model is examined. Both the receive filter and the equalizer are included in the analysis. In the analysis to follow the following notation is assumed: N_{rx} and T/M_{rx} denote, respectively, the total number of taps, the tap spacing of the receive filter, while N_{eq} and T/M_{eq} represent the total number of taps and tap spacing of the equalizer. In evaluating the computational complexity, the number of real multiplications needed to process a block of length N_{eq} is taken into account.

Since suboptimum receive filtering was used in the form of a root raised cosine, the receive filter is real valued. For a real input signal, filtering a block of N samples requires N^2 real multiplications. For QPSK transmission, this number is doubled as the input data is complex, resulting in $2N^2$ real multiplications per block. Thus to process a block of N_{eq} symbols, the receive filter requires

$$2 \frac{N_{eq}/M_{eq}}{N_{rx}/M_{rx}} N^2 \quad (3.20)$$

real multiplications for a real input data and

$$2 \frac{N_{eq}/M_{eq}}{N_{rx}/M_{rx}} N^2 \quad (3.21)$$

for complex input.

As discussed in Section 2.3.1, for the case of BPSK signaling, an adaptive equalizer implemented using the LMS algorithm requires total of $2N_{eq}^2$ real multiplications to process a block of N_{eq} samples. For QPSK transmission, the computational complexity increases by a factor of 4, resulting in $8N_{eq}^2$ multiplications per block of data.

Computational complexity of the FBLMS algorithm was analyzed in Section 2.6.3. Recall that a constrained algorithm requires a total of 5 FFT's of length $2N_{eq}$. For complex input data, each FFT can be computed using $2N_{eq}\log 2N_{eq}$ multiplications, while for real input, the computational load decreases by half to $N_{eq}\log 2N_{eq}$. In addition, the FBLMS algorithm requires $2N_{eq}$ complex multiplications for computing the gradient and for updating the equalizer taps. The total number of real multiplications required to process a block of N_{eq} data is $10N_{eq}\log 2N_{eq} + 16N_{eq}$ and $20N_{eq}\log 2N_{eq} + 16N_{eq}$ for BPSK and QPSK signaling respectively. As described in Section 2.6.3, the normalized FBLMS algorithm requires $10N_{eq}\log 2N_{eq} + 20N_{eq}$ real multiplications for real input and $20N_{eq}\log 2N_{eq} + 20N_{eq}$ for complex input.

Table 3.1 summarizes the total computational complexity for a T/M_{rx} -spaced receiver with a total number of taps N_{rx} and a T/M_{eq} -spaced linear equalizer of total length N_{eq} .

Table 3.1: Complexity of the LMS, FBLMS and NFBLMS Algorithms

	BPSK	QPSK
LMS	$\frac{N_{eq}/M_{eq}}{N_{rx}/M_{rx}} N_{rx}^2 + 2N_{eq}^2$	$2\frac{N_{eq}/M_{eq}}{N_{rx}/M_{rx}} N_{rx}^2 + 8N_{eq}^2$
FBLMS	$\frac{N_{eq}/M_{eq}}{N_{rx}/M_{rx}} N_{rx}^2 + 10N_{eq}\log 2N_{eq} + 16N_{eq}$	$2\frac{N_{eq}/M_{eq}}{N_{rx}/M_{rx}} N_{rx}^2 + 20N_{eq}\log 2N_{eq} + 16N_{eq}$
NFBLMS	$\frac{N_{eq}/M_{eq}}{N_{rx}/M_{rx}} N_{rx}^2 + 10N_{eq}\log 2N_{eq} + 20N_{eq}$	$2\frac{N_{eq}/M_{eq}}{N_{rx}/M_{rx}} N_{rx}^2 + 20N_{eq}\log 2N_{eq} + 20N_{eq}$

Table 3.2 shows tabulated results for a $T/8$ - spaced receiver spanning a duration of 8 symbols followed by a T -spaced equalizer of length 16.

Table 3.2: Complexity of a $T/8$ -spaced Receiver and a T -spaced Equalizer

	BPSK	QPSK
LMS	8704	18432
FBLMS	9248	18240
NFBLMS	9312	18304

The above results demonstrate that when the receive filter is followed by a linear equalizer, the FBLMS and NFBLMS algorithms require approximately the same amount of computation to process a block of data. For BPSK signaling, the frequency domain algorithms are computationally heavier by approximately 7%, while in the case of QPSK, they are less intensive by 1%. However, the relatively large improvement in convergence properties as demonstrated in Section 3.5 justifies the use of the NFBLMS algorithm.

Removing the receive filter from the system and increasing the tap spacing of the equalizer allows for a receiver consisting of only one structure. Complexity for an arrangement where $N_{rx} = 0$, $N_{eq} = 16 \times 8$ and $M_{eq} = 8$ is tabulated in Table 3.3.

Table 3.3: Complexity of a $T/8$ -spaced Equalizer

	BPSK	QPSK
LMS	32768	131072
FBLMS	12288	22528
NFBLMS	12800	23040

For a system with no receive filter, where the equalizer consists of fractionally-spaced taps, the LMS algorithm requires a considerably large amount of computation. Implementing the equalizer using either the FBLMS or NFBLMS algorithms results in a comparable computational complexity. The increase of approximately 30% can

be justified for the normalized step size algorithm as it does exhibit fast convergence while allowing for a single piece receiver.

Chapter 4

Performance over High Rate Channels

4.1 Introduction

Simulation results for high data rate multipath channels are presented here. Since such channels are characterized by a relatively long dispersion length, the elimination of intersymbol interference requires equalizers with a significantly larger number of taps compared to that used in the case of low data rate transmission. In such applications one expects the efficiency of frequency domain equalization to be advantageous.

The length of the equalizers employed in high data rate systems greatly exceeds the length of the receive filter. Based on the analysis from Chapter 3, a single structure receiver consisting of a fractionally spaced equalizer is not practical from the computational efficiency point of view. For this reason we examine only the conventional receiver consisting of a cascade of a receive filter and a T -spaced equalizer.

Studies were performed on two channels, referred to here as Channel C and Channel D, characterized by long delay spreads often encountered in urban environments [28]. Channel C, whose delay profile is shown in Figure 4.1, represents a high rate channel with a maximum delay spread of $8\mu s$. Reference [29] conducts a statistical

Figure 4.1: Channel C Delay Profile

analysis of wireless channel impulse responses at 1900 MHz. The shape of Channel D, whose delay profile is shown in Figure 4.2, was chosen based on the analysis performed by [29] in several Quebec City locations. The maximum delay spread of Channel D is $5\mu s$. Both channels were modeled by uncorrelated $T/8$ -spaced multipath components. The impulse response was obtained by varying the phase according to a uniform distribution from 0 to 2π and held constant for every symbol. Typical frequency spectra of Channels C and D are shown in Figures 4.3 through 4.5. The spectrum for Channel D was quite constant relative to the choice of the random phase per symbol. However, in the case of Channel C, the spectrum varied with the phase

Figure 4.2: Channel D Delay Profile

Figure 4.3: Channel C (Case 1) Amplitude Spectrum

samples. For instance, from Figure 4.3 we see that the channel contains a an inband null at DC. Such nulls are known to result in poor performance in the case of a linear equalizer [18][25]. In order to examine the effects of the null on the equalizer performance, a second instance of Channel C, one with a different phase characteristic, was also included in the BER analysis. The spectrum for case 2 of Channel C, which does not contain an inband null, is shown in Figure 4.4. Assuming that the bandwidth of a channel is defined by the frequency at an amplitude drop of 5 dB relative to 0 dB, the bandwidth of Channel C for cases 1 and 2 is approximately 200 kHz and 100 kHz, respectively. Channel D, with a shorter delay profile, has a larger bandwidth of

Figure 4.4: Channel C (Case 2) Amplitude Spectrum

Figure 4.5: Channel D Amplitude Spectrum

400 kHz. The simulations were performed using BPSK and QPSK transmission. The data rates were varied from 0.1 to 2 Mbps for BPSK and 0.2 to 4 Mbps for QPSK. We see that since these data rates exceed the Nyquist rate defined as twice the coherence bandwidth of the channel, such transmission results in significant distortion [23][30].

The receive filter used was of the root raised cosine shape sampled at a rate of $T/8$ and spanning 8 symbols. We note that the timing phase at the detector was optimized to give the lowest BER in all the examples to follow. The choice of the sampling phase was done on a per-channel basis. The signal-to-noise ratio E_b/N_0 was computed as given by (3.9) and (3.10). For data rates of 0.1 to 1 Mbps BPSK (0.2 to 2 Mbps for QPSK), the receive filter was followed by a T -spaced linear equalizer of length 32. For higher rates, the number of taps was increased to 64.

The chapter is divided as follows. First, the bit error rate performance for both channels is examined for BPSK and QPSK modulation methods. Following the BER results, we present the convergence characteristics of the LMS, FBLMS and NFBLMS algorithms. Adaptation learning curves are shown along with the analysis of the eigenvalue spreads of input data correlation matrices. To support the experimental results, we present an analysis of the input data correlation matrix for long equalizer responses. We show that in such cases [9][31][32], the correlation matrix is approximately circular and thus can be diagonalized by the discrete Fourier transform. This property results in a vastly improved convergence rate of the frequency domain algorithm, provided step size normalization is performed. Finally in the last section, the computational complexity for the LMS, FBLMS and NFBLMS algorithms is compared.

4.2 Bit Error Rate Performance

The bit error rate (BER) was computed using the series technique described in Section 3.4.2. The LMS, FBLMS and NFBLMS adaptation algorithms produced nearly identical sets of equalizer taps thus yielding the same BER performance. The BER results presented here are those corresponding to equalizer taps calculated using the NFBLMS algorithm.

4.2.1 BER Performance as a Function of SNR

Figures 4.6 and 4.7 show the BPSK and QPSK bit error rate results for case 1 of Channel C as a function of the signal-to-noise ratio (SNR) as defined by (3.9). In the case of BPSK signaling over Channel C, in order to achieve a bit error rate of 10^{-4} for 0.5 Mbps BPSK (1.0 Mbps QPSK), a signal-to-noise ratio of approximately 11.5 dB is required. In order to maintain a steady bit error rate, the signal power must be increased by approximately 4 dB for each 0.5 Mbps increase in transmission rate between 0.5 and 1.5 Mbps. The performance drop at BER of 10^{-4} between 1.5 and 2.0 Mbps is approximately 1.5 dB. A similar trend occurs for QPSK modulation where the same increase in SNR is needed for each 1.0 Mbps increase between 1.0 and 3.0 Mbps. Once again the performance drop between 3.0 and 4.0 Mbps is on the order of 1.5 dB.

Bit Error Rate results for case 2 of Channel C are shown in Figures 4.8 and 4.9. We see that the bit error probability at data rates of 0.5, 1.0 and 2.0 Mbps BPSK is nearly identical to that for case 1. The difference occurs for 1.5 Mbps BPSK where the performance in comparison to case 1 drops by approximately 1 dB. We see that despite the lack of an inband null, the BER performance for case 2 does not improve. This may be attributed to the fact that the bandwidth of the channel in case 2 is

Figure 4.6: Channel C (Case 1) BER Results (BPSK)

Figure 4.7: Channel C (Case 1) BER Results (QPSK)

Figure 4.8: Channel C (Case 2) BER Results (BPSK)

Figure 4.9: Channel C (Case 2) BER Results (QPSK)

Figure 4.10: Channel D BER Results (BPSK)

significantly smaller than that of case 1.

Finally, the results for Channel D are presented in Figures 4.10 and 4.11. They demonstrate that for Channel D nearly the same performance can be achieved at data rates of 0.5, 1.0 and 1.5 Mbps for BPSK and 1.0, 2.0 and 3.0 Mbps for QPSK modulation. Signal-to-noise ratio of 10 dB is sufficient to obtain BER of 10^{-4} . In order to maintain the error rate of 10^{-4} at 2 and 4 Mbps respectively for BPSK and QPSK, an increase of 2.5 dB in signal power is necessary.

Figure 4.11: Channel D BER Results (QPSK)

4.2.2 BER Performance as a Function of Data Rate

We now examine the relationship between the BER performance and the transmission rate. Shown in Figures 4.12 and 4.13 are the BER results (for SNR=15 dB) as a function of the data rate. The following are the findings.

- BER performance for Channel C is much poorer than that for Channel D, with a difference of approximately 7 orders of magnitude at 1 Mbps BPSK (2 Mbps QPSK).
- After an initial spike, the BER for case 1 of Channel C experiences a sudden drop at the data rate of 0.4 Mbps BPSK (0.8 Mbps QPSK). Similar drop occurs at 0.6 Mbps BPSK (1.2 Mbps QPSK) for case 2.
- Following the dip in BER, the performance for both cases of Channel C continues to degrade until 1.5 Mbps BPSK (3.0 Mbps for QPSK), when the BER becomes steady at a non-practical value of 10^{-2} .
- As expected from Figures 4.10 and 4.11, the BER over Channel D increases slowly with data rate between 0.5 and 1.5 Mbps BPSK (1.0 and 3.0 Mbps QPSK). The performance degrades drastically at rates greater than 1.5 Mbps BPSK (3.0 Mbps QPSK).

4.2.3 Channel Distortion Analysis

The large difference in performance between Channels C and D can be explained by examining the amount of intersymbol interference introduced by the channels. Figure

Figure 4.12: High Rate Channel BER as a Function of Data Rate (BPSK)

Figure 4.13: High Rate Channel BER as a Function of Data Rate (QPSK)

4.14 shows the input signal-to-interference ratio (SIR), defined by

$$SIR_{input} = \frac{f_0 f_0^*}{\sum_{i \neq 0} f_i f_i^*}. \quad (4.1)$$

where f_i denotes the impulse response of the overall (unequalized) channel. To include the contribution of noise, the input signal-to-noise-and-interference ratio (SNIR) defined by

$$SNIR_{input} = \frac{f_0 f_0^*}{\sum_{i \neq 0} f_i f_i^* + \sigma_v^2}. \quad (4.2)$$

where σ_v^2 denotes the total noise variance at the channel output, is shown in Figure 4.15. From Figures 4.14 and 4.15 it is clear that Channel C introduces significantly more interference than Channel D, which can be expected since Channel C has a longer delay spread. The difference in SIR and SNIR predicts the large difference in BER performance.

In order to study the sudden drop in BER at 0.4 Mbps for case 1 and 0.6 Mbps for case 2 of Channel C, we have also calculated the equalized SNIR, defined by

$$SNIR_{output} = \frac{g_0 g_0^*}{\sum_{i \neq 0} g_i g_i^* + \sigma_z^2}. \quad (4.3)$$

where σ_z^2 denotes the total noise variance at the equalizer output. Output SNIR is an indication of the equalizer's ability to combat the intersymbol interference introduced by the channel. Plots of SNIR as a function of data rate for BPSK and QPSK are shown in Figures 4.16 and 4.17. The results show consistency with the bit error rate plots for all cases considered. The dip in BER for Channel C at 0.4 Mbps BPSK is predicted by a sudden increase in the equalized SNIR. The relatively slow degradation in performance over Channel D with increasing data rate is also predicted by the shape

Figure 4.14: High Rate Channel Input SIR

Figure 4.15: High Rate Channel Input SNIR

Figure 4.16: High Rate Channel Output SNIR (BPSK)

of the SNIR curves which show a slow decrease in the amount of equalized distortion.

4.3 Convergence Properties

4.3.1 Equalizer Learning Curves

In this section we compare the convergence characteristics of the LMS, FBLMS and NFBLS algorithms for high data rate channels. The analysis is performed on Channel C (case 1) and Channel D at data rates of 2 Mbps BPSK and 4 Mbps QPSK with SNR fixed at 15 dB. As required for an adequate bit error rate performance, the

Figure 4.17: High Rate Channel Output SNIR (QPSK)

Figure 4.18: Channel C Equalizer Learning Curves (BPSK)

equalizer length at these data rates was set to 64 taps. The results were obtained by averaging an ensemble of 100 independent trials. In order to fairly compare the learning characteristics, individual curves in each graph were obtained using a step size chosen such as to result in the same steady state misadjustment. The step size was selected according to (2.26), (2.80) and (2.81) for a misadjustment value of 5%.

Figures 4.18 and 4.19 show the learning curves of the various algorithms as applied to Channels C and D for BPSK modulation. Corresponding results for QPSK are plotted in Figures 4.20 and 4.21.

As indicated by Figures 4.18 through 4.21, the NFBLS algorithm offers a signif-

Figure 4.19: Channel D Equalizer Learning Curves (BPSK)

Figure 4.20: Channel C Equalizer Learning Curves (QPSK)

Figure 4.21: Channel D Equalizer Learning Curves (QPSK)

Table 4.1: High Rate Channel Eigenvalue Spread Comparison

$\frac{\lambda_{max}}{\lambda_{min}}$	Channel C	Channel D
LMS, FBLMS	190.39	24.80
NFBLMS	12.02	4.01

icant improvement in the speed of convergence. In all cases examined, the LMS and FBLMS algorithms required over 3000 iterations to reach the steady state solution. On the other hand, the NFBLMS algorithm converged to the optimum tap values within 1500 iterations, or approximately 25 blocks.

The fast convergence of the NFBLMS algorithm also allowed for shorter training sequences. In all cases, stability was ensured with training sequences of only 10 blocks. The LMS and FBLMS algorithms required in excess of 50 training blocks in order to converge to the optimum solution.

4.3.2 Correlation Matrix Eigenvalue Analysis

As stated in Chapter 2, the rates of convergence of the LMS and NFBLMS algorithms are governed by the ratio of the largest and smallest eigenvalues of their input correlation matrices. Table 4.1 presents a comparison of the eigenvalue disparity of the transformed and non-transformed correlation matrices. The results shown indicate that the eigenvalue spread reduction is greater in the case of Channel C. This is consistent with the learning curves presented in Figures 4.18 through 4.21, which show that the improvement in the convergence is greater in the case of Channel C.

As the tabulated results show, the Fourier transform operation significantly reduces the eigenvalue disparity of the input process. In the cases examined, the eigenvalue spread is reduced by about an order of magnitude thus supporting the learning

curve results previously presented.

The reduction of the eigenvalue disparity for large N can be explained by close examination of the structure of the input correlation matrix. The following discussion is based on the work presented in [9],[31] and [32]. Defining the auto-correlation of the input x_n as $\rho_l = E[x_n^* x_{n-l}]$ for $l = 0, 1, \dots, N-1$, we have the auto-correlation matrix

$$\mathbf{R} = \begin{bmatrix} \rho_0 & \rho_1 & \rho_2 & \cdots & \rho_{N-1} \\ \rho_1^* & \rho_0 & \rho_1 & \cdots & \rho_{N-2} \\ \rho_2^* & \rho_1^* & \rho_0 & \cdots & \rho_{N-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho_{N-1}^* & \rho_{N-2}^* & \rho_{N-3}^* & \cdots & \rho_0 \end{bmatrix}. \quad (4.4)$$

From (4.4) we see that the correlation matrix $\mathbf{R}$ is Toeplitz and Hermitian. It can be shown [33] that when ρ_l approaches zero for lag values l greater than m (m much smaller than $N/2$), the eigenvalue distribution of $\mathbf{R}$ is asymptotically equivalent to the eigenvalue distribution of a circular matrix $\mathbf{C}$, defined as

$$\mathbf{C} = \begin{bmatrix} \rho_0 & \rho_1 & \cdots & \rho_m & & \rho_m^* & \cdots & \rho_2^* & \rho_1^* \\ \rho_1^* & \rho_0 & \ddots & \vdots & \rho_m & & \ddots & & \rho_2^* \\ \vdots & \rho_1^* & \ddots & \rho_1 & \vdots & \ddots & & \mathbf{0} & \rho_m^* \\ \rho_m^* & \vdots & \ddots & \rho_0 & \rho_1 & \rho_m & & & \rho_m^* \\ & \rho_m^* & & \rho_1^* & \rho_0 & \ddots & \vdots & \ddots & \\ & & \ddots & \vdots & \rho_1^* & \ddots & \rho_1 & \ddots & \\ \rho_m & & & \rho_m^* & \vdots & \ddots & \rho_0 & \ddots & \rho_m \\ \vdots & \ddots & \mathbf{0} & & \rho_m^* & \rho_1^* & \ddots & \ddots & \vdots \\ \rho_2 & & \rho_m & & \ddots & \vdots & \ddots & \rho_0 & \rho_1 \\ \rho_1 & \rho_2 & \cdots & \rho_m & & \rho_m^* & \cdots & \rho_1^* & \rho_0 \end{bmatrix}. \quad (4.5)$$

The circular matrix $\mathbf{C}$ is obtained from $\mathbf{R}$ by appropriately filling in its upper-right and lower-left corners. When the correlation matrix $\mathbf{R}$ is approximately circular, corresponding frequency domain correlation matrix $\mathbf{R}_{\mathcal{F}}$ is approximately diagonal

[15]

$$\begin{aligned}
\mathbf{R}_{\mathcal{F}} &= \mathcal{F}\mathbf{C}\mathcal{F}^H \\
&= N(\mathcal{F}\mathbf{C}\mathcal{F}^{-1}) \\
&= N\text{diag}[r_{\mathcal{F}}],
\end{aligned} \tag{4.6}$$

where $r_{\mathcal{F}} = \mathcal{F}r$, that is, the Fourier transform of the first column of $\mathbf{R}$. This shows that when the decay time of the input auto-correlation is small compared to filter length N , the Fourier-transformed correlation matrix is approximately diagonal. As described in Section 2.4, proper step normalization is equivalent to obtaining a correlation matrix close to identity, which in turn has a small eigenvalue spread.

4.4 Computational Complexity

We now compare the computational complexity of the LMS, FBLMS and NFBLS equalizers applied to the high data rate channels. Since all cases considered thus far in this chapter included a receive filter of constant complexity, only the number of real multiplications to process a block of data of length N at the equalizers is included in the calculations.

Table 4.2 shows the total number of multiplications required to process a block of data of length N using the LMS, FBLMS and NFBLS algorithms for real input sequence (BPSK modulation). As discussed in Chapter 2, the total number of real multiplications used by the LMS, FBLMS and NFBLS algorithms for a block of N symbols are $2N^2$, $10N\log 2N + 16N$ and $10N\log 2N + 20N$ respectively. For data rates less than 1 Mbps (BPSK), equalizers of length $N = 32$ were used, with N increasing to 64, for rates of 1 to 2 Mbps (BPSK). We see that for equalizer length of 64 both the non-normalized and normalized frequency domain algorithms offer computational

Table 4.2: Equalizer Complexity for High Rate Channels (BPSK)

	$N = 32$	$N = 64$
LMS	2048	8192
FBLMS	2432	5504
NFBLMS	2560	5760

savings over the time domain implementation.

For QPSK signaling, the total number of real multiplications used by the LMS, FBLMS and NFBLMS algorithms for a block of N symbols are $8N^2$, $20N\log_2 N + 16N$ and $20N\log_2 N + 20N$ respectively. In the case QPSK modulation, where the input data is complex, the reduction in complexity is noticeable for all cases examined here. The results are tabulated in Table 4.3. The increase in the computational savings for

Table 4.3: Equalizer Complexity for High Rate Channels (QPSK)

	$N = 32$	$N = 64$
LMS	8192	32768
FBLMS	4352	9984
NFBLMS	4480	10240

complex input data is due to the fact that the time domain implementation requires 4 times the number multiplications compared to real data. The frequency domain algorithm, which operate with complex numbers regardless of the type of input, only have a two-fold increase in computations resulting from the loss of symmetry of the FFT algorithm. Thus, the frequency domain implementations offer savings of 45%, and 69% for filter lengths of $N = 32$ and $N = 64$.

As stated in [14], many DSP's are capable of performing up to 1.6×10^9 real operations per second. Assuming a transmission rate of 2×10^6 complex symbols per

second, the DSP will be capable of performing approximately 800 real operations per symbol. Taking into account that the values given in Tables 4.2 and 4.3 are for real operations for blocks of 32 and 64 symbols, it can be seen that, at the above data rate, modern DSP's should be well suitable for equalization applications.

Chapter 5

Conclusion

5.1 Conclusions

This thesis presented a comparison of time domain and frequency domain equalizers. Specifically, we examined the BER performance, convergence characteristics and computational complexity of LMS, FBLMS and NFBLS algorithms as applied to static multipath channels with short and long delay spreads. The conclusions of this study are as follows:

1. The block LMS algorithm can be efficiently implemented in the frequency domain using the overlap-save data sectioning and a gradient constraint. The resulting FBLMS and NFBLS algorithms converge to the same sets of equalizer taps as their LMS counterpart. The bit error rate performance of all of these equalizers is thus identical.
2. A fractionally-spaced FBLMS and NFBLS equalizers have the capability of performing the function of a receive filter. In cases when the span of the receive filter is approximately equal to that of the equalizer, the same BER performance can be obtained by eliminating the former and increasing the sampling rate of the equalizer. Due to the low complexity growth with an increase of the number

of taps, the computational complexity of the fractionally-spaced equalizer is comparable to the combined complexity of the receive filter and a symbol-spaced equalizer.

3. The BER performance was studied for two high data rate multipath channels with random phase characteristics. The first channel, with maximum delay spread of $8\mu s$, was examined for two different phase profiles. In the first case, for which the channel spectrum contained an inband null, a signal-to-noise ratio of 11.5 dB was required to achieve a BER of 10^{-4} at data rate of 0.5 Mbps BPSK (1.0 Mbps QPSK). Such performance can be maintained with an increase in SNR of 4 dB for each increase in data rate of 0.5 Mbps BPSK until 1.5 Mbps. Performance drop between 1.5 and 2.0 Mbps BPSK (3.0 and 4.0 Mbps QPSK) was approximately 1.5 dB. The second case of channel one, one without an inband null, resulted in similar BER performance. The lack of performance improvement for case 2 was attributed to the smaller channel bandwidth. The second high data rate channel studied, one with a maximum delay spread of $5\mu s$, required an SNR of approximately 10 dB to achieve BER of 10^{-4} for the rates of 0.5 to 1.5 BPSK and 1.0 to 3.0 Mbps QPSK. In order to increase the data rate to 2.0 and 4.0 Mbps, a SNR increase of 2.5 dB was needed.
4. By varying the step size in the adaptation process according to the power distribution of the input signal, the resulting NFBLS algorithm exhibits a much greater convergence rate than the LMS and FBLMS implementations. The improvement is most noticeable for long equalizers where the input data correlation matrix is approximately circular and thus diagonalized by the Fourier transform. When an appropriate normalization technique is implemented, the

eigenvalue disparity is greatly reduced, resulting in a uniform convergence across all modes. As a result, for both high data rate channels examined, the NFBLS algorithm converged to a steady state after about 25 data blocks. In all of the cases examined the LMS and FBLMS algorithms failed to converge after processing 50 data blocks.

5. For long equalizer responses, the NFBLS and FBLMS algorithms offer significant savings in computational complexity over the LMS counterpart. For BPSK signaling, the frequency domain implementations yield an improvement of 35% for block lengths of 64. In the case of QPSK transmission, gains of 50%, 65% are obtained for block lengths of 32 and 64.

5.2 Suggestions for Further Research

The frequency domain block algorithms offer significant computational savings for long equalizers. The algorithm studied in this thesis is based on a constrained gradient approach. The advantage of this adaptation technique is that the steady state equalizer coefficients are equal to the Wiener solution. The gradient constraint, however, can be omitted in favor of eliminating two of the FFT operations. It has been shown by Mansour and Gray [34] that when the filter length is sufficiently large, such an algorithm will converge to a similar tap configuration as the constrained counterpart. As pointed out by [13] and [15], while this algorithm is much simpler to implement, it does come at a cost of a higher misadjustment for a given step size. Despite the above shortcoming, it may be worthwhile to examine the performance of the unconstrained FBLMS equalizer for high data rate channels. Apart from reduced computational load, the convergence rate should also be improved over the LMS algorithm since, as shown by [31], the frequency domain correlation matrix of such an

algorithm is also approximately diagonal.

One of the traits of fractionally-spaced equalizers is their ability to compensate for a poor choice of timing phase at the receiver [35]. While in this thesis we have in all cases optimized the timing phase, it would be of interest to examine the ability of the fractionally-spaced frequency domain equalizers to compensate for delay distortion introduced by the system.

While the convergence rate of the NFBLMS equalizers was shown to be much better than that of the LMS and FBLMS implementations, it still remains on the order of 25 data blocks in the cases of long equalizer responses. A further improvement in the learning curve can be obtained by implementing a frequency domain RLS algorithm, as these types of equalizers are usually superior in their convergence properties in comparison to the LMS equalizer [18].

In examining high data rate channels, we have limited ourselves to static impulse responses. Consequently, tracking properties of the frequency domain equalizers were not analyzed. One may perhaps wish to analyze the combined effects of the delay introduced by block processing, as well as the relatively fast rate of convergence, on the ability of the frequency domain equalizer to track temporal variations in the channel impulse response.

Finally, it is known [18] that the linear equalizer suffers great performance degradation in cases in which the involved channel exhibits bad spectral characteristics (deep spectral nulls). The most common solution to this problem is the Decision Feedback Equalization (DFE). In the recent literature, several authors have worked on developing frequency implementations of block decision feedback equalizers [36] [37]. It may be worthwhile to examine the performance of the developed structures as applied to the high data rate wireless environment.

Bibliography

- [1] R. W. Lucky. Automatic equalization for digital communications. *Bell System Technical Journal*, 44:547–588. April 1965.
- [2] R. W. Lucky. Techniques for adaptive equalization of digital communications systems. *Bell System Technical Journal*, 45(2):255–286. February 1966.
- [3] B. Widrow and M. E. Hoff Jr. Adaptive switching circuits. *IRE WESCON Convention Record*, pages 563–583. 1960.
- [4] B. Widrow. Adaptive filters I: Fundamentals. *Stanford Electronics Laboratory, Stanford, CA*, Tech Report No. 6764-6. December 1966.
- [5] T. Waltzman and M. Schwartz. Automatic equalization using the discrete frequency domain. *IEEE Transactions on Information Theory*, IT-19(1):59–68. January 1973.
- [6] M. Dentino, J. McCool, and B. Widrow. Adaptive filtering in the frequency domain. *Proceedings of the IEEE*, 66(12):1658–1659. December 1978.
- [7] E. R. Ferrara. Fast implementation of LMS adaptive filters. *IEEE Transactions on Acoustics, Speech, and Signal Processing*, ASSP-28(4):474–475. August 1980.
- [8] S. S. Narayan and A. M. Peterson. Frequency domain least-mean-square algorithm. *Proceedings of the IEEE*, 69(1):124–126. January 1981.

- [9] J. C. Lee and C. K. Un. Performance of transform-domain LMS adaptive algorithms. *IEEE Transactions on Acoustics, Speech, and Signal Processing*, ASSP-34:499-510, 1986.
- [10] D. F. Marshall, W. K. Jenkins, and J. J. Murphy. The use of orthogonal transforms for improving performance of adaptive filters. *IEEE Transactions on Circuits and Systems*, CS-36(4):474-483, April 1989.
- [11] B. Farhang-Boroujeny and S. Gazor. Selection of orthonormal transforms for improving the performance of the transform domain normalised LMS algorithm. *IEE Proceedings-F*, 139(5):1992, October 1992.
- [12] B. Widrow and S. D. Stearns. *Adaptive Signal Processing*. Prentice-Hall, Englewood Cliffs, NJ, 1985.
- [13] J. J. Shynk. Frequency-domain and multirate adaptive filtering. *IEEE Signal Processing Magazine*, 9(1):14-37, January 1992.
- [14] M. V. Clark. Adaptive frequency-domain equalization and diversity combining for broadband wireless communications. *IEEE Journal on Selected Areas in Communications*, 16(8):1385-1395, October 1998.
- [15] B. Farhang-Boroujeny. *Adaptive Filters: Theory and Applications*. John Wiley & Sons Ltd., Toronto, ON, 1998.
- [16] S. Haykin. *Adaptive Filter Theory*. Prentice Hall, Upper Saddle River, NJ, 1996.
- [17] J. G. Proakis and D. G. Manolakis. *Digital Signal Processing, Principles, Algorithms, and Applications*. Prentice-Hall, Toronto, ON, 1996.
- [18] J. G. Proakis. *Digital Communications*. McGraw-Hill, New York, NY, 1995.

- [19] A. Papoulis. *Probability, Random Variables, and Stochastic Processes*. McGraw-Hill, Toronto, ON, 1991.
- [20] A. V. Oppenheim and R. W. Schaffer. *Discrete-Time Signal Processing*. Prentice-Hall, Englewood Cliffs, NJ, 1989.
- [21] J. W. Cooley et al. The fast Fourier transform algorithm: Programming considerations in the calculation of sine, cosine and Laplace transforms. *Journal of Sound and Vibrations*, 12:315–337, July 1970.
- [22] J. E. Smee. Error-rate evaluation of linear equalization and decision feedback equalization. Master's thesis, Queen's University, 1995.
- [23] T. S. Rappaport. *Wireless Communications, Theory and Practice*. Prentice Hall, Upper Saddle River, NJ, 1996.
- [24] E. A. Lee and D. G. Messerschmitt. *Digital Communication*. Kluwer Academic Publishers, Norwell, MA, 1994.
- [25] J. F. Hayes R. D. Gitlin and S. B. Weinstein. *Data Communications Principles*. Plenum Press, New York, NY, 1992.
- [26] N. C. Beaulieu. The evaluation of error probabilities for intersymbol and cochannel interference. *IEEE Transactions on Communications*, 39(12):1740–1749, December 1991.
- [27] J. E. Smee and N. C. Beaulieu. Error-rate evaluation of linear equalization and decision feedback equalization with error propagation. *IEEE Transactions on Communications*, 46(5):656–664, May 1998.

- [28] G. L. Stuber. *Principles of Mobile Communication*. Kluwer Academic Publishers. Norwell, MA. 1996.
- [29] T. J. Hicks. Time spread characterization of outdoor microcellular and indoor wireless 1.9 GHz radio channels. Master's thesis. Queen's University. 1997.
- [30] L. W. Couch II. *Digital and Analog Communication Systems*. Macmillan Publishing Co., Inc., New York, NY. 1983.
- [31] J. C. Lee and C. K. Un. Performance analysis of frequency-domain block LMS adaptive digital filters. *IEEE Transactions on Circuits and Systems*, 36(2):173-189, February 1989.
- [32] B. Farhang-Boroujeny and K. S. Chan. Analysis of the frequency-domain block LMS algorithm. *IEEE Transactions on Signal Processing*, 48(8):2000, August 2000.
- [33] R. M. Gray. On the asymptotic eigenvalue distribution of Toeplitz matrices. *IEEE Transactions on Information Theory*, IT-18:725-730, November 1972.
- [34] D. Mansour and A. H. Gray Jr. Unconstrained frequency domain adaptive filter. *IEEE Transactions on Acoustics, Speech, and Signal Processing*, ASSP-30:726-734, October 1982.
- [35] R. D. Gitlin and S. B. Weinstein. Fractionally-spaced equalization: An improved digital transversal equalizer. *The Bell Technical Journal*, 60(2):275-298, February 1981.

- [36] K. Berberidis and J. Palicot. A frequency-domain decision feedback equalizer for multipath echo cancellation. *Proceedings from 1995 IEEE GLOBECOMM Conference*, November 1995.
- [37] K. Berberidis and J. Palicot. Efficient block implementation of the decision feedback equalizer. *IEEE Signal Processing Letters*, 5(6):1998, June 1998.