GENERALIZED SINGULARITY ANALYSIS OF MECHANISMS

by

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ABSTRACT

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This thesis investigates a general class of mechanism configurations, usually referred to as kinematic singularities. The study of such configurations is of major practical and theoretical importance. Indeed, the kinematic properties of mechanisms change significantly in a singular configuration, and these changes can prove to be either beneficial or undesirable for different applications. On the other hand, the theoretic significance of singularities in mechanism theory is well-known and related to the fact that singular points play a prominent role in the theory of differentiable mappings.

The central objective of this dissertation is to address the problems of mechanism singularity in a most general setting, namely, to consider arbitrary singular configurations of both non-redundant and redundant mechanisms with arbitrary kinematic chains, with a special emphasis on the study of mechanical devices with complex kinematic chains and non-serial, high-degree-of-freedom architectures. To this goal, a rigorous general mathematical definition of kinematic singularity for arbitrary mechanisms is introduced. This is achieved by means of a mathematical model of mechanism kinematics formulated in terms of differentiable mappings between manifolds. When the mathematical model is applied to the relationship between the joint and output velocities, a new unifying framework for the interpretation and classification of mechanism singularities is obtained. This framework, based on the newly introduced six singularity types, is applicable to arbitrary non-redundant as well as redundant mechanisms. Mathematical tools, such as singularity criteria and identification methods, are developed for the study of the singularity sets of both non-redundant and redundant systems with lower kinematic pairs.

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NOMENCLATURE

$\mathcal{A}(W, \gamma)$	An articulated system with kinematic chain $W = (\Gamma, \rho)$ and link geometry γ.
A_j	The set of arcs in the graph of a HCM corresponding to the n_j active joints in the j-th subchain.
$\mathcal{A}_j$	The subspace of $\mathcal{T}$ spanned by the active joint screws in the j-th subchain of a HCM.
$\mathbf{A} \circ \mathbf{B}$	The reciprocal product of two twists.
$\mathcal{B}$	A rigid body.
C	Cylindrical joint type.
c	Maximum number of independent loops.
D	The configuration space of a mechanism.
$d(\boldsymbol{x}, \boldsymbol{y})$	The distance between points $\boldsymbol{x}$ and $\boldsymbol{y}$ in E^3.
d_I, d_O	Dimensions of the null-spaces of $\boldsymbol{p}_I$ and $\boldsymbol{p}_O$, respectively.
d_{IO}	Dimension of the intersection of the null-spaces of $\boldsymbol{p}_I$ and $\boldsymbol{p}_O$.
dof	Degree of freedom.
E^3	The affine Euclidian space in three dimensions.
FIKP	Forward instantanoeus kinematic problem.
f_I	The input map of a mechanism.
f_O	The output map of a mechanism.
$\boldsymbol{G}$	Rotation matrix of a displacement g.
$\mathcal{G}(M, N, \nu)$	A directed graph with M vertices and N arcs.
Γ	A directed graph.
$\gamma : A \rightarrow SE(3)^2$	A link geometry map.

$\gamma(a) = (\gamma_-, \gamma_+)$ — Location of the joint of arc a in the first and second bodies.

H(p) — Helical joint type of pitch p.

HCM — Hybrid-chain manipulator.

$I = \prod_{a \in A_I} Q_a$ — The input space of a mechanism with active joints A_I.

$\mathcal{I}$ — The set of input velocities, i.e., $T_q O$.

IIKP — Inverse instantanoeus kinematic problem.

II — Impossible input.

IIM — Increased instantaneous mobility.

IO — Impossible output.

I_n — The $n \times n$ unit matrix.

J_F, P_F — Matrices solving the forward instantanoeus kinematic problem.

J_I, P_I — Matrices solving the inverse instantanoeus kinematic problem.

$\mathcal{K}(C, Q)$ — Kinematic system on Q with configuration space C.

$\mathcal{K}(Q)$ — The kinematic system $\mathcal{K}(Q, Q)$.

$\mathcal{K}(C, SE(3)^n)$ — A kinematic system of n rigid bodies with configuration space C.

k — Number of serial subchains in a HCM.

$L(q)$ — The matrix of the velocity equation.

$\bar{L}(q)$ — The matrix of a simplified velocity equation where some passive velocities have been eliminated.

L_I, L_O and L_p — The submatrices of L obtained by removing the columns corresponding to the input, output, and both the input *and* output, respectively.

L_T — The matrix formed by the columns of L which correspond only to the output velocities.

L_a — The matrix formed by the columns of L which correspond only to the input velocities.

$\bar{L}_I$, $\bar{L}_O$ and $\bar{L}_p$ — Submatrices of $\bar{L}$ obtained by removing the columns corresponding to the input, output, and both the input *and* output, respectively.

$\mathcal{M}$, $\mathcal{N}$	Mechanical systems.
$\overline{\boldsymbol{M}}$	A velocity vector in which k passive velocities have been eliminated.
$\mathfrak{M}$	A mechanism.
$\mathcal{M}_I$	The input system of a mechanism.
$\mathcal{M}_O$	The output system of a mechanism.
$\mathcal{M}_q$	The motion space at $\boldsymbol{q}$ of a mechanism.
$\boldsymbol{m} = (\boldsymbol{T}, \boldsymbol{\Omega})$	A velocity vector.
μ	The mobility of a mechanism.
$\mu(\mathcal{M})$	The global mobility of a mechanical system $\mathcal{M}$.
N	Number of particles in a particle system.
N	Number of joints in a mechanism with 1-dof lower-pair joints.
$\overline{N}$	The number remaining joint velocities after k passive joint velocities have been eliminated, i.e., $\overline{N} = N - k$.
$n(V)$	The smallest $n_x(V)$.
n_I	The dimension of the input space, I, of a mechanism.
n_O	The dimension of the output space, O, of a mechanism.
n_j	Number of active joints in the j-th subchain of a HCM.
n_q	Instantaneous mobility in configuration $\boldsymbol{q}$.
$n_x(V)$	The dimension of the Zariski-tangent space at $\boldsymbol{x}$.
Nrg $\mathcal{M}$	The set of configurations of $\mathcal{M}$ which are not regular.
Nsg V	The set of nonsingular points of an algebraic set V.
O	The output space of a mechanism.
$\boldsymbol{O}$	The set of output velocities, i.e., T_qO.
$C^{(e)}$	The set of configurations with local mobility e.
$o\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$	A reference frame in E^3.
$o\ (\mathbf{x}, \mathbf{y}, \mathbf{z})$	The orientations of an array of three vectors in $\mathbf{R}^3$ (or four points in E^3).
$\boldsymbol{\Omega} = (\boldsymbol{\Omega}^a, \boldsymbol{\Omega}^p)$	A tangent vector of the joint space manifold, Q.

Ω^a — An input vector, i.e., an element of $T_q I$.

Ω^p — A vector composed of the passive-joint velocities of a mechanism.

$\overline{\Omega}^p$ — A vector with components $(N - n - k)$ of the passive-joint velocities of a mechanism.

ω_P or v_P — Joint velocity at joint P.

P — Prismatic joint type.

$\mathcal{P}$ — The set of passive velocities.

$P(v)$ — Unique elementary path from vertex 0 to vertex v in a tree-structured graph.

$\mathcal{P}_j$ — The subspace of $\mathcal{T}$ spanned by the passive joint screws in the j-th subchain of a HCM.

$\boldsymbol{p}_I, \boldsymbol{p}_O$ — Projection mappings which map any motion vector into the vector of its input or output, respectively.

$\boldsymbol{q}$ — A configuration of a mechanical system.

Q — The joint space manifold of a mechanism.

R — Revolute joint type.

$\boldsymbol{R}(\boldsymbol{q}), \boldsymbol{H}(\boldsymbol{q})$ — Matrices multiplying the output and input, respectively, in the input-output equation of a HCM.

$\mathbf{R}[\mathbf{x}]$ — The ring of polynomials.

RI — Redundant input.

RO — Redundant output.

RPM — Redundant passive motion.

Reg $\mathcal{M}$ — The set of regular configurations of $\mathcal{M}$.

$\mathrm{Rot}_o(E^3)$ — The set of all rotations about an arbitrary fixed point $\boldsymbol{o}$.

r_I, r_O — The ranks of the maps $\boldsymbol{p}_I$ and $\boldsymbol{p}_O$.

$\rho : A \rightarrow \mathcal{P}$ — A joint distribution map.

$\rho(a)$ — Joint type of joint corresponding to arc a.

S	Spherical joint type.
S	An n-dimensional subspace $\mathcal{T}$, containing all the joint screws of a HCM of mobility n.
$S(M)$	The graph of a serial chain with M joints.
$\mathbf{S}_P$	Joint screw at joint P.
$\mathbf{S}_i^j$	The joint screw of joint i in subchain j of a HCM.
$SE(3)$	The group of displacements of the affine Euclidian space E.
$SO(3)$	The special orthogonal group in three dimensions.
Sing V	The set of the singular points of an algebraic set V.
T	A joint type.
$\boldsymbol{T}$	An output vector, i.e., an element of T_qO.
$\mathcal{T}$	The 6-dimensional vector space of all twists.
$\mathbf{T} = (\omega, \mathbf{v})$	The output twist of a mechanism.
$\mathcal{T}_j$	The subspace of $\mathcal{T}$ spanned by all the joint screws in the j-th subchain of a HCM.
$T_x^{\mathrm{Zar}}(V)$	The Zariski-tangent space at x of an algebraic set V.
$T_x X$	The tangent space of a manifold X at x.
$\mathbf{t}_g$	Translation vector of a displacement g.
$\mathrm{Tr}(E^3)$	The set of all translations in E^3.
$\mathcal{V}$	The velocity space of a mechanism.
V^0	The set of all x in V with a minimal $n_x(V)$.
V^-	The complement of V^0 in V.
(V, A, ν)	A directed graph with vertices V and arcs A.
$\nu(a) = (\nu_-, \nu_+)$	The starting and terminating vertices of the arc a.
$W = (\Gamma, \rho)$	A kinematic chain with graph Γ and joint-type distribution ρ.
$\boldsymbol{x}$	A point in E^3.
$\mathbf{x}$	A vector in $\mathbf{R}^3$ equal to $\boldsymbol{x} - \boldsymbol{o}$, where $\boldsymbol{o}$ is a chosen origin in E^3.

CHAPTER 1

INTRODUCTION AND PRELIMINARIES

1.1. Introduction

This thesis investigates a class of mechanism configurations generally known as singularities. Mechanisms with arbitrary kinematic chains are considered, the emphasis being on complex, multi-loop closed chains. It is assumed that the primary purpose of a mechanism is to move an end-effector: a rigid body which is identified with one of the links of the mechanism, referred to as the output link. It will be further assumed that some, but not necessarily all, of the mechanism's joints are actuated.

In such a general context, there is no standard definition of mechanism singularity. However, it is well known that the study of the kinematics of mechanical systems in robotics and mechanism theory cannot be considered complete unless the problems related to a certain class of configurations, usually referred to as "singular", "special" or "critical" configurations, have been addressed. In these configurations, the kinematic (and static) properties of mechanisms change dramatically. The purpose of this thesis is to contribute to the theory of mechanism kinematics by studying these configurations.

1.1.1. Motivation for research on kinematic singularity

The significant changes in the kinematic properties of mechanisms, which occur in a singular configuration, may be undesirable or potentially beneficial. Hence, the study of such configurations is of significant importance for the application, control and design of mechanisms.

The common objective of numerous researchers who studied kinematic singularities has been the desire to avoid such configurations during the operation of robotic manipulators. For a serial manipulator, for example, the Jacobian is not invertible at a singular configuration. This causes local control methods to fail at singularities. Also, at near-singular configurations, very large joint motions may be required to produce relatively small end-effector displacements. Singularities of parallel manipulators are also highly undesirable, since, if such a configuration were to be encountered, the acquisition of extra unwanted freedoms would transitorily put the end-effector out of control (Hunt 1986).

Sometimes it may, however, be possible to take advantage of singularities. In a singular configuration, a serial manipulator can withstand, in principle, infinitely large forces and torques about the screw axes of the impossible end-effector motions (Hunt 1978, 1986). It has been suggested that this property may be used in applications like drilling, grinding or handling of heavy objects (Hunt 1986, Wang and Waldron 1987).

Singularities are also of major theoretical importance. It would be impossible to understand the kinematics of mechanisms without a profound study of their "special" or singular configurations.

In the case of serial chains, the significant role that singular configurations play in manipulator kinematics can be deduced from the importance of singularities in the theory of differentiable mappings. This connection is due to the fact that manipulator kinematics is fully described by a smooth map (the so-called manipulator map, or the output map) between two smooth manifolds: the joint space and the workspace (Burdick, 1988). The singular configurations are the critical (or singular) points of this map. Hence, the theory of singularities of smooth maps (Arnold, et al. 1985, Golubitsky and Guillemin 1983) is well suited to provide tools for the investigation of both the local properties of manipulator kinematics near singularities and the global properties of the manipulator map.

Indeed, a number of researchers have applied ideas from singularity theory for the classification of serial manipulators and their singularities. The concepts of generic and transversal mappings have been applied to the kinematic map of a manipulator by Pai and Leu (1992) and Tsai et al. (1993). (A manipulator kinematic map, f, is said to be *generic* when its differential, Df, is transversal to the collection $\{\mathcal{L}_i\}$ of all rank-i submanifolds of the space, $\mathcal{L}$, of all linear maps from the tangent space of the configuration space to the tangent space of the task space). Tchon (1991) and Burdick (1991) apply the concepts of generic maps to propose classifications of certain robots and their singular configurations.

Furthermore, a number of studies have demonstrated that singularities play a key role in determining the global kinematic properties of serial manipulators. Burdick (1988, 1992) uses singularity submanifolds to partition the joint space into singularity-free regions and analyzes the global properties of the workspace resulting from the mapping of these regions by the manipulator map. The special role played by singularities in the study of the manipulator workspace is also illustrated by the works of Borrel and Liegeois (1986), Hsu and Kohli (1987) and Wenger (1992). Moreover, the insight gained through such topological methods can be particularly useful in path planning and design. Indeed, Borrel and Liegeois (1986) as well as Luck and Lee (1994) have applied singularity analysis to motion planning, while Burdick (1988) discusses applications to design.

1.1.2. Previous approaches to the study of singularities

In most of the existing literature, singularity analysis has been restricted to specific, narrowly defined classes of mechanisms, with an emphasis on lower-degree-of-freedom problems. Thus, the identification and avoidance of singularities has been investigated extensively for manipulators with non-redundant open-loop kinematic chains (Waldron, et al. 1985, Hunt 1986, Wang and Waldron 1987, Lipkin and Pohl 1991, and Burdick 1991). For redundant serial manipulators with one extra degree of freedom, singularities have been classified with respect to their avoidability with self-motion (Bedrossian 1990, Shamir 1990, Bedrossian and Flueckiger 1991, Flueckiger and Bedrossian 1992). Kiefer (1992, 1994) analyzed singularities of a non-redundant manipulator following a fixed end-effector path, and revealed that this problem is equivalent to the classification of the special configurations of single-loop chains (Hunt 1978, Sugimoto, et al. 1982) as well as to the aforementioned avoidability problem for redundancy-1 manipulators. Classifications of singularities and criteria for their occurrence have been developed for classes of parallel manipulators (Agrawal 1990, Kumar 1990, Merlet 1989, Shi and Fenton 1992). More recently, some authors have studied the geometry of the singularity sets of some parallel manipulators (Sefrioui and Gosselin 1994, 1995, Collins and McCarthy 1996, Mayer St-Onge and Gosselin 1996), while others have addressed the issue of continuous singular motion (or self-motion) (Husty and Zsombor-Murray 1994, Karger and Husty 1996, 1997). However, there has been no general approach to singularity analysis which would allow the study, in a single framework, of *all* singularities of an *arbitrary* mechanism.

A necessary first step in singularity analysis is the proper definition of singular configurations and the understanding of the way they affect mechanism kinematics. Existing studies, however, provide only specific and limited definitions for kinematic singularity. Although the singularity of serial manipulators has been well defined, studies of closed-loop kinematic chains do not provide corresponding explicit definitions. In these works singularity is usually said to be present when a Jacobian matrix, relevant to the specific mechanisms under investigation, is rank deficient.

For a serial manipulator, a configuration is defined as singular when the end-effector loses one or more degrees of freedom and the Jacobian becomes rank-deficient, i.e., when the input-output map $x=f(\theta)$ is singular. For parallel manipulators, the usual definition of singularity is dual to the one for serial chains: a configuration is singular when the end-effector acquires one or more additional degrees of freedom and the Jacobian of the inverse kinematics becomes rank-deficient, i.e., the inverse input-output map $\theta=f(x)$ is singular. However, this duality is incomplete since parallel manipulators can also have configurations where the end-effector has reduced degrees of freedom and it is natural to consider such configurations as singular as well. Thus, for a closed chain mechanism, singularity cannot be solely associated with the degeneracy of the derivative of an input-output map.

To surmount this obstacle, one can analyze the singular configurations of both open and closed chains using the derivative of a more general input-output relationship of the type $f(x, \theta)=0$ (Gosselin and Angeles 1990). However, as it will be shown later, in Chapter 3, this approach overlooks certain configurations in which the instantaneous motion of part of the mechanism is indeterminate, and the end-effector's degrees of freedom may be reduced.

Hunt (1978), and later Sugimoto et al. (1982), analyzed single-loop chains and defined two types of "special" configurations: "stationary" and "uncertainty" configurations. In this approach, the mechanism is not considered as an input-output device and some special configurations cannot be considered as singular from a control viewpoint.

1.1.3. Objectives of the thesis

The lack of a proper definition of kinematic singularities can lead to imperfect methods for singularity identification and incomplete classification schemes, especially when the analyzed mechanisms consist of complex, multi-loop kinematic chains. In this thesis, a general approach to singularity analysis is developed in order to achieve the following objectives:

(a) Definition of singularity. The first question that must be answered is "What is a singularity of a general input-output mechanism?" It is important to have a meaningful general definition from which specific definitions for singularity, for particular classes of mechanisms, can be obtained. The proposed approach is based on a mathematical model of mechanism kinematics formulated in terms of differentiable mappings between manifolds. This formulation allows a rigorous general mathematical definition of kinematic singularity for arbitrary mechanisms.

(b) Classification of singularities. The next goal is to reveal the structure of the singularity set. Classification of singularities serves this goal. Classification seeks to disclose what different kinds of singular configurations are possible and divide the singularity set into subclasses consisting of different singularity types. The study of the infinitesimal and local properties of the model yields a comprehensive classification of singularity, based on the type of degeneracy of the velocity kinematics.

(c) Criteria for singularity. Once the phenomenon is defined, the next task is to provide methods for answering the questions: "Is a given configuration singular? To what singularity class does it belong?" Such methods can be developed by establishing effective analytical or geometrical criteria for singularity. The generalized approach to singularity proposed in this thesis allows the development of new improved singularity tests. This is especially true in the case of hybrid-chain manipulators.

(d) Identification of singularities. It is not sufficient to be able to determine whether a specific (though arbitrary) configuration is singular. For both practical and theoretical reasons it is important to have means of obtaining the set of *all* singular configurations. The large majority of the existing methods and algorithms for finding singularities have been developed for serial manipulators. The formulation of the infinitesimal model by means of a velocity equation allows the development of methods for determining the singularities of closed-loop mechanisms.

1.1.4. Overview of results

Herein, we summarize the contents of each chapter of the thesis, and outline how the goals, described in Sub-section 1.1.3, are achieved.

Chapter 1. The introductory chapter consists of two sections. This first one, i.e., the present Section 1.1, clarifies the motivation and background of this work. In the following section (Section 1.2), we introduce the basic mathematical terminology which will be used to formulate the kinematic models in Chapters 2 and 3.

Chapter 2. The main task achieved in this chapter is the definition of kinematic singularity of a general mechanism in terms of the position kinematics. To obtain this result, we reformulate mechanism kinematics in terms of a novel mathematical model. After defining a very general class of kinematic systems as families of smooth motions on a manifold, we proceed to consider articulated systems with their two equivalent models: as motions in joint space; and in link space, respectively. The configuration space of an articulated system has the structure of an algebraic set, when considered in either of the equivalent joint and link formulations. The dimension of this set is the mobility of the articulated system. Furthermore, we propose a new conception for the definition of a mechanism. Rather than consider it as a medium for an input-output mapping, which cannot, in general, be given a proper global definition, we create a symmetrical model by introducing two subsystems of the given articulated system: the input and the output systems, and two well-defined mappings, the input and output maps, which map the configuration space into the input- and output-space manifolds, respectively. Finally, it is postulated that near a nonsingular configuration, the configuration space is a smooth manifold and the two mappings are of maximum rank. At a singularity, either the smoothness of the configuration space or the regularity of the two maps are violated.

Chapter 3. This chapter introduces a new and general framework for the classification and interpretation of singular configurations, which is obtained by the examination of the interdependence of six singularity types. The chapter starts with the interpretation of the symmetric two-map model of mechanism kinematics, developed in Chapter 2, for the study of the relationship between the joint velocities and the output twist. This amounts to modelling the instantaneous kinematics at a given configuration. Then, the definition of singularity is re-stated in terms of the instantaneous model, the six singularity types are defined, and a classification theorem is established. All these definitions and propositions are stated for arbitrary kinematic chains, but. at this stage, are restricted to non-redundant mechanisms, with equal dimensions of the input and output spaces. As shown later, in Chapter 6, the classification framework can be formulated for redundant mechanisms as well. However, in the non-redundant case the statements have a simple symmetry which becomes obscured in the more general case, when redundancy is possible. The singularity types and their interaction are illustrated by numerous examples.

Chapter 4. The approach developed in Chapter 3 is applied to the study of a class of non-redundant mechanisms, which are referred to as hybrid-chain manipulators (HCMs). For these parallel-like manipulators, we simplify the velocity kinematic equations by eliminating a maximum number of passive-joint velocities. We do that using an improved method for "annihilation" of the passive-joint screws with reciprocal screws. Unlike previous approaches based on reciprocity of screws, the technique described in Chapter 4 does not fail at singular configurations. We then proceed to develop singularity criteria for HCMs, i.e., we provide necessary and sufficient conditions for a configuration to belong to each of the six singularity types. We finish the chapter by proving a classification theorem which describes the possible singularity classes for HCMs.

Chapter 5. The issue of singularity identification is addressed, for the case of a general non-redundant mechanism. After deriving singularity criteria on the basis of the formulations and propositions in Chapter 3, we develop methods that can compute the singularity set and reveal its division into singularity classes. The application of these methods is exemplified by the detailed singularity analysis of a six-degree-of-freedom parallel manipulator with a complex singularity set.

Chapter 6. In this chapter, we revisit the formulations and derivations of Chapters 3 and 5 and demonstrate that both the classification framework and the identification procedures can be generalized and made applicable to mechanisms with redundancy. Mechanisms with kinematic and dynamic redundancy are considered. The resulting classification tables reveal the effects that redundancy has on the possible and impossible singularity classes. We introduce some modifications in the identification methods of Chapter 5 so that the singularity set of redundant mechanisms can be revealed.

Chapter 7. This final chapter summarizes the contributions and conclusions of the thesis and points out possible areas of extension.

1.2. Mathematical Preliminaries

The purpose of this section is to introduce a number of mathematical concepts that will be needed in the thesis and which do not frequently appear in the mechanisms literature. The propositions which we state are given without proof, since detailed proofs can be found in the quoted literature.

1.2.1. Groups and rings

In this sub-section, we state some fundamental definitions and facts in abstract algebra. They will be referred to in later chapters as well as in the rest of the present section.

1.1. ***Definition.*** *Let G be a set and let* $p : G \times G \to G$ *be a binary operation on G. For any pair (a, b) of elements of G, let p(a, b) be denoted by ab. G is said to be a* **group** *with* **group product** *p, if:*

(i) *p is associative, i.e.,* $(ab)c = a(bc)$ *for any a, b, and c in G.*

(ii) *p has a* **unit element**, *i.e., there exists an element e in G such that, for any* $a \in G,\ ea = ae = a$.

(iii) *p admits inverses, i.e., for every a in G there exists an* **inverse element**, *denoted* a^{-1}, *such that* $aa^{-1} = a^{-1}a = e$.

It can be shown that the unit element, e, must be unique. If the group product is commutative, i.e., if $ab = ba$ for any a and b, then $p(a, b)$ is usually denoted $a + b$ and referred to as a sum, while G is called an *additive group*. For such groups, the unit element is denoted by 0, while the inverse element is written as $-a$.

1.2. ***Definition.*** *A* **group map** $f : G \to H$ *is a map between groups G and H that respects the products of G and H, that is, is such that, for all* $a, b \in G$,

$$f(ab) = f(a)f(b).$$

A group map $f : G \to H$ *is said to be a (group)* **isomorphism**, *if it is bijective.*

It can be shown that the inverse of a group map must also be a group map.

1.3. ***Definition and Proposition.*** *Let G be a group and let* $F \subset G$. *Then, F is a group with the group product of G if, and only if, the following two conditions are satisfied:*

(i) $FF \subset F$, *i.e.,* $a, b \in F$ *implies* $ab \in F$,

(ii) $F^{-1} \subset F$, *i.e.,* $a \in F$ *implies* $a^{-1} \in F$.

A subset, F, with these properties is said to be a **subgroup** *of G.*

1.4. ***Definition.*** *A* **ring** *is an additive group, R, in which, apart from the sum, a + b, there is a product, ab, which is distributive over addition, i.e., for all a,b,c in R,*

$$c(a + b) = ca + cb \quad and \quad (a + b)c = ac + bc.$$

R is said to be **commutative**, *if the product is commutative, and/or* **with unity**, *if the product has unity.*

Examples of rings are: the set of integers, $\mathbf{Z}$; the ring of remainders modulo r, $\mathbf{Z}_r$; the ring of polynomials in n variables with coefficients in a field k, $k[x_1, \ldots, x_n]$. An important case, to which we will return later in this section, is the ring of polynomials over the reals, $\mathbf{R}[x_1, \ldots, x_n]$.

1.5. ***Definition.*** *Let R be a commutative ring with unity and let $I \subset R$.*

(1) *A subset, I, is referred to as an* **ideal** *of R, if it has the two properties:*

- *(i)* *for any $a, b \in I$, we have $a - b \in I$,*
- *(ii)* *$IR \subset R$, i.e., for any $a \in I, x \in R$, it is true that $ax \in I$.*

(2) *The ideal I is said to be (***finitely***)* **generated** *by the elements $a_1, \ldots, a_n \in I$, if every element, $b \in I$, can be written as $b = b_1a_1 + \cdots + b_na_n$ for some $b_1, \ldots, b_n$ in R. We denote: $I = (a_1, \ldots, a_n)$.*

Every ring contains at least two ideals: the *zero ideal*, which consists solely of the zero in R; and the *unit ideal*, which is R itself. Every ideal must contain 0 (the zero of R). The only ideal that contains 1 (the unity of R) is the unit ideal.

1.6. ***Proposition.*** *Every ideal in $k[x_1, \ldots, x_n]$ is finitely generated.*

A proof of this fact is given in (Hodge and Pedoe, 1954).

1.7. ***Definition.*** *Let I be an ideal in the ring R.*

(1) *I is said to be* **prime**, *if $ab \in I$ implies $a \in I$ or $b \in I$.*

(2) *I is said to be* **primary**, *if $ab \in I$ and $a \notin I$ implies $b^k \in I$, for some integer k.*

1.2.2. Affine spaces

From a geometrical or a mechanical point of view, it is important to distinguish the affine Euclidian space E^3 from the underlying 3-dimensional linear space $\mathbf{R}^3$. An affine space is a linear space without a fixed origin. The elements of E^3 are points, while the elements of $\mathbf{R}^3$ are vectors. While E^3 is a bijective image of $\mathbf{R}^3$, it is not a vector space, and the elements of an affine space are not subject to the linear operations (vector addition and scalar multiplication).

An affine space is rigorously defined as follows, (Porteous 1981, Arnold 1979):

1.8. ***Definition.*** *Let X be a non-empty set and V a vector space. An* **affine structure** *on X with vector space V is a map*

$$c : X \times X \rightarrow V, \; c(\boldsymbol{x}, \mathbf{v}) = \boldsymbol{x} - \mathbf{v},$$

which satisfies two axioms:

(i) *for all* $\boldsymbol{o}$ *in X the map* $c_o : X \rightarrow V, \; c_o(\boldsymbol{x}) = \boldsymbol{x} - \boldsymbol{o}$, *is bijective,*

(ii) *for all* $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{o}$ *in X,* $\boldsymbol{x} - \boldsymbol{y} = (\boldsymbol{x} - \boldsymbol{o}) - (\boldsymbol{y} - \boldsymbol{o})$.

The set X, equipped with the affine structure is an **affine space**.

E^3 will denote a three-dimensional affine space over the linear space $\mathbf{R}^3$.

The inverse of the map c_o is denoted by $\boldsymbol{x} = \boldsymbol{o} + \mathbf{x}$, where $\mathbf{x} = \boldsymbol{x} - \boldsymbol{o}$. This defines the sum of a point, $\boldsymbol{x}$, and a vector, $\mathbf{v}$. The result, $\boldsymbol{x} + \mathbf{v}$, is the unique point $\boldsymbol{y}$ in E^3, such that $\mathbf{v} = \boldsymbol{y} - \boldsymbol{x}$.

By the use of reference frames (coordinate systems) the elements of both E^3 and $\mathbf{R}^3$ can be described by triples of real numbers. A coordinate system in E^3 is given by a point, $\boldsymbol{o}$, in E^3 and a basis, $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ in $\mathbf{R}^3$. For a fixed choice of the reference frame, $\boldsymbol{o}\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$, every point, $\boldsymbol{x}$, in E^3 is described by the coordinates of the vector $\mathbf{x} = \boldsymbol{x} - \boldsymbol{o}$ in the basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$.

E^3 is referred to as an affine *Euclidean space* when its vector space, $\mathbf{R}^3$, is a real orthogonal space with a positive-definite scalar product. The scalar product in $\mathbf{R}^3$ defines a *distance* function, d, on E^3.

1.9. Definition.

(1) *The* **distance** *between two points in E^3 is the function:*

$$d\colon E^3 \times E^3 \to \mathbf{R},\ d(x, y) = \sqrt{(x-y).(x-y)},$$

where "." is the standard scalar product in $\mathbf{R}^3$.

(2) *The* **orientation** *function, o, is defined on frames* $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ *in* $\mathbf{R}^3$ *(or E^3), or, equivalently, on arrays of four points in E^3,* (o, x, y, z)*:*

$$\sigma(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \begin{cases} 1 & \textit{if} \quad \det(\mathbf{x}, \mathbf{y}, \mathbf{z}) > 0 \\ 0 & \textit{if} \quad \det(\mathbf{x}, \mathbf{y}, \mathbf{z}) = 0 \\ -1 & \textit{if} \quad \det(\mathbf{x}, \mathbf{y}, \mathbf{z}) < 0 \end{cases}$$

The so-defined distance and orientation functions are, in general, dependant on the choice of a reference frame. This is so, since the scalar product and the determinant function are not invariant with respect to an arbitrary change of basis in $\mathbf{R}^3$. To ensure that both distance and orientation are invariant with respect to reference frame it is sufficient to restrict the choice of bases in $\mathbf{R}^3$ to only such triples $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ for which $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$ and $\det(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) = 1$ (δ_{ij} is the Kronecker symbol). This assumption restricts the allowable coordinate systems in E^3. While the origin can be arbitrary, the coordinate vectors must satisfy the above conditions. Such reference frames are referred to as Cartesian.

1.2.3. Topological spaces and smooth manifolds

1.10. Definition. *A* **topology** *on a set X is a collection, $\mathcal{T}$, of subsets of X, which includes the empty set, ∅, and the whole set, X, and which is such that:*

- *(i)* *the intersection of any finite number of elements of $\mathcal{T}$ belongs to $\mathcal{T}$;*
- *(ii)* *the union of any set of elements of $\mathcal{T}$ belongs to $\mathcal{T}$.*

A set X together with a fixed topology $\mathcal{T}$ is called a **topological space**. *The elements of $\mathcal{T}$ are referred to as the* **open sets** *in X. Subsets of X whose complements are open are said to be* **closed**.

A subset, Y, of a topological space, X, is a *topological subspace* of X, if it is endowed with the induced topology $\mathcal{T}_s = \{U \cap Y \mid U \in \mathcal{T}\}$, where $\mathcal{T}$ is the topology of X.

An *open neighbourhood* of a point $x \in X$ is an open subset of X containing x. A *neighbourhood* is a subset of X with an open neighbourhood of x as a subset. In *Hausdorf spaces*, every two distinct points have non-intersecting neighbourhoods.

Thus, topological spaces generalize familiar concepts about Euclidian affine spaces, such as open and closed sets, and nearness to a point. The space E^n, is itself a topological set with the usual open sets, which can be defined as: the balls with radii $\varepsilon > 0$ (i.e., $B_\varepsilon(x) = \{y \mid d(x, y) < \varepsilon\}$) and all their possible unions.

1.11. Definition. *A map, $f : X \to Y$, between topological spaces is said to be* **continuous**, *if the inverse image of any open set in Y is an open set in X. A bijective continuous map with a continuous inverse map is referred to as a* ***homeomorphism***.

Topological spaces connected with a homeomorphism are considered equivalent from a topological point of view.

1.12 Definition. *An m-dimensional* **smooth manifold** *is a Hausdorf space provided with a family of pairs $\{(U_i, \phi_i)\}$, such that,*

(i) *$\{U_i\}$ is a family of open sets which covers X, i.e., $X = \cup_i U_i$, while ϕ_i is a homeomorphism from U_i onto an open subset of $\mathbf{R}^m$.*

(ii) *For any $U_i \cap U_j \neq \varnothing$, the map: $\phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j)$, is infinitely differentiable.*

Each pair (U_i, ϕ_i) is referred to as a *chart* of X, while the whole family, $\{(U_i, \phi_i)\}$, is called an *atlas*. The subset U_i is referred to as a *coordinate neighbourhood* and the maps ϕ_i are the *coordinate maps*. Two atlases are said to be *equivalent* or to define the same *smooth structure*, if their union is also an atlas, i.e., it satisfies condition (ii) in Definition 1.12. Manifolds with the same underlying set, X, and equivalent atlases are considered equivalent.

1.13. Definition. *Let $f : X \rightarrow Y$ be a map between smooth manifolds, and let $y = f(x)$. The map f is said to be* **smooth** *at the point x, if for some pair of charts, (U, ϕ) and (V, ψ) with $x \in U$ and $y \in V$, the map $\psi \circ f \circ \phi^{-1}$ is smooth. If a smooth map f is invertible and the inverse map, f^{-1}, is also smooth, then f is referred to as a* **diffeomorphism** *and the manifolds X and Y are said to be* **diffeomorphic**.

A diffeomorphism is a homeomorphism which preserves the smooth structure. Diffeomorphic manifolds are considered geometrically equivalent.

A *submanifold*, Y, of a manifold X, is a subset of X, which is a manifold.

A *curve* at $x \in X$ is a smooth map $g : I \rightarrow X$, where I is an open interval of the real line, $0 \in I \subset \mathbf{R}$, such that $g(0) = x$. Two curves at x, g and h, are said to be *tangent* if, for some chart, (U, ϕ) with $x \in U$, we have:

$$\left.\frac{dx^i(g(t))}{dt}\right|_{t=0} = \left.\frac{dx^i(h(t))}{dt}\right|_{t=0}, \quad i = 1, \ldots, m ,$$

where x^i are the coordinate functions of the map ϕ. This is, in fact, an equivalence relation for the space of curves. An equivalence class of curves at x under this relation is referred to as a *tangent vector* at x. The set of all tangent vectors at x form the *tangent space* at x and is denoted by $T_x X$. It can be seen that $\dim T_x X = \dim X$. The dual space of $T_x X$ is referred to as the *cotangent space* and is denoted by $T_x^* X$.

A smooth map between two manifolds, $f : X \rightarrow Y$, induces a linear mapping, $D_x f : T_x X \rightarrow T_{f(x)} Y$, between the two tangent spaces at $x \in X$ and $f(x) \in Y$, defined by: $D_x f([g]) = [f \circ g]$, where $[h]$ denotes the equivalence class (i.e., the tangent vector) of the curve h.

The above facts and further details on smooth and differentiable manifolds can be found in (Lang 1962) or (Sternberg 1964).

1.14. Definition. A **Lie group** *G is a smooth manifold which is endowed with a group structure, such that the following two maps, defined by the group operations, are smooth:*

(i) $p : G \times G \to G, \quad p(a, b) = ab$,

(ii) $q : G \to G, \quad q(a) = a^{-1}$.

A *Lie subgroup* of a Lie group, G, is a subgroup of G, which is also a submanifold of G.

1.2.4. Real algebraic sets

1.2.4.1. Basic definitions. We consider the ring of polynomials $\mathbf{R}[x_1, \ldots, x_N]$. We will abbreviate $\mathbf{x} = (x_1, \ldots, x_N)$. The array of n indeterminates, $\mathbf{x}$, will be interpreted as the coordinates of a point, x, of the affine Euclidian space, E^N, in some fixed Cartesian reference frame. If $J \subset \mathbf{R}[\mathbf{x}]$ is a set of polynomials, then we denote the *vanishing set* of J as:

$$\mathcal{V}(J) = \{\ x \in E^N \mid f(\mathbf{x}) = 0 \text{ for all } f \bullet \mathbf{R}[\mathbf{x}]\ \}.$$

On the other hand, for every subset $V \subset E^N$, we define the *ideal of V*:

$$\mathcal{J}(V) = \{f \in \mathbf{R}[\mathbf{x}] \mid f(\mathbf{x}) = 0 \text{ for all } x \in V\ \}.$$

*1.15. Definition. A subset, $V \subset E^N$, is referred to as a (***real***)* **algebraic set**, *if it is the vanishing set of a collection of polynomials, i.e., $V = \mathcal{V}(J)$ for some $J \subset$* $\mathbf{R}[\mathbf{x}]$.

It can be assumed that the set J from Definition 1.15 is an ideal. (Indeed, if a collection of polynomials vanishes on V, then the ideal generated by them vanishes on V as well).

1.16. Definition. *An algebraic set, V, is* **reducible** *if it is the union of two other algebraic sets, i.e., there exist two algebraic sets, V_1 and V_2, $V_1 \neq V \neq V_2$, such that $V = V_1 \cup V_2$. Otherwise, the set V is said to be* **irreducible**.

1.17. Proposition. *Every algebraic set, V, has a unique decomposition into* **irreducible components**, *that is, V can be written uniquely as $V = V_1 \cup \cdots \cup V_m$, where each V_i is an irreducible algebraic set and no V_i is contained in another V_j.*

1.18. Proposition. *Let J be an ideal in $\mathbf{R}[\mathbf{x}]$, and let $V = \mathcal{V}(J)$.*

(1) *V is irreducible, if and only if $\mathcal{J}(V)$ is a prime ideal.*

(2) *If V is irreducible, then J is a primary ideal.*

(3) *If J is a prime ideal then V is irreducible.*

Proofs of Propositions 1.17 and 1.18 can be found in (Shafarevich 1977 and Hodge and Pedoe 1954).

1.19. Definition. *A mapping, $f: U \to W$ is an (*__entire__*)* **rational map**, *if there exist polynomials p and q, such that $0 \notin q(U)$ (i.e., $q(x) \neq 0$ for all $x \in U$) and $f(x) = p(x)/q(x)$. A rational map, f, is said to be a* **birational isomorphism**, *if it has a rational inverse, i.e., if there exists a rational map, $g : W \to U$, such that $f \circ g$ and $g \circ f$ are identities.*

Two algebraic sets are said two be *birationally isomorphic*, if they are connected with a birational isomorphism. Birational isomorphism is a homeomorphism which preserves all algebraic structures.

1.2.4.2. Singularities of real algebraic sets. There is no general agreement in the literature on the proper way to define singular and nonsingular points on algebraic sets. The following approach, proposed in (Akbulut and King, 1992) appears to be best suited to applications in mechanism theory, since it emphasizes the local geometric and topological properties of the algebraic set.

1.20. Definition. *Let $V \subset E^N$ be an algebraic set. We say that $x \in V$ is* **nonsingular of dimension** *d* **in** *V, if there exists a neighbourhood, U, of x in E^N, and $N-d$ polynomials $f_1, \ldots, f_{N-d}$ such that:*

(i) $U \cap V = U \cap \mathcal{V}(f_1, \ldots, f_{N-d})$,

(ii) *The gradients, $\nabla f_i(x)$, $i = 1, \ldots, N-d$ are linearly independent.*

1.21. Definition. *Let V be an algebraic set. The* **dimension of** *V,* dim *V, is defined to be the largest integer, d, such that there exists a point in V, which is nonsingular of dimension d. The set of all nonsingular points of dimension $d = \dim V$ is denoted by* Nsg *V and its elements are referred to as the* **nonsingular points** *of V. The complement of* Nsg *V, denoted by* $\operatorname{Sing} V = V - \operatorname{Nsg} V$, *is the set of the* **singular points** *of V.*

1.22. Definition. *Let V be a real algebraic set and let $\mathcal{J}(V) = (f_1, \ldots, f_k)$. Let x be in V. Then, the* **Zariski-tangent space at** x, $T_x^{\mathrm{Zar}}(V)$, *is a vector space defined by*

$$T_x^{\mathrm{Zar}}(V) = \{\mathbf{v} \mid \nabla f_i(x) \cdot \mathbf{v} = 0,\ i = 1, \ldots, k\}.$$

We note that Definition 1.22 does not depend on the choice of basis in $\mathcal{J}(V)$.

1.23. Notations. We denote:

$$n_x(V) = \dim T_x^{\mathrm{Zar}}(V),$$

$$n(V) = \min\{n_x(V) \mid x \in V\},$$

$$V^0 = \{x \in V \mid n_x(V) = n(V)\},$$

$$V^- = V - V^0.$$

An alternative to Definition 1.21 is the use of V^0, V^-, as definitions of nonsingular and singular points, respectively (Bochnak et al., 1987).

1.24. Proposition. *Let $V \subset E^N$ be an irreducible algebraic set.*

(1) Nsg V *is not empty.*

(2) $\dim V = n(V)$.

(3) $\text{Nsg } V = V^0$.

(4) *If $W \subset V$ is an irreducible algebraic set and $W \neq V$, then* $\dim W < \dim V$.

1.25. Proposition. *Let $V \subset E^N$ be an algebraic set, and let $V = V_1 \cup \cdots \cup V_m$, where V_i, $i = 1, \ldots, m$, are the irreducible components of V.*

(1) Nsg V *is not empty.*

(2) *$x \in V$ is nonsingular of dimension d if and only if: for some j, $x \in$ Nsg V_j; $x \notin V_i$, for all $i \neq j$; and* $\dim V_j = e$.

(3) $$\text{Nsg } V = \bigcup_{\dim V_i = \dim V} (\text{Nsg } V_i - \bigcup_{i \neq j} V_j).$$

(4) Sing V *is is an algebraic set and* $\dim(\text{Sing } V) < \dim V$.

All statements in Propositions 1.24 and 1.25 are proved in (Akbulut and King 1992).

CHAPTER 2

MECHANISM KINEMATICS

2.1. Introduction

This chapter introduces a novel mathematical model of mechanism kinematics. The model evolves from a natural definition of mechanical systems, and kinematic systems in particular, as families of smooth mappings of the unit interval, Δ, into a smooth manifold, Q. This basic idea allows for a rigorous and consistent derivation of the central concepts and facts of mechanism theory, including the precise mathematical definition of fundamental notions like configuration space, kinematic model, mobility, redundancy and singularity.

The chapter begins with a brief discussion of the properties of the group of displacements of the affine Euclidian space, $SE(3)$, (Section 2.2) which are derived from the properties of affine spaces, reviewed in Chapter 1. In the subsequent sections, the basic elements and facts of multi-body kinematics are derived from the properties of the Euclidian group. All formulations are coordinate-free and underline the intrinsic geometric and topological nature of kinematic systems.

Kinematic systems and their kinematic models are introduced in most general terms in Section 2.3. Then, we consider as narrowing subsets of abstract kinematic systems, consecutively: systems of particles in Section 2.4; rigid-body systems in Section 2.5; articulated systems in Section 2.6; and serial-chain articulated systems in Section 2.7. The derivations in Sections 2.8 and 2.9 are, for the most part, valid for general abstract kinematic systems, but the natural emphasis is on articulated systems with smooth joints (e.g., lower pairs). The goal, achieved in Section 2.9, is to provide rigorous formulations of mechanism mobility, singularity and redundancy, based on the local and global topological properties of a kinematic system.

2.2. The Euclidian Group

It is assumed that a three-dimensional affine Euclidian space†, E^3, is given.

2.1. ***Definition.*** *A* **displacement** *of E^3 is a transformation of E^3 (i.e., a map $g: E^3 \to E^3$), which preserves the distance and the orientation in E^3. The set of all displacements in three-dimensional space is denoted by $SE(3)$ and is known as the* **Euclidian group** *in three dimensions.*

It can be shown that a displacement is a bijective affine map and an automorphism (i.e., a homeomorphism of E^3 onto itself). The set $SE(3)$ is, indeed, a group. The group product of two displacements is defined as their composition as maps, $g_1 g_2(\boldsymbol{x}) = g_1 \circ g_2(\boldsymbol{x}) = g_1(g_2(\boldsymbol{x}))$. The unit element of the group is the identity map on E^3, $e = id_{E^3}$. The inverse element of g is given by the inverse map, g^{-1}.

2.2. ***Example.*** A simple example of a displacement is the *translation* map, $g_t(\boldsymbol{x}) = \boldsymbol{x} + \mathbf{t}$, where $\mathbf{t}$ is a constant vector in $\mathbf{R}^3$. It maps a point $\boldsymbol{x}$ into the unique point $\boldsymbol{y}$ in E^3 such that $\boldsymbol{y} - \boldsymbol{x} = \mathbf{t}$. Since a translation is defined uniquely by a vector $\mathbf{t}$, the set of all translations, $\mathrm{Tr}(E^3)$, can be identified with $\mathbf{R}^3$ by means of the bijective map

† All statements in this chapter can be made for a Euclidian space of arbitrary dimension. Of practical importance are mainly the cases $n = 2$ and $n = 3$. The theory for the plane ($n = 2$) can be derived from the spatial case ($n = 3$), since planar displacements are a subgroup of spatial displacements.

$$\tau : \mathrm{Tr}(E^3) \to \mathbf{R}^3,\ \tau(g_t) = \mathbf{t}. \tag{2.1}$$

The addition of vectors in $\mathbf{R}^3$ turns the space of translations into an additive group. The unit element of this group is the translation by the zero vector, $\mathbf{o}$. In fact, τ is a group isomorphism between $\mathrm{Tr}(E^3)$ and $\mathbf{R}^3$. It can be seen that the group operation and the unit element in $\mathrm{Tr}(E^3)$ are identical with those in $SE(3)$, and therefore the group of translations is a sub-group of $SE(3)$.

2.3. Example. A displacement which maps at least one point, $o \in E^3$, into itself is called a *rotation*. The set of all rotations about an arbitrary fixed point o, $\mathrm{Rot}_o(E^3)$, is a subgroup of $SE(3)$. The group $\mathrm{Rot}_o(E^3)$ can be identified with $SO(3)$, the Special Orthogonal Group. $SO(3)$ consists of the so-called orthogonal linear maps in $\mathbf{R}^3$, i.e., the maps which preserve the scalar product and the determinant function. (When a Cartesian basis is fixed in $\mathbf{R}^3$, each element of $SO(3)$ is given by an orthogonal matrix with a positive determinant).

The isomorphism between $\mathrm{Rot}_o(E^3)$ and $SO(3)$ is given by the map

$$\rho\colon \mathrm{Rot}_o(E^3) \to SO(3),\ \rho(g) = \boldsymbol{G}, \tag{2.2}$$

where $\boldsymbol{G} : \mathbf{R}^3 \to \mathbf{R}^3$ is defined by

$$\boldsymbol{G}\mathbf{x} = g(o + \mathbf{x}) - o. \tag{2.3}$$

The inverse map is $\rho^{-1}(\boldsymbol{F}) = f$, where f is given by $f(x) = o + \boldsymbol{F}(x - o)$.

To show that ρ is indeed an isomorphism, consider two rotations about o, f and g, and denote $fg = h$, $\rho(f) = \boldsymbol{F}$, $\rho(g) = \boldsymbol{G}$, $\rho(h) = \boldsymbol{H}$. We need to show that $\rho(fg) = \rho(f)\rho(g)$, i.e., $\boldsymbol{H} = \boldsymbol{F}\boldsymbol{G}$. From the definition of ρ (Equations 2.2 and 2.3), we have:

$$\begin{aligned} \boldsymbol{F}\boldsymbol{G}\mathbf{x} &= \boldsymbol{F}(g(o + \mathbf{x}) - o) = f(o + (g(o + \mathbf{x}) - o)) - o = \\ &f(g(o + \mathbf{x})) - o = fg(o + \mathbf{x}) - o = h(o + \mathbf{x}) - o = \boldsymbol{H}\mathbf{x}. \end{aligned} \tag{2.4}$$

The third equality in (2.4) follows from the definition of the operations "+" and "–" in the space E^3 (discussed in Sub-section 1.2.2).

Consider the set $\mathbf{R}^3 \times SO(3)$, which has as its elements the pairs of the type $(\mathbf{v}_f, \boldsymbol{F})$. We define a product operation by

$$(\mathbf{v}_f, \boldsymbol{F})(\mathbf{v}_g, \boldsymbol{G}) = (\boldsymbol{F}\mathbf{v}_g + \mathbf{v}_f, \boldsymbol{F}\boldsymbol{G}). \tag{2.5}$$

It can be shown that with this product $\mathbf{R}^3 \times SO(3)$ is a group with unit element $(\mathbf{o}, \boldsymbol{I})$, where $\boldsymbol{I}$ is the 3×3 unit linear map. This group will be denoted by $\mathbf{R}^3 \times_s SO(3)$. (Note: the symbol $G \times H$, where G and H are groups, denotes a group with a product operation different from the one in Equation (2.5), namely, $(g_1, h_1)(g_2, h_2) = (g_1 g_2, h_1 h_2)$.)

2.4. Theorem. *(Arnold, 1980). The Euclidian group of all displacements in E^3, $SE(3)$, is isomorphic to the group $\mathbf{R}^3 \times_s SO(3)$.*

Proof. We will construct a map, $\phi : SE(3) \rightarrow \mathbf{R}^3 \times_s SO(3)$, and show that it is a group isomorphism.

First, for every dispacement, f, and an arbitrary fixed point, $\boldsymbol{o}$, we define, a translation, f_t, and a rotation, f_r, as:

$$\begin{aligned} f_t(\boldsymbol{x}) &= \boldsymbol{x} + (f(\boldsymbol{o}) - \boldsymbol{o}), \\ f_r(\boldsymbol{x}) &= f(\boldsymbol{x}) + (\boldsymbol{o} - f(\boldsymbol{o})). \end{aligned} \tag{2.6}$$

The second equation in (2.6) defines a rotation, since $f_r(\boldsymbol{o}) = \boldsymbol{o}$.

We now define the map ϕ by:

$$\phi(f) = (\tau(f_t), \rho(f_r)). \tag{2.7}$$

The maps τ and ρ are the ones defined in Examples 2.1 and 2.2 (Equation (2.1) and Equations (2.2–3)), respectively. We denote

$$\mathbf{t}_f = \tau(f_t) = f(\boldsymbol{o}) - \boldsymbol{o}, \quad \boldsymbol{F} = \rho(f_r). \tag{2.8}$$

We also consider the map $\psi : \mathbf{R}^3 \times SO(3) \rightarrow SE(3)$, defined as

$$\psi(\mathbf{t}_f, \boldsymbol{F}) = \tau^{-1}(\mathbf{t}_f)\rho^{-1}(\boldsymbol{F}). \tag{2.9}$$

The image of ψ is a displacement, f, which is a composition of the translation, $f_t = \tau^{-1}(\mathbf{t}_f)$, given by the vector $\mathbf{t}_f$, and the rotation, $f_r = \rho^{-1}(\boldsymbol{F})$, which corresponds to the orthogonal linear map $\boldsymbol{F}$.

It can be checked that $\phi \circ \psi(\mathbf{t}_f, \boldsymbol{F}) = (\mathbf{t}_f, \boldsymbol{F})$ and $\psi \circ \phi(f) = f$, i.e., $\psi = \phi^{-1}$. Thus, it is established that ϕ is bijective. This implies that every displacement, f, can be written as the product of a translation, f_t, and a rotation, f_r, which are obtained from f as shown in Equations (2.6).

It remains to show that the map ϕ preserves the group product, i.e., for any f, g in $SE(3)$, $\phi(fg) = \phi(f)\phi(g)$.

We denote $h = fg$ and $(\mathbf{t}_h, \boldsymbol{H}) = \phi(fg)$. By the definition of the group product in $\mathbf{R}^3 \times_s SO(3)$ (Equation 2.5), we have: $\phi(f)\phi(g) = (\mathbf{t}_f, \boldsymbol{F})(\mathbf{t}_g, \boldsymbol{G}) = (\boldsymbol{F}\mathbf{t}_g + \mathbf{t}_f, \boldsymbol{FG})$. It must be proven that: (i) $\mathbf{t}_h = \boldsymbol{F}\mathbf{t}_g + \mathbf{t}_f$, and (ii) $\boldsymbol{H} = \boldsymbol{FG}$:

(i) The definition of the translation vector, $\mathbf{t}_f$, and the orthogonal map of a displacement f (Equation 2.8) and the fact that $f = f_t f_r$ for every f, allow us to write the following sequence of equalities:

$$\begin{aligned}
\mathbf{t}_h &= h(o) - o = fg(o) - o = fg_t g_r(o) - o = fg_t(o) - o = \\
& f_t f_r g_t(o) - o = (f_r g_t(o) + \mathbf{t}_f) - o = (f_r g_t(o) - o) + \mathbf{t}_f = \\
& ((o + \boldsymbol{F}(g_t(o) - o)) - o) + \mathbf{t}_f = ((o + \boldsymbol{F}\mathbf{t}_g) - o) + \mathbf{t}_f = \boldsymbol{F}\mathbf{t}_g + \mathbf{t}_f.
\end{aligned} \tag{2.10}$$

In (2.10), we also use the definitions of f_t and f_r in Equations (2.6).

(ii) Similarly to the proof of (i) above, we write:

$$\begin{aligned}
\boldsymbol{H}\mathbf{x} &= (h_r(o + \mathbf{x}) - o = (h(o + \mathbf{x}) + (-\mathbf{t}_h)) - o = \\
& (fg(o + \mathbf{x}) + (-\mathbf{t}_h)) - o = (f(g_t g_r(o + \mathbf{x})) + (-\mathbf{t}_h)) - o = \\
& (f(o + \boldsymbol{G}\mathbf{x} + \mathbf{t}_g) - (-\mathbf{t}_h)) - o = \\
& ((f_r(o + \boldsymbol{G}\mathbf{x} + \mathbf{t}_g) + \mathbf{t}_f) - (-\mathbf{t}_h)) - o = \\
& ((o + \boldsymbol{F}(\boldsymbol{G}\mathbf{x} + \mathbf{t}_g)) + (\mathbf{t}_f - \mathbf{t}_h)) - o = \\
& ((o + \boldsymbol{FG}\mathbf{x} + \boldsymbol{F}\mathbf{t}_g) + (\mathbf{t}_f - \boldsymbol{F}\mathbf{t}_g - \mathbf{t}_f)) - o = (o + \boldsymbol{FG}\mathbf{x}) - o = \boldsymbol{FG}\mathbf{x}.
\end{aligned} \tag{2.11}$$

In the third-last equality in (2.11) we use the result of (i) (Eq. 2.10).

Thus, by proving (i) and (ii), it is established that ϕ is an isomorphism and the theorem is proven. □

2.5. Remarks.

(1) Theorem 2.4 implies that for a fixed choice of the origin, $\boldsymbol{o}$, every displacement, g, of E^3 can be achieved in a unique way as a composition of a translation, g_t, and a rotation, g_r, $g = g_t g_r$. Moreover, every g is described by a pair $(\mathbf{t}_g, G)$, where G is an orthogonal linear map and $\mathbf{t}_g$ is a vector. The image of each point $\boldsymbol{x}$ with coordinates $\mathbf{x} = \boldsymbol{x} - \boldsymbol{o}$ is the point $g(\boldsymbol{x})$ with coordinates $G\mathbf{x} + \mathbf{t}$. Furthermore, it follows that each displacement is uniquely defined by a Cartesian coordinate system, $\boldsymbol{o}_g\mathbf{e}_{g1}\mathbf{e}_{g2}\mathbf{e}_{g3}$, which is the image of the initial reference frame, $\boldsymbol{o}\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$. The new origin, $\boldsymbol{o}_g$, is the image of $\boldsymbol{o}$ under g, $\boldsymbol{o}_g = g(\boldsymbol{o}) = \boldsymbol{o} + \mathbf{t}_g$, while the new coordinate vectors, $\mathbf{e}_{g1}$, $\mathbf{e}_{g2}$ and $\mathbf{e}_{g3}$, are the images of $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ under the orthogonal map G, $\mathbf{e}_{gi} = G\mathbf{e}_i$.

(2) The elements of $SE(3)$ are conveniently described by homogeneous 4×4 matrices of the type:

$$H_g = \begin{bmatrix} G & \mathbf{t}_g \\ 0 & 1 \end{bmatrix}.$$

The image of a point in E^3 under a displacement, g, with matrix H_g is obtained by pre-multiplying the column vector of the homogeneous coordinates of the point $\boldsymbol{x}$, $(\mathbf{x}, 1)^{\mathsf{T}}$, by the matrix: $H_g\begin{bmatrix} \mathbf{x} \\ 1 \end{bmatrix}$. The composition of two displacements is given by the matrix product.

2.6. Remark. Both the sets $\mathbf{R}^3$ and $SO(3)$ are at the same time three-dimensional smooth manifolds and groups under vector addition and matrix multiplication, respectively. The group $\mathbf{R}^3 \times_s SO(3)$ is, therefore, a smooth manifold of dimension six. The group operations in $SE(3)$ are smooth, since they are given by matrix multiplication of 4×4 matrices. Therefore, $SE(3)$ is a 6-dimensional Lie group which is a subgroup of $GL(4)$ (The group of nonsingular 4×4 matrices). $\mathrm{Tr}(E^3)$ and $\mathrm{Rot}_{\boldsymbol{o}}(E^3)$ are 3-dimensional Lie subgroups of $SE(3)$.

2.3. Kinematic Systems

Consider the space E^{3n}, $E^{3n} = E^3 \times \cdots \times E^3$, i.e., the Cartesian product of n copies of the Euclidian space E^3. A point of E^{3n}, $x = (x_1, \ldots, x_n)$, can be thought of as describing the positions of n particles in E^3. In this interpretation, E^{3n} is referred to as the *configuration space of an unconstrained system of n particles*. The points of E^{3n} are called *(feasible) configurations* of the (unconstrained) system of n particles. (Note that in this system all configurations are feasible, even those where different particles occupy the same point in E^3.) A *(feasible) motion* of the system is defined as a smooth path in E^{3n}. The set of all such paths forms the *space of (feasible) motions* of the system.

The concept of an (unconstrained) particle system, outlined in the preceding paragraph, can be generalized by replacing the affine space, E^{3n}, by an arbitrary smooth manifold. On the other hand, one can define different mechanical systems on a single manifold by imposing restrictions on the motions which are considered feasible. In the statements which follow, we introduce a mathematical formalism for the description of mechanical systems. Our approach is based on the understanding that a mechanical system is, mathematically, nothing more than its space of feasible motions.

Let Q be a smooth manifold and let Δ be the interval [0, 1] of the real line. A *(smooth) motion on Q* is understood to be a smooth map, $f : \Delta \to Q$. More precisely, $f(t)$ is a continuous mapping of the unit interval of the real line into the manifold Q, such that $f(t)$ is smooth on the interior of Δ. The set of all such smooth maps will be denoted by $C^\infty(\Delta, Q)$.

*2.7. **Definitions.** Let Q be a smooth manifold and let $C \subset Q$.*

(1) An **(abstract) mechanical system on** *Q is a subset, $\mathcal{M}$, of the space $C^\infty(\Delta, Q)$. The elements of $\mathcal{M}$ are referred to as* **feasible motions** *of the mechanical system.*

(2) *An* **(abstract) kinematic system on** Q **with configuration space** C, *is the set:*

$$\mathcal{K}(C, Q) = \{f \in C^\infty(\Delta, Q) \mid \operatorname{Im} f \subset C\},$$

i.e., the set of all smooth motions on Q, *which are contained in* C. *When* C *is a proper subset of* Q, *the system is said to be* **constrained**. *When* $C = Q$, *the system is* **unconstrained**. *(*$\mathcal{K}(Q, Q) = C^\infty(\Delta, Q)$ *is abreviated* $\mathcal{K}(Q)$.*) The points in* C *are the* **feasible configurations** *of* $\mathcal{K}(C, Q)$, *while the elements of* $Q - C$ *are referred to as* **non-feasible configurations** *of the system.*

(3) *An abstract mechanical system on* E^{3n} *is referred to as a* **mechanical system of** n **particles** *in* E^3.

(4) *An abstract kinematic system on* E^{3n} *with configuration space* C, *i.e., the system*

$$\mathcal{K}(C, E^{3n}) = \{f \notin C^\infty(\Delta, E^{3n}) \mid f(\Delta) \subset C\},$$

is said to be a **kinematic system of** n **particles with configuration space** C.

2.8. Remark.

(1) We emphasize that in Definition 2.7(2) the system $\mathcal{K}(C, Q)$ is *not* defined as the space of mappings $C^\infty(\Delta, C)$. Such a definition would require the configuration space C to be a manifold and would be too restrictive. Instead, we define a kinematic system with a configuration space C with the help of a manifold Q containing C. This ensures that the feasible motions are well defined as smooth mappings into a manifold, even though the set C may not have any global differential structure.

(2) Analogously to Definitions 2.7.(3–4), we can define a system of particles in E^k, for any integer k. Among the cases with $k \neq 3$ of particular importance are systems in E^{2n}, referred to as *planar* (particle) systems, which will be used in numerous examples. Although planar systems can be thought of as a special case of particle systems in E^3, it is more convenient to remove the third coordinate and think of the motions of such a system as defined in a $2n$ -dimensional space.

In this thesis, we are interested mainly in mechanical systems that can be described by Definition 2.7(4). A kinematic system of finitely-many particles with a configuration space, as defined in Definition 2.7(4), is a very important special case of a general mechanical system of n particles (Definition 2.7(3)). Unlike the general case, the system $\mathcal{K}(C, E^{3n})$ is fully defined by specifying a subset of E^{3n}. Other special cases are discussed in the following remark.

2.9. Remark. The space $\mathcal{K}(C, E^{3n})$, used in Definitions 2.7, is a special sub-space of $\mathcal{K}(E^{3n})$, described only by constraints on the *values* of the functions. There are subsets of motions that cannot be described by a subset of E^{3n}, e.g., the space of solutions of a system of differential equations. Mechanical systems that are described by second (or higher) order differential equations are referred to as *dynamic*. However, systems described only with first-order differential equations are usually referred to as kinematic. Such systems are called *non-holonomic kinematic systems*, and they can be described by specifying a subset of the tangent bundle, TE^{3n} (or, of the cotangent bundle, T^*E^{3n}), i.e., by specifying constraints on the positions *and* velocities (or momenta) of the particles. The systems described in Definition 2.7(4) are referred to as *holonomic*.

2.10. Definitions. *Let* $\mathcal{M}_1$ *and* $\mathcal{M}_2$ *be abstract mechanical systems on* X_1 *and* X_2. *We denote by* $\mathrm{Im}\ \mathcal{M}_i$ *the sets* $\bigcup_{f \in M_i} \mathrm{Im} f$, $i = 1, 2$. *If* $\mathcal{M}_i = \mathcal{K}(C_i, X_i)$, *then* $\mathrm{Im}\ \mathcal{M}_i = C_i$.

(1) $\mathcal{M}_1$ *and* $\mathcal{M}_2$ *are said to be* **diffeomorphic** *when* $\mathrm{Im}\ \mathcal{M}_2$ *is the image of* $\mathrm{Im}\ \mathcal{M}_1$ *under a homeomorphism which can be extended, at least locally, to a diffeomorphism of submanifolds of* X_1 *and* X_2.

More precisely, $\mathcal{M}_1$ *is said to be diffeomorphic to* $\mathcal{M}_2$ *when there exists a map* $\phi : \mathrm{Im}\ \mathcal{M}_1 \to \mathrm{Im}\ \mathcal{M}_2$, *such that:*

(a) ϕ *is a homeomorphism.*

(b) For every $x_1 \in X_1$ denote $x_2 = \phi(x_1)$. For some neighbourhoods O_{x_i}, $x_i \in O_{x_i} \subset X_i$, there exist submanifolds M_{x_i}, $x_i \in M_{x_i} \subset X_i$, and a diffeomorphism, $\Phi : O_{x_1} \cap M_{x_1} \to O_{x_2} \cap M_{x_2}$, such that $\Phi(x) = \phi(x)$ for all x for which both maps are defined.

We will say that the diffeomorphism of $\mathcal{M}_1$ and $\mathcal{M}_2$ is induced by the map ϕ or that $\mathcal{M}_1$ and $\mathcal{M}_2$ are diffeomorphic with map ϕ.

(2) *Let $\mathcal{M}_1 = \mathcal{K}(C_1, E^{3n})$, $\mathcal{M}_2 = \mathcal{K}(C_2, E^{3n})$. $\mathcal{M}_1$ and $\mathcal{M}_2$ are said to be* **congruent**, *if there is a displacement, $g \in SE(3)$, such that $C_2 = gC_1$, i.e.,*

$$C_2 = \{(gx_1, \ldots, gx_n) \mid (x_1, \ldots, x_n) \in C_1\}.$$

(3) *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be diffeomorphic with ϕ. Then, the pair $(\mathcal{M}_1, \phi)$ is said to be a* (**kinematic**) **model** *of $\mathcal{M}_2$.*

(4) *Let $(\mathcal{M}_1, \phi_1)$ and $(\mathcal{M}_2, \phi_2)$ be models of systems $\mathcal{N}_1$ and $\mathcal{N}_2$. The models are said to be* **congruent**, *if $\mathcal{N}_1$ and $\mathcal{N}_2$ are congruent. The models are* **equivalent** *when $\mathcal{N}_1 = \mathcal{N}_2$.*

2.11. Remarks.

(1) A sufficient but not a necessary condition for the systems $\mathcal{M}_i = \mathcal{K}(C_i, X_i)$, $i = 1, 2$, to be diffeomorphic is the existence of a homeomorphism from C_1 onto C_2 that can be extended *globally* to a diffeomorphism of two submanifolds of X_1 onto X_2. More precisely, if there exist submanifolds $C_i \subset M_i \subset X_i$, and a diffeomorphism Φ, $\Phi : M_1 \to M_2$, such that $\phi(C_1) = C_2$, then the systems $\mathcal{K}(C_i, X_i)$ are diffeomorphic.

(2) It can be shown that Definition 2.10(1) introduces an equivalence relation in the set of all mechanical systems. (Similarly, the property of congruence, introduced in Definition 2.10(2) is an equivalenece relation in the class of the systems of the type $\mathcal{K}(C, E^{3n})$). However, the property defined in part (1) of the present remarks (i.e., the existence of a diffeomorphism of submanifolds containing the configuration spaces) is not an equivalence relation.

We note that a kinematic system is described entirely by its model (as defined in Definition 2.10 (4)). A congruent model is obtained by a change of the reference frame in the ambient space. Other diffeomorphic models allow, when studying the behaviour of the system, to substitute the motions in E^{3n} (or E^{2n}) with motions on lower-dimensional manifolds. In particular, whenever the configuration space, C, of some kinematic system $\mathcal{K}(C, Q)$, is a submanifold of the ambient manifold, Q, the system is diffeomorphic to $\mathcal{K}(C)$.

2.12. Example. We examine the planar kinematic system of two particles, $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$, with a configuration space C, defined by the following three conditions: (i) the first particle, $\boldsymbol{p}_1$, must remain at a constant distance, l_1, from a fixed point $\boldsymbol{o}$; (ii) The distance between the two particles must always be equal to l_2; (iii) the second particle must always remain on a fixed line through $\boldsymbol{o}$. This system, denoted by $\mathcal{M}$, is shown in Figure 2.1.

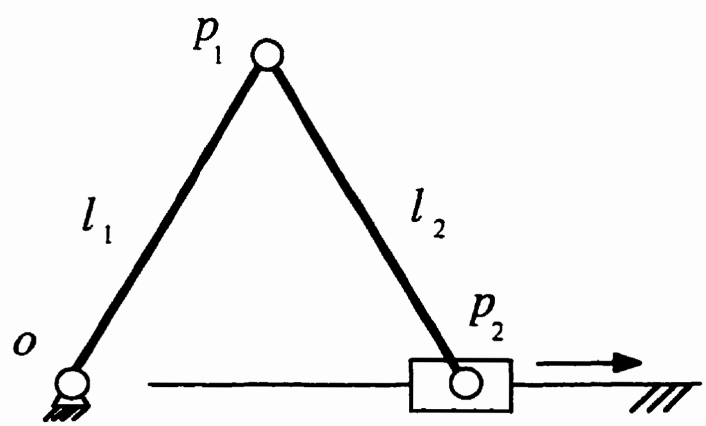

Figure 2.1. A two-particle kinematic system.

Let the coordinate system in E^2 be chosen with an origin at $\boldsymbol{o}$ and the x axis along the line containing $\boldsymbol{p}_2$. Then, the kinematic system is $\mathcal{M} = \mathcal{K}(C, E^4)$, where C is the set of points in E^4 with coordinates, (x_1, y_1, x_2, y_2), which satisfy the following constraints:

$$x_1{}^2 + y_1{}^2 = l_1{}^2,$$

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 = l_2^2, \tag{2.12}$$
$$y_2 = 0.$$

This system is diffeomorphic to $\mathcal{M}' = \mathcal{K}(C', E^3)$, where E^3 is the set of points with coordinates (x_1, y_1, x_2) and C' is the vanishing set of the two equations:

$$\begin{aligned} x_1^2 + y_1^2 &= l_1^2 , \\ (x_2 - x_1)^2 + y_1^2 &= l_2^2. \end{aligned} \tag{2.13}$$

The map of this diffeomorphism is the inclusion, i, of E^3 into E^4, which maps E^3 into the hyperplane $\{y_2 = 0\}$. ($i(x_1, y_1, x_2) = (x_1, y_1, x_2, 0)$).

Furthermore, this system is diffeomorphic to the system $\mathcal{M}'' = \mathcal{K}(C'', X)$, where X is a cylinder with radius l_1, and C'' is the vanishing set of the equation:

$$h^2 - 2hl_1\cos\theta = l_2^2 - l_1^2 \tag{2.14}$$

where h and θ are the cylindrical coordinates on X. The diffeomorphic map in this case is the inclusion map, j, which identifies X with the cylinder in E^3 with equation $x_1^2 + y_1^2 = l_1^2$. ($j(h, \theta) = (l_1\cos\theta, l_1\sin\theta, h)$).

When $l_2 \neq l_1$, the set C'', given by Equation (2.14), is the disjoint union of two smooth closed curves. Therefore, the system is diffeomorphic to $\mathcal{K}(C'')$ and, furthermore, to $\mathcal{K}(C''')$, $C''' = S^1 \cup S^1$, i.e., to the set of motions of a point on a pair of circles. The diffeomorphic relation is established by a map denoted, respectively, k_1 for the case $l_1 > l_2$, and k_2 when $l_2 > l_1$. Each map is defined for an element, φ, of either the first or the second circle denoted $(S^1)_1$ and $(S^1)_2$, respectively. When $l_1 > l_2$:

$$k_2 \colon S^1 \cup S^1 \to M, \quad k_2(\varphi) = \begin{cases} (h(\varphi), \varphi) & \text{if } \varphi \in (S^1)_1 \\ (h(\varphi), \varphi + \pi) & \text{if } \varphi \in (S^1)_2 \end{cases} \tag{2.15}$$

where

$$h(\varphi) = l_1\cos\varphi + \sqrt{l_2^2 - (l_1\sin\varphi)^2}.$$

When $l_2 > l_1$:

$$k_1\colon S^1 \cup S^1 \to E^3, \qquad k_1(\varphi) = \begin{cases} (x(\varphi), y(\varphi), z(\varphi)) & \text{if } \varphi \in (S^1)_1 \\ (-x(\varphi), -y(\varphi), z(\varphi)) & \text{if } \varphi \in (S^1)_2 \end{cases} \tag{2.16}$$

where

$$x(\varphi) = \sqrt{l_1^2 - l_2^2\cos^2\varphi}\,,$$
$$y(\varphi) = l_2\cos\varphi,$$
$$z(\varphi) = l_2\sin\varphi + \sqrt{l_1^2 - l_2^2\cos^2\varphi}\,.$$

When $l_2^{\,2} = l_1^{\,2} = l$, C'' becomes the union of a pair of intersecting closed curves on the cylinder: the circle $S = \{h = 0\}$ and the ellipse $E = \{h - 2l\cos\theta = 0\}$, Figure 2.2.

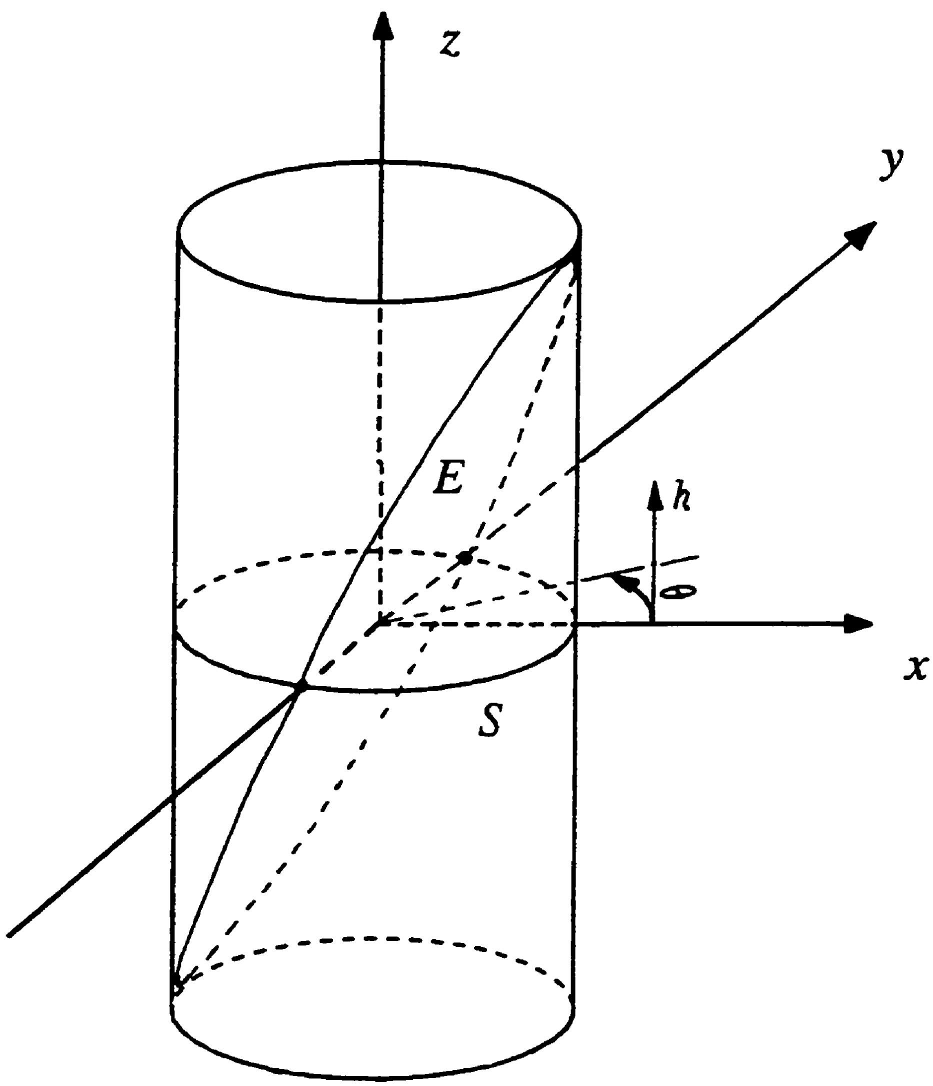

Figure 2.2. The configuration space of a 2-particle system.

In this case, the configuration space is not a smooth manifold and the kinematic system cannot be modelled by a system on a one-dimensional manifold. The simplest possible representation is as a system, such as $\mathcal{M}''$, defined on a two-dimensional manifold. An open submanifold of X containing C'', e.g. $\{-2l - \varepsilon < h < 2l + \varepsilon\}$, $\varepsilon > 0$, can be mapped onto an annular region of the plane and thus $\mathcal{M}$ can be shown to be diffeomorphic to a system of the type $\mathcal{K}(S, E^2)$, where S is a set of two intersecting circles in E^2.

In conclusion, the pairs $(\mathcal{M}', i)$, $(\mathcal{M}'', i \circ j)$, $(\mathcal{M}''', i \circ k)$ are diffeomorphic kinematic models of the same kinematic system, $\mathcal{M}$.

2.13. Example. Consider the set $C \subset E^3$, $C = S \cup I$, where S is the unit circle in the Oxy plane, $S = \{x^2 + y^2 = 1,\ z = 0\}$, and I is the straight-line interval $\{x = 0,\ y = 1,\ -0.5 < z < 0.5\}$, Figure 2.3.

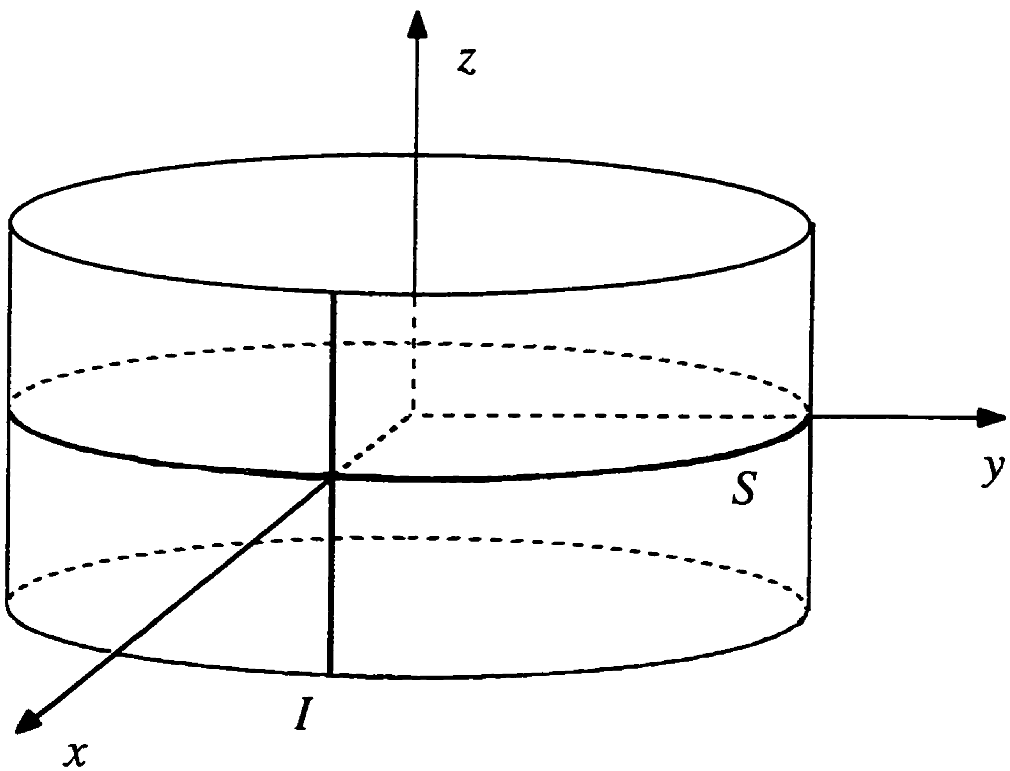

Figure 2.3. A singular configuration space, $C = S \cup I$, on a cylindrical surface.

We note that C is a subset of a cylindrical strip, Q, (with axis the z coordinate axis). Simultaneously, the same set, C, can be considered as a subset of a Möbius strip, M, obtained by cutting, twisting and re-attaching the cylindrical strip Q. The systems $\mathcal{K}(C, Q)$ and $\mathcal{K}(C, M)$ are diffeomorphic, but they do not satisfy the global diffeomorphic condition described in Remark 2.11(1). Indeed, if this were the case, it would follow that a cylinder is homeomorphic to a Möbius strip, which is known to be incorrect. This example illustrates that the condition (which we used in Example 2.12) requiring the existence of a "global diffeomorphism" (as described in Remark 2.11(1)) is not always satisfied when the configuration space is not a smooth manifold.

2.4. Rigid Body

Let us consider a system of $N > 1$ moving particles subject to the condition that in every feasible configuration the distances between the particles remain the same. Let $x_1^0, \ldots, x_N^0$ be $N > 1$ distinct points in E^3. We have the system:

$$\mathcal{B} = \mathcal{K}(C_N, E^{3n}),\ C_N = \{(x_1, \ldots, x_N) |\ d(x_i, x_j) = d(x_i^0, x_j^0)\}. \qquad (2.17)$$

2.14. Proposition. *The set C_N, defined in Equation (2.17) is a smooth manifold described by the following statements:*

(1) If all the points x_i^0 lie along one line, then C_N is diffeomorphic to the product of E^3 and the 2-sphere, S^2, $E^3 \times S^2$.

(2) If all the points x_i^0 do not belong to any single line but lie in one plane, then C_N is diffeomorphic to $SE(3)$.

(3) Otherwise, if the points x_i^0 do not belong to any single plane, then C_N is diffeomorphic to the disjoint union of two copies of $SE(3)$.

Proof. We denote $\mathbf{v}_{ij} = x_j - x_i$, $\mathbf{v}_{ij}^0 = x_j^0 - x_i^0$, $d_{ij} = d(x_i^0, x_j^0)$.

(1) Consider the set C_2, $C_2 = \{(x_1, x_2) \mid d(x_1, x_2) = d_{12}\}$.

C_2 is diffeomorphic to $E^3 \times S^2$ by the map: $(x_1, x_2) \to (x_1, o + (1/d_{12})(x_2 - x_1))$, which is obviously bijective and smooth. It remains to show that C_2 is diffeomorphic to C_N when the points of the initial configuration are co-linear. When all points of the initial configuration lie along a single line, then this will be true for any configuration. This is due to the fact that, in E^3, three points, x_1, x_2, x_3, are co-linear if, and only if,

$$d_{12} = \pm(d_{13} \pm d_{23}).$$

If λ_i are scalars such that

$$x_i^0 = x_1^0 + \lambda_i(x_2^0 - x_1^0),$$

then the diffeomorphism of C_2 onto C_N is given by the map:

$$x_i = x_1 + \lambda_i (x_2 - x_1).$$

(2) Without loss of generality, let x_1^0, x_2^0 and x_3^0 be non-collinear. We denote by C_3 the configuration space of a rigid body which consists of only the first three particles of the body C_N, i.e.,

$$C_3 = \{(x_1, x_2, x_3) \mid d(x_i, x_j) = d_{ij},\ 1 \le i < j \le 3\} \tag{2.18}$$

First, we show that C_3 is diffeomorphic to $SE(3)$. Denote $x^0 = (x_1^0, x_2^0, x_3^0)$ and consider the mapping

$$\phi_{x^0} : SE(3) \to C_3, \quad \phi_{x^0}(g) = gx^0 = (gx_1^0, gx_2^0, gx_3^0). \tag{2.19}$$

To show that ϕ_{x^0} is bijective, we will choose a reference frame attached to x^0.

For any given three non-colinear points, (x_1, x_2, x_3), one can define a Cartesian reference frame in E^3, $o\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$, in a unique way by specifying: the origin at x_1, $o = x_1$; the $\mathbf{e}_1$ axis along $x_2 - x_1$; the second axis, $\mathbf{e}_2$, in the plane of x_1, x_2 and x_3 in such a way that x_3 has a positive second coordinate and $\mathbf{e}_1 \cdot \mathbf{e}_2 = 0$; and $\mathbf{e}_3$ so that it completes the Cartesian frame (i.e., $\mathbf{e}_3$ must be orthogonal to $\mathbf{e}_1$ and $\mathbf{e}_2$, and $\det(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) = 1$).

We fix the frame defined by (x_1^0, x_2^0, x_3^0) in this manner.

The map ϕ_{x^0} is injective. Indeed, if $gx^0 = fx^0$, we will show that $g = f$. Let g and f be given by $(G, \mathbf{t}_g)$ and $(F, \mathbf{t}_f)$. $gx_1^0 = fx_1^0$ implies $\mathbf{t}_g = \mathbf{t}_f$. Since $\mathbf{e}_1$ and $\mathbf{e}_2$ are linear combinations of $\boldsymbol{x}_2 - \boldsymbol{x}_1$ and $\boldsymbol{x}_3 - \boldsymbol{x}_1$, it follows that $\boldsymbol{G}\mathbf{e}_1 = \boldsymbol{F}\mathbf{e}_1$ and $\boldsymbol{G}\mathbf{e}_2 = \boldsymbol{F}\mathbf{e}_2$. Finally, $\boldsymbol{G}\mathbf{e}_3 = \boldsymbol{F}\mathbf{e}_3$ since $\boldsymbol{G}$ and $\boldsymbol{F}$ preserve the scalar product and orientation.

The map $\phi_{\boldsymbol{x}^0}$ is also surjective. Given an arbitrary $\boldsymbol{x} = (\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3) \in C_3$, we chose $\mathbf{t}_g$ equal to $\boldsymbol{x}_1 - \boldsymbol{x}_1^0$, and $\boldsymbol{G}$ such that the axes of the frame attached to $\boldsymbol{x}^0$ are mapped along the ones attached to $\boldsymbol{x}$. Then, the displacement defined by $(\mathbf{t}_g, \boldsymbol{G})$ is mapped by $\phi_{\boldsymbol{x}^0}$ into $\boldsymbol{x}$.

$\phi_{\boldsymbol{x}^0}$ is smooth, since it is linear with respect to the matrix components of g. Therefore, $\phi_{\boldsymbol{x}^0}$ is a diffeomorphism of $SE(3)$ and C_3.

To complete the proof of (2), we show that C_3 is diffeomorphic to C_N. The conditions (2.17) imply that the coordinates of all points are determined by the coordinates of the three points. If we attempt to find the coordinates of a point $\boldsymbol{x}_k$, $k > 3$, in the frame $o\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$, we find a unique solution for the first two coordinates, $\boldsymbol{x}_{k1}$ and $\boldsymbol{x}_{k2}$, and two solutions for the third coordinate, $x_{k3} = \pm\sqrt{d_{1k}^2 - x_{k1}^2 - x_{k2}^2}$.

These two solutions coincide if, and only if, the point $\boldsymbol{x}_k$ is in the plane of $\boldsymbol{x}_1$, $\boldsymbol{x}_2$ and $\boldsymbol{x}_3$. Therefore, when all points of the rigid body are in one plane, the configuration space C_N is diffeomorphic to C_3 and to $SE(3)$. The theorem is therefore proven for the case of a "flat" body.

(3) Without loss of generality, let $\boldsymbol{x}_1^0$, $\boldsymbol{x}_2^0$ and $\boldsymbol{x}_3^0$ be non-colinear and let $\boldsymbol{x}_4^0$ be outside of the plane $o\mathbf{e}_1\mathbf{e}_2$ (as above, $o\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ is the frame attached to $\boldsymbol{x}_1^0$, $\boldsymbol{x}_2^0$ and $\boldsymbol{x}_3^0$). Then, for each of the two solutions for $\boldsymbol{x}_4$ there is a unique solution for the coordinates of every $\boldsymbol{x}_k$, $k > 4$: $\boldsymbol{x}_{k3} = (1/2x_{43}^2)(d_{4k}^2 - d_{1k}^2 + x_{k1}^2 + x_{k2}^2 - x_{43}^2)$. Therefore, given $\boldsymbol{x}_1$, $\boldsymbol{x}_2$ and $\boldsymbol{x}_3$, and the conditions (2.17), there are two solutions for the set of points $(\boldsymbol{x}_3, \ldots, \boldsymbol{x}_N)$. (The two solutions coincide if and only if all points lie in the same plane). Hence, there are two one-to-one smooth mappings, Φ^+ and Φ^-, of $SE(3)$ into C_N, defined by the two solutions and therefore $C_N = \Phi^+(SE(3)) \cup \Phi^-(SE(3))$. The image space of each of these two mappings, $\Phi^+(SE(3))$ or $\Phi^-(SE(3))$, is a manifold diffeomorphic to C_3 and to $SE(3)$. In particular, this implies that $\Phi^+(SE(3))$ and $\Phi^-(SE(3))$ are path-connected. The two image spaces do not intersect unless they coincide (i.e, unless the rigid body is flat). Indeed, if we

assume that they do intersect, (in view of the path-connectedness of the two image sets) it would follow that there exists a continuous path, γ, connecting two configurations with the same positions of x_1, x_2 and x_3 but two different (mirror-image) locations of x_4. In these two configurations the values of the orientation of the four points, $\sigma(x_1, x_2, x_3, x_4)$, have opposite signs. Since the determinant function is continuous, it follows that there is a configuration along γ where the orientation function equals zero, i.e., the points $x_1, \ldots, x_4$ lie in one plane. This, however, contradicts the conditions (2.17) and our assumption that the points are not co-planar in the initial configuration. □

2.15. *Definition.* *A* **rigid body**, *$\mathcal{B}$, is a kinematic system of $N \geq 3$ particles in E^3 such that the distance and orientation functions, computed for the positions of the particles, do not change during the motion of the system. Thus, $\mathcal{B} = \mathcal{K}(C_B, E^{3n})$, with*

$$C_B = \{(x_1, x_2, \ldots, x_N) \mid d(x_i, x_j) = d(x_i^0, x_j^0),$$
$$\sigma(x_i, x_j, x_k, x_l) = \sigma(x_i^0, x_j^0, x_k^0, x_l^0)\},$$

where $(x_1^0, x_2^0, \ldots, x_N^0)$ is a given array of points in E^3.

2.16. *Proposition.* *Let $\mathcal{B}$ be a rigid body with a feasible configuration x^0, $x^0 = (x_1^0, x_2^0, \ldots, x_N^0)$. Then, the pair $(\mathcal{K}(SE(3)), \phi_{x^0})$, where*

$$\phi_{x^0} : SE(3) \rightarrow E^{3n}, \quad \phi_{x^0}(g) = g(x^0),$$

is a model of $\mathcal{B}$.

Proof. The proposition follows from the proof of Proposition 2.14. Indeed, in part *(2)* of that proof it was shown that the mapping ϕ_{x^0}, defined first in Equation (2.19), is a diffeomorphism between $SE(3)$ and C_3. However, since the orientation in the rigid body is fixed, C_3 can be identified with C_B. □

2.17. *Remark.*

(1) Usually, when the rigid-body concept is introduced in the literature, only the conditions (2.17) are used. Rigorously speaking, the configuration space of such a system

is homeomorphic to $\mathbf{R}^3 \times O(3)$. ($O(3)$ is used to denote the space of linear maps in $\mathbf{R}^3$ which preserve the scalar product, but not necessarily the det function, i.e., matrices with a determinant of ± 1). The manifold $SE(3) = \mathbf{R}^3 \times SO(3)$ is only one of the two connected components of $\mathbf{R}^3 \times O(3)$, each of which corresponds to a fixed orientation of the body. Since a change of orientation cannot be achieved by continuous rigid-body motion, one of the components of the configuration space can be disregarded. To specify the component it is sufficient to provide one (initial) feasible configuration.

(2) In view of Proposition 2.16 and Remark 2.17(1), a rigid body containing three non-colinear points can be imagined as another Euclidian space co-located with E^3. The relative position and orientation of these two copies is given by an element of $SE(3)$ and their relative motion is modelled by the motion of a point in $SE(3)$. It is thus common to substitute the system $\mathcal{K}(C_B, E^{3n})$ with $\mathcal{K}(SE(3))$.

(3) Proposition 2.13 proves that C_B is a 6-dimensional smooth manifold and therefore a local coordinate system can be chosen at each of its points. Hence, the relative motion of two rigid bodies can be *locally* described by six scalar parameters. However, there is no system of six coordinates that can be used *globally*, i.e., on the entire rigid-body configuration space. Indeed, a global coordinate space would imply that $SE(3)$ is diffeomorphic to $\mathbf{R}^6$. However, one of the components of $SE(3)$ is the manifold $SO(3)$ (homeomorphic to the real projective space $\mathbf{RP}^3$), which is topologically different from $\mathbf{R}^3$.

2.5. Rigid-Body Systems: Kinematic Joints

In this section, systems of rigid bodies are defined formally, in a way analogous to the introduction of systems of particles by Definition 2.7. This approach is justified since, by Remark 2.17(2), we can "ignore" that a rigid body is composed of particles and treat it as a point in $SE(3)$. Nevertheless, the definitions in this section can easily be shown to be compatible with those in Section 2.4 in the sense that rigid-body systems are well defined as systems of particles as well.

Denote $SE(3)^n = SE(3) \times \cdots \times SE(3)$. Just as an element of $SE(3)$ determines the location of a rigid body in a Euclidian space, a point in $SE(3)^n$ can be thought of as describing the location of n rigid bodies in their common ambient space, E^3. A *kinematic system of n rigid bodies* (or, in short, a *rigid-body system*) is a subset of $\mathcal{K}(SE(3)^n)$.

2.18. Definitions.

(1) An abstract mechanical system on $SE(3)^n$, *i.e., a subset of* $\mathcal{K}(SE(3)^n)$, *is referred to as a* **mechanical system of n rigid bodies.**

(2) An abstract kinematic system on $SE(3)^n$ *with configuration space C, i.e.,* $\mathcal{K}(C, SE(3)^n)$, *is referred to as a* **kinematic system of n rigid bodies with configuration space** C.

2.19. Remark. A system of n rigid bodies, $\mathcal{K}(C, SE(3)^n)$, can be modeled as a system of particles. Indeed, we recall that in the proof of Proposition 2.14(2) it was shown that $SE(3)$ is diffeomorphic to a set, $C_3 \subset E^9$, defined by the (arbitrary) choice of a triple of non-colinear points, $\boldsymbol{x}^0 = (\boldsymbol{x}_1^0, \boldsymbol{x}_2^0, \boldsymbol{x}_3^0)$, $\boldsymbol{x}_i^0 \in E^3$. Therefore, $SE(3)^n$ is diffeomorphic to a subset of E^{9n}, $C_3{}^n = C_3 \times \cdots \times C_3$. The diffeomorphism in question is:

$$\phi_{\boldsymbol{x}^0} : SE(3)^n \to E^{9n}, \ \ \phi_{\boldsymbol{x}^0}(g_1, \ldots, g_n) = (g_1\boldsymbol{x}^0, \ldots, g_n\boldsymbol{x}^0),$$

where $g_i\boldsymbol{x}^0 = (g_i\boldsymbol{x}_1^0, g_i\boldsymbol{x}_2^0, g_i\boldsymbol{x}_3^0)$. Thus, $\mathcal{K}(C, SE(3)^n)$ is diffeomorphic to a system of $6n$ particles, $\mathcal{K}(D, E^{9n})$, where D is the image of the configuration space C, $D = \phi_{\boldsymbol{x}^0}(C)$.

Similarly to Remark 2.9 (regarding systems of particles), we note here that the kinematic systems of rigid bodies with a configuration space, introduced in Definition 2.18(2), are commonly referred to as holonomic, while systems in which not all motions inside the configuration space are feasible are called non-holonomic. We will be dealing with holonomic systems.

Definition 2.10(1) (where diffeomorphic systems were defined) applies to systems of rigid bodies as well. (Note that two diffeomorphic rigid-body systems need not be composed of the same number of bodies). Furthermore, some stronger equivalence relations between systems of an equal number of rigid bodies can be introduced.

2.20. Definitions. *Let* $\mathcal{M}_1 = \mathcal{K}(S_1, SE(3)^n)$, $\mathcal{M}_2 = \mathcal{K}(S_2, SE(3)^n)$.

(1) $\mathcal{M}_1$ *and* $\mathcal{M}_2$ *are said to be* **conjugated**, *if there is a displacement,* $g \in SE(3)$, *such that* $S_2 = gS_1g^{-1}$, *i.e.,* $S_2 = \{gfg^{-1} \mid f \in S_1\}$.

(2) *Let* $(\mathcal{N}_1, \phi_1)$ *and* $(\mathcal{N}_2, \phi_2)$ *be models of the rigid-body systems* $\mathcal{M}_1$ *and* $\mathcal{M}_2$. *The models are said to be* **conjugated**, *if* $\mathcal{M}_1$ *and* $\mathcal{M}_2$ *are conjugated. The models are* **equivalent** *when* $\mathcal{M}_1 = \mathcal{M}_2$.

2.21. Definitions.

(1) *Let* Q *be a path-connected subset of* $SE(3)$ *containing the unit element,* e. *Then, the system* $\mathcal{K}(Q, SE(3))$ *is referred to as a (***kinematic***)* **joint** *with* **configuration space** Q.

(2) *Let* $C \subset SE(3)$. *The system of two rigid bodies,* $\mathcal{K}(B, SE(3)^2)$, *with configuration space* $B = \{e\} \times C$ *is referred to as a* **kinematic pair with output space** C.

(3) *Let* $\mathcal{J}$ *be a joint with configuration space* Q, *and let* γ_- *and* γ_+ *be a pair of fixed displacements. Then, the system of two rigid bodies,* $\mathcal{K}(B, SE(3)^2)$, *with configuration space*

$$B = \{e\} \times \gamma_- Q \gamma_+^{-1} = \{(e, \gamma_- g \gamma_+^{-1}) \mid g \in Q\}$$

is referred to as a **kinematic pair with joint** $\mathcal{J}$. *The displacements* γ_- *and* γ_+ *are said to give the* **location of the joint** $\mathcal{J}$ *in the first and second body of the kinematic pair, respectively.*

2.22. *Remarks.*

(1) The configuration space of the joint, Q, is the set of the possible displacements of a rigid body *relative* to one chosen possible location of the body, i.e., Q consists of the elements of $SE(3)$, which map one chosen possible configuration of the rigid body into all its possible configurations. If A, a path-connected subset of $SE(3)$, is the set of the possible displacements of the body in the ambient space, then Q would be the set Au^{-1}, where u is the chosen configuration. The reference frame attached to u is referred to as the *fixed joint-frame*, while a frame attached to a variable displacement $v \in A$ is referred to as the *moving joint-frame*.

(2) A kinematic pair with output space C is also a kinematic pair with joint $\mathcal{J}$, if the joint $\mathcal{J}$ is specified as $\mathcal{J} = \mathcal{K}(Q, SE(3))$, $Q = Cu^{-1}$, where u is any chosen element of B. The location of the joint is then given by $(\gamma_-, \gamma_+) = (e, u^{-1})$. Conversely, a kinematic pair with joint $\mathcal{J}$ has an output space equal to $\gamma_- Q \gamma_+^{-1}$.

(3) A kinematic pair describes the displacement of two rigid bodies with respect to a frame fixed in one of them. The displacement of the second body with respect to the first one is given by the product $\gamma_- f \gamma_+^{-1}$. In this expression: γ_- is the displacement from the frame associated with the first (fixed) body onto the fixed joint-frame; f is the joint displacement measured in the fixed joint frame; and γ_+ is the displacement mapping the frame of the second (moving) body onto the moving joint-frame, measured in the frame of the second body. (If all the displacement were measured in the fixed-body frame, the product displacement would be $\gamma_+^{-1} f \gamma_-$).

Clearly, a kinematic joint is a kinematic system diffeomorphic to a kinematic pair with this joint. Thus, $(\mathcal{J}, f \to (e, \gamma_- f \gamma_+^{-1}))$ is a kinematic model of $\mathcal{K}(Q, SE(3)^2)$.

2.23. Definition. *A set of joints,* T, *is referred to as a* **joint type**, *if it consists of all joints conjugated with some joint,* $\mathcal{J}$.

The conjugacy of joints is an equivalence relation and the joint types are its equivalence classes. Definition 2.23 provides a criterion for comparison of different joints while disregarding the reference frame in which the joint displacements are being calculated.

2.24. Example. The present example discusses a category of joint types that are of special practical and theoretical importance. These are the so-called ***Reuleaux pairs*** (Reuleaux 1875), also known as *lower pairs*, listed in Table 2.1.

Name of joint	Notation	Surface	Configuration Space	Dimension
Spherical joint	S	Sphere	$SO(3)$	3
Planar joint	F	Plane	$SE(2)$	2
Cylindrical joint	C	Cylinder	$\mathbf{R} \times SO(2)$	2
Revolute joint	R	of Revolution	$SO(2)$	1
Prismatic joint	P	of Translation	$\mathbf{R}$	1
Helical joint of pitch p	H(p)	Helicoidal	$Sp(1, \mathbf{R})_p$	1

Table 2.1. The Reuleaux pairs.

In the fourth column of Table 2.1, **R** denotes the group of translations parallel to a fixed line (isomorphic to the set of real numbers). $SO(2)$ is the group of rotations in the 2-dimensional space. The manifold $SO(2)$ is diffeomorphic to the 1-dimensional circle, S^1.The group $SE(2)$, the Euclidian group in two dimensions, is defined in a way similar to $SE(3)$: $SE(2)$ is obtained from the set $\mathbf{R}^2 \times SO(2)$ analogically to Theorem 2.4. The notation $Sp(1, R)_p$ is understood as the symplectic subgroup of $SE(3)$ for pitch p, $p \in (0, \infty)$. This group consists of all helical displacements of pitch p and is isomorphic to (but not a conjugate of) **R**.

Physically, the Reuleaux pairs are defined as pairs of identical surfaces in E^3, which can move relative to each other while remaining in surface contact. Most practical mechanisms have only Reuleaux pairs, since they provide stable contact and are relatively easy to implement as two parts with mating surfaces.

According to a mathematical definition, a Reuleaux pair is a kinematic pair whose joint type consists of the symmetry Lie groups of a 2-dimensional (smooth) submanifold of E^3 (Selig and Rooney, 1989). In other words, a joint, J, is a joint of a Reuleaux pair when: (i) its configuration space, Q_J, is not only a submanifold but also a subgroup of $SE(3)$ (i.e. Q_J is closed under the composition of displacements); and (ii) there exists a surface in E^3, such that: (a) every displacement in Q_J maps the surface into itself and (b) all displacements in $SE(3)$ with this property are elements of Q_J.

A classification of the subgroups of $SE(3)$ (up to conjugacy class) can be found in Hervé (1978). There are eight different subgroups of dimension 2 or higher. Only four of these, however, preserve some surface in E^3. These are: $SO(3)$, $SE(2)$, $\mathbf{R} \times SO(2)$ and $\mathbf{R}^2$. The notation $\mathbf{R} \times SO(2)$ denotes a subgroup generated by the rotations and translations about one and the same line in E^3. The group denoted by $\mathbf{R}^2$, which is generated by the translations in two directions, does not satisfy our definition, since it does not contain all the symmetries of its invariant surface (a plane parallel to both translations). This condition is satisfied by a larger group, $SE(2)$, which has $\mathbf{R}^2$ as its subgroup. Therefore, there are three Reuleaux joint types of dimension greater than one, and they are listed in the first three rows of Table 2.1.

The subgroups of $SE(3)$ of dimension 1 are the so-called symplectic subgroups of $SE(3)$, denoted $Sp(1, \mathbf{R})_p$ in general, symplectic groups are groups which preserve antisymmetric forms (Weyl 1946). In the case of $SE(3)$, a symplecic group preserves a 4×4 antisymmetric form (when the elements of the group are interpreted as the 4×4 matrices used to change coordinates in $P\mathbf{R}^3$, cf. Remark 2.5). A classification of the symplectic groups (by conjugacy class) can be identified with a classification of the space of vector forms in the tangent space of $SE(3)$. This vector-form classification identifies elements that can be mapped into each other by means of a coordinate change in E^3 or a multiplication by a scalar factor. This is, in fact, a classification of screws according to their pitch*, p. Thus, there are ∞^1 different 1-dimensional subgroups of $SE(3)$, one for every value of p, from 0 to ∞. All these groups correspond to a different Reuleaux pair, which has an invariant helicoidal surface in E^3. The most practical joints are given by the groups with $p = 0$ and $p = \infty$, and these are the groups $SO(2)$ and $\mathbf{R}$, where the helicoid degenerates into a surface of rotation or translation. The one-dof Reuleaux joints are listed in the last three rows of Table 2.1. It should be noted that the "helical joint type" actually consists of an infinite number of distinct joint types with pitch p, $0 < p < \infty$.

2.6. Articulated Systems

Kinematic pairs describe the possible relative motions of two bodies. When we say that two bodies are connected with a joint, $\mathcal{J}$, it is understood that the relative displacements of the two bodies are restricted to the configuration space of a kinematic pair with joint $\mathcal{J}$. If we imagine that the two bodies are part of a system of rigid bodies then a joint describes a restriction on the feasible configurations of the system. Systems of rigid bodies, where the configuration spaces are defined solely by specifying kinematic pairs, are referred to as *articulated systems*. The main purpose of this section is to define such systems and demonstrate some of their basic properties. To achieve this, we will need some basic concepts from graph theory, which we gradually introduce as we proceed. Our graph-theoretic notation is closest to Wittenburg (1994).

* The pitch is a projective number (i.e. an element of $P\mathbf{R}$) and an invariant in the 6-dimensional twist space. For a twist, $\mathbf{A}$, p is defined as $p = (\mathrm{Ki}(\mathbf{A}, \mathbf{A}) : 2\mathrm{Kl}(\mathbf{A}, \mathbf{A}))$, where $\mathrm{Ki}(\mathbf{A}, \mathbf{B})$ and $\mathrm{Kl}(\mathbf{A}, \mathbf{B})$ are the Klein and Killing forms–the only invariant scalar products in a twist space (Karger and Novak 1985). As a projective number, p does not change when multiplied by a scalar such as -1 and therefore p can be thought of as having a value from 0 to ∞.

A directed graph, $\Gamma = \mathcal{G}(M, N, v) = (V, A, v)$, consists of $M + 1$ vertices labeled $v \in V = \{0, \ldots, M\}$ and N connecting arcs labeled $a \in A = \{1, \ldots, N\}$, together with a mapping $v = (v_-, v_+)$,

$$v : A \to V \times V, \quad v(a) = (v_-(a), v_+(a)),$$

which specifies the starting and terminating vertex of the arc a.

We shall always assume that Γ has the following properties:

(i) For any arc, a, $v_-(a) \neq v_+(a)$.

(ii) Any two vertices are connected with at most one arc, i.e., the map v is injective;

(iii) Any two different vertices are connected by either an arc or by a sequence of arcs and vertices (i.e., Γ is a *connected* graph).

2.25. Definitions. *Let $\Gamma = \mathcal{G}(M, N, v) = (V, A, v)$ be a directed graph and let $\mathcal{P}$ be a collection of joints such that no two elements of $\mathcal{P}$ are of the same joint type.*

(1) *A map ρ, $\rho : A \to \mathcal{P}$, is referred to as a* **joint distribution** *for the graph Γ.*

(2) *A map γ,*

$$\gamma : A \to SE(3)^2, \quad \gamma(a) = (\gamma_-(a), \gamma_+(a)),$$

is referred to as a **link geometry** *for the graph Γ.*

2.26. Remark. The set $\mathcal{P}$, used in Definition 2.25 is, in fact, a collection of representatives of joint types, and hence a joint distribution, ρ, assigns a joint type to each arc of the graph. Each vertex of the graph is associated with a rigid body. In mechanism theory, these bodies are referred to as *links*. Then, the pair of maps (ρ, γ) defines a kinematic pair for each arc of the graph. The first body of the kinematic pair corresponds to $v_-(a)$ and the second body—to $v_+(a)$. The displacements $\gamma_-(a)$ and $\gamma_+(a)$ determine the location of the joint in the first and second bodies, respectively, while $\rho(a)$ is the joint of the kinematic pair.

2.27. Definitions. *Let $\Gamma = \mathcal{G}(M, N, v)$ be a directed graph.*

(1) *Let ρ be a joint-type distribution for Γ. The pair, (Γ, ρ), is referred to as a* **kinematic chain** *with graph Γ and joint-type distribution ρ. Let Q_a be the configuration space of the joint assigned to the arc a, i.e. $\rho(a) = \mathcal{K}(Q_a, SE(3))$. Then, the space $\prod_{a \in A} Q_a$ is referred to as the* **joint space** *of the kinematic chain.*

(2) *Let $W = (\Gamma, \rho)$, where $\rho(a) = \mathcal{K}(Q_a, SE(3))$, be a kinematic chain and let γ be a link geometry for Γ. Then, an* **articulated system** *with kinematic chain W and link geometry γ, $\mathcal{A}(W, \gamma)$, is defined as the kinematic system of M + 1 rigid bodies with configuration space C (i.e., $\mathcal{A}(W, \gamma) = \mathcal{K}(C, SE(3)^{M+1})$) where*

$$C = \{(e, g_1, \ldots, g_M) \mid g_{v_+(a)} = g_{v_-(a)}\gamma_-(a)f_a\gamma_+(a)^{-1}, f_a \in Q_a\}.$$

2.28. Remark. A kinematic chain specifies the bodies that are connected with kinematic pairs and the joint types of these pairs. However, the kinematic pairs are not fully described since the location of the joints in the adjacent bodies is unknown. These locations are given by the displacements $\gamma_-(a)$ and $\gamma_+(a)$. As we mentioned in Remark 2.26, the rigid bodies (associated with the vertices of a kinematic chain) are referred to as links, which accounts for the term "link geometry" adopted for γ. Knowing γ, one can calculate the relative displacement between the joint-frames of two different joints in one and the same body. It will be convenient to define a transformation of $SE(3)^N$, $F : SE(3)^N \to SE(3)^N$, for a given link geometry γ. F is given by:

$$F = (F_1, \ldots, F_a, \ldots, F_N), \quad F_a(h) = \gamma_-(a)^{-1}h_a\gamma_+(a).$$

It can be seen that F is an automorphism of $SE(3)^N$. This map transforms an array of N displacements, $(h_1, \ldots, h_N)$, which are thought of as the relative displacements of pairs of bodies, into $(F_1, \ldots, F_N)$, which can be regarded as an array of joint displacements. The inverse map, $F^{-1} = (F_1^{-1}, \ldots, F_a^{-1}, \ldots, F_N^{-1})$, is given by

$$F_a^{-1}(f) = \gamma_-(a)f_a\gamma_+(a)^{-1}.$$

An (ordered) sequence of arcs in which every arc (except perhaps the first and the last) is connected to the preceding and following arcs (i.e., it shares a vertex with them) will be referred to as a *path*. In other words, a sequence, $P = (a_1, a_2, \ldots, a_k)$, is a path if there exists a (necessarily unique) sequence of vertices, $(v_1, v_2, \ldots, v_{k+1})$, such that

$$\{ v_i, v_{i+1} \} = \{ v_-(a_i), v_+(a_i) \}, \quad i \in \{ 1, \ldots, k \} .$$

A path is called *elementary* if all its vertices, with the possible exception of the first and last, are distinct, i.e., $v_i \neq v_j$ whenever $|i - j| < k$. We note that all arcs in an elementary path must be distinct.

An elementary path for which the first and last vertices coincide ($v_1 = v_{k+1}$) is called a *circuit* (or *loop*†). A set of loops $\{L_1, \ldots, L_l\}$ is said to consist of *independent loops* if every loop, L_j, has at least one arc, $a^{(j)}$, which belongs to no other loop in the set. In every connected graph, there exists a set of c, $c = N - M$, independent loops, but there is no set with $c + 1$ independent loops. A set of $N - M$ independent loops is referred to as a *fundamental system of loops*. The arcs $a^{(j)}, j \in \{1, \ldots, c\}$, are called *chords* of the graph. A graph with no loops, i.e., with $c = 0$ is called a *tree*. If all chords were eliminated from a graph, Γ, the remaining graph would be a tree. The subgraph of Γ obtained by removing the chords is called a *spanning tree* of Γ. For a given graph, the choice of a fundamental system of loops, a system of chords (and the spanning tree) is not unique.

Kinematic chains, as well as articulated systems, are classified according to the topology of their graph, Γ. When Γ is a tree (i.e., it contains no closed loops and therefore $M = N$), the kinematic chain is referred to as *open*. A kinematic chain is *closed* when every arc of Γ is part of a closed loop. A *simple* chain has a graph where every vertex has

† In graph theory, the term *loop* refers to an arc which begins and ends in the same vertex, i.e. a circuit with only one arc. On the other hand, in mechanism theory circuits are usually called loops. Since circuits with one arc are not present in the graphs we address in this thesis, we shall use the term loop instead of circuit, thus complying with the usual terminology in kinematics.

at most two adjacent arcs (i.e., for every vertex, v, the set $v_-^{-1}(v) \cup v_+^{-1}(v)$ has at most two elements). Simple, open chains are referred to as *serial* chains.

2.29. Notations. Herein, we introduce some notations which will be used in the subsequent propositions. If a graph, Γ, is a tree, then $M = N$. Moreover, for every vertex, v, there is a *unique minimal path* (i.e., a sequence of distinct arcs and vertices) connecting v with any fixed vertex, e.g., the vertex 0. Therefore, there exists a correspondence, $P(v)$, which gives for every v, a unique elementary path, $P(v) = (a_1, \ldots, a_{k(v)})$, such that, for the vertices of this path, $v_1 = 0$ and $v_{k(v)+1} = v$.

Let $P = (a_1, a_2, \ldots, a_k)$ be a path in Γ. Let the function $\delta_P : A \to \{0, 1, -1\}$ be given by:

$$\delta_P(a) = \begin{cases} 0 & \text{if } a \neq a_i \ \forall i \\ 1 & \text{if } a = a_i \text{ and } v_-(a) = v_i \\ -1 & \text{if } a = a_i \text{ and } v_+(a) = v_i \end{cases} \tag{2.20}$$

For every path, P, we denote by $\Lambda_P : SE(3)^N \to SE(3)$ the map

$$\Lambda_P(h) = h_{a_1}^{\delta_P(a_1)} h_{a_2}^{\delta_P(a_2)} \ldots h_{a_k}^{\delta_P(a_k)}. \tag{2.21}$$

where $h = (h_1, \ldots, h_a, \ldots, h_N)$. If h_a is the relative displacement of the rigid body associated with $v_+(a)$ with respect to the body-frame associated with $v_-(a)$, calculated in the body frame of body $v_-(a)$, then $\Lambda_P(h)$ is the relative displacement of body v_k with respect to body frame v_1, measured in body-frame v_1. The value of $\Lambda_P(h)$ can be presented as a function of the joint displacements, f_a, by expressing $h = F^{-1}(f)$. (The map F depends on γ and was defined in Remark 2.28.) We denote $\Sigma_{P,\gamma}(f) = \Lambda_P(F^{-1}(f))$. When the link geometry, γ, is clear from the context, we will write simply $\Sigma_P(f)$. When P is a loop the equation $\Sigma_P(f) = e$ is referred to as the *loop equation* for loop P.

In what follows, we show how an articulated system can be described as a set of motions in the joint space, $\prod_{a \in A} Q_a$, rather than in $SE(3)^{M+1}$.

We shall assume that the sets Q_a, used in the definition of the joint-distribution map, ρ, are smooth submanifolds of $SE(3)$. A system satisfying this condition will be said to be with *smooth joints.*

2.30. Theorem. *Let $\mathcal{M} = \mathcal{A}(\Gamma, \rho, \gamma)$ be an articulated system and let $\mathcal{L} = \{L_1, \ldots, L_c\}$ be a fundamental system of loops in Γ. Then, $\mathcal{M}$ is diffeomorphic to $\mathcal{K}(D, Q)$, where*

$$Q = \prod_{a \in A} Q_a, \text{ and } D = \{(f_1, \ldots, f_N) \in Q \mid \Sigma_L(f) = e, \forall L \in \mathcal{L}\}.$$

Proof. The articulated system, $\mathcal{A}(\Gamma, \rho, \gamma)$, defined as $\mathcal{K}(\{e\} \times C, SE(3)^{M+1})$ (Definition 2.27(2)), is obviously diffeomorphic to $\mathcal{K}(C, SE(3)^M)$, where

$$C = \{(g_1, \ldots, g_M) \mid g_{v_+(a)} = g_{v_-(a)}(F^{-1})_a(f_a), f_a \in Q_a\}.$$

It is therefore sufficient (and necessary) to prove that the systems $\mathcal{K}(C, SE(3)^M)$ and $\mathcal{K}(D, Q)$ are diffeomorphic.

We denote by A'' a set of chords for Γ, and let the elements of A'' be $a^{(j)} \in L_j$, $j \in \{1, \ldots, c\}$. We consider the spanning tree, $\Gamma' = \mathcal{G}(V, A', v)$, obtained from Γ by removing A'' $(A' = A - A'')$. Then, the articulated system $\mathcal{M}' = \mathcal{A}(\Gamma', \rho, \gamma)$ is an open-chain system. (Note that, for simplicity, we use the same notation for a map defined on A, such as v, ρ or γ, and its restrictions on A'). We will first prove the statement of the theorem for the open-chain system $\mathcal{M}'$.

2.31. Lemma. *Let $\mathcal{M}' = \mathcal{A}(\Gamma', \rho, \gamma)$ be an articulated system with an open kinematic chain. Then $\mathcal{M}'$ is diffeomorphic to $\mathcal{K}(Q')$, where $Q' = \prod_{a \in A'} Q_a$.*

Proof. We need to show that $\mathcal{K}(C', SE(3)^M)$ is diffeomorphic to $\mathcal{K}(Q)$, where

$$C' = \{(g_1, \ldots, g_M) \mid g_{v_+(a)} = g_{v_-(a)} F a^{-1}(f_a), f_a \in Q_a, a \in A'\}. \quad (2.22)$$

Let the map $\Phi : SE(3)^M \to SE(3)^M$, be given by

$$\Phi(f') = (\Phi_1, \ldots, \Phi_v, \ldots, \Phi_M), \quad \Phi_v = \Sigma_{P(v)}(f'). \quad (2.23)$$

If we denote $G : SE(3)^M \to SE(3)^M$ as:

$$G(h) = (G_1, \ldots, G_v, \ldots, G_M), \quad G_v(h) = \Lambda_{P(v)}(h), \quad (2.24)$$

then we have: $\Phi = G \circ F^{-1}$. (We recall that the maps Λ_P and Σ_P were introduced in Notations 2.29).

Let the map $\Psi : SE(3)^M \to SE(3)^M$, be given by

$$\Psi(g) = (\Psi_1, \ldots, \Psi_a, \ldots, \Psi_N), \ \ \Psi_a(g) = F_a(g_{v_-(a)}{}^{-1} g_{v_+(a)}).$$

If we denote $H : SE(3)^M \to SE(3)^M$ as:

$$H(g_1, \ldots, g_v, \ldots, g_M) = (H_1, \ldots, H_a, \ldots, H_M), \ \ H_a(g) = g_{v_-(a)}{}^{-1} g_{v_+(a)},$$

we have $\Psi = F \circ H$.

We show that $G = H^{-1}$ and therefore $\Phi = \Psi^{-1}$. We need to prove that $G \circ H$ and $H \circ G$ are identity maps. Consider $G \circ H(g)$, $G_v(H(g)) = \Lambda_{P(v)}(H)$. Let the path $P(v)$ be $(a_1, \ldots, a_k)$ with vertices $(v_1, \ldots, v_k)$. Then, from Equation (2.21) and the definition of $P(v)$ (see Notations 2.29) we have:

$$\begin{gathered} G_v(H(g)) = (g_{v-(a_1)}{}^{-1} g_{v+(a_1)})^{\delta_{P(a_1)}} \cdots (g_{v-(a_k)}{}^{-1} g_{v+(a_k)})^{\delta_{P(a_k)}} = \\ (g_{v_1}{}^{-1} g_{v_2})(g_{v_2}{}^{-1} g_{v_3}) \cdots (g_{v_{k-1}}{}^{-1} g_{v_k}) = g_{v_k} = g_v \ . \end{gathered} \tag{2.25}$$

Therefore, $G(H(g)) = g$. For $H \circ G(h)$, we have

$$H_a(G(h)) = G_{v-(a)}{}^{-1} G_{v+(a)} = (\Lambda_{P(v_-(a))}(h))^{-1} (\Lambda_{P(v_+(a))}(h)). \tag{2.26}$$

Exactly one of the two paths, $P(v_-(a))$ and $P(v_+(a))$, contains the arc a as its last element. Either $P(v_+(a)) = (P(v_-(a)), a)$ and $\delta_{P(v_+(a))}(a) = 1$, or alternatively $P(v_-(a)) = (P(v_+(a)), a)$ and $\delta_{P(v_-(a))}(a) = -1$. In both cases, the right-hand side of Equation (2.26) equals h_a and hence $H_a(G(h)) = h$.

Therefore, Φ is invertible. Both Φ and its inverse, $\Psi = \Phi^{-1}$, are smooth maps. Indeed, both G and H are smooth maps since the group product and the inverse on $SE(3)$ are smooth. Thus, Φ is a diffeomorphism.

It remains to prove that $\Phi(Q') = C'$. Let $f' \in Q'$ and consider $\Phi(f') = (\Phi_1, \ldots, \Phi_M)$. Since $\Phi_v = G(F^{-1}(f'))$, we have, similarly to Equation (2.26):

$$\Phi_{v_-(a)}{}^{-1} \Phi_{v_+(a)} = G_{v_-(a)}{}^{-1}(F^{-1}(f')) G_{v_+(a)}(F^{-1}(f')) = (F^{-1})_a(f'). \tag{2.27}$$

Therefore, $(\Phi_1, \ldots, \Phi_M)$ satisfies the equations defining C', i.e., $\Phi(Q') \subset C'$.

On the other hand, if $g \in C'$ then $\Psi_a(g) = F_a(g_{v_-(a)}{}^{-1} g_{v_+(a)})$. Since, for every $g \in C'$, we must have $g_{v_-(a)}{}^{-1} g_{v_+(a)} \in F_a{}^{-1}(C')$, we conclude that $C' \subset \Phi(Q')$.

This shows that the restriction of Φ on Q' is a diffeomorphism between smooth manifolds and the lemma is proven. □

Proof of Theorem 2.30. (Continuation). To prove the statement for a kinematic chain with closed loops, where $N > M$, we construct a set , Q^*, diffeomorphic to Q', such that $D \subset Q^* \subset SE(3)^N$. We will show that there exists a diffeomorphism $\sigma : Q^* \to C'$ such that $\sigma(D) = C$. This will prove that $\mathcal{K}(D, SE(3)^N)$is diffeomorphic to $\mathcal{K}(C, SE(3)^M)$. Since $\mathcal{K}(D, Q)$ is diffeomorphic to $\mathcal{K}(D, SE(3)^N)$ by inclusion, the statement of the theorem would follow. (We recall that the diffeomorphism of mechanical systems is an equivalence relation and hence transitive, see Remark 2.11(2)).

Without loss of generality, we can assume that the arcs in Γ are numbered in such a way that $A = (A', A'')$. Let Q^* be given by:

$$Q^* = \{(f', f'') \mid f' \in Q', f_a'' = F_a((\Phi_{v_-(a)}(f'))^{-1}(\Phi_{v_+(a)}(f'))), \ a \in A''\}. \quad (2.28)$$

The set Q^* is a smooth bijective image of Q' and we denote by $\pi : Q^* \to Q'$ the diffeomorphism $\pi(f', f'') = f'$. Now we set $\sigma = \Phi \circ \pi$, defining a diffeomorphism between the M-dimensional manifolds $Q^* \subset SE(3)^N$ and $C' \subset SE(3)^M$.

It remains to prove that $D \subset Q^*$ and $\sigma(D) = C$. If $f \in D \subset Q, f = (f', f'')$, then for every fundamental loop L, we have $\Sigma_L(f) = e$. Therefore, for every chord, $a \in A''$, we can write:

$$(\Sigma_{P(v_-(a))}(f))(F_a{}^{-1}(f))(\Sigma_{P(v_+(a))}(f))^{-1} = e, \quad (2.29)$$

which implies

$$F_a{}^{-1}(f_a) = F_a{}^{-1}(f) = (\Sigma_{P(v_-(a))}(f))^{-1}(\Sigma_{P(v_+(a))}(f)) = (\Phi_{v_-(a)}(f'))^{-1}(\Phi_{v_+(a)}(f')), \quad (2.30)$$

and therefore,

$$f_a = F_a((\Phi_{v_-(a)}(f'))^{-1}(\Phi_{v_+(a)}(f'))), \ \forall \ a \in A''. \quad (2.31)$$

Equation (2.31) is equivalent to $f \in Q^*$ (cf. Equation (2.29)), and this proves that $D \subset Q^*$.

Finally, we show that if $f = (f', f'') \in D$ then $\sigma(f) = \Phi(f') \in C$, i.e., $\sigma(D) = C$. From the proof of Lemma 2.31, we know that $\Phi(f') \in C'$, since $\Phi(f')$ satisfies Equation (2.27) for all $a \in C'$. However, Equation 2.31 shows that this condition is satisfied for $a \in A''$ as well. Thus, we have

$$(\Phi_{v_-(a)}(f'))^{-1}(\Phi_{v_+(a)}(f')) \in F_a^{-1}(Q),\ \forall\ a \in A. \tag{2.32}$$

and therefore, $\sigma(D) \subset C$.

If $g \in C$, then $\sigma^{-1}(g) = \pi^{-1}(\Psi(g)) = (\Psi(g), \Psi^*(g))$, where Ψ^* has c components:

$$\Psi^*_a(g) = F_a(\Phi_{v_-(a)}(\Psi(g)))^{-1}(\Phi_{v_+(a)}(\Psi(g))) = F_a(g_{v_-(a)}{}^{-1}g_{v_+(a)}),\ a \in A''. \tag{2.33}$$

Let $L \in \mathcal{L}$ and let a be the chord in L. Then,

$$\Sigma_L(\sigma^{-1}(g)) = \Sigma_L(\Psi, \Psi^*) = \Sigma_{P(v_-(a))}(\Psi))(F_a^{-1}(\Psi^*_a))(\Sigma_{P(v_+(a))}(\Psi))^{-1}. \tag{2.34}$$

We recall that $\Phi_v = \Sigma_{P(v)}$ and we substitute Ψ^*_a from Equation (2.33) to obtain:

$$\Sigma_L(\sigma^{-1}(g)) = \Phi_{v_-(a)}(\Psi)F_a^{-1}(F_a(g_{v_-(a)}{}^{-1}g_{v_+(a)}))(\Phi_{v_+(a)}(\Psi))^{-1}. \tag{2.35}$$

We have $\Phi = \Psi^{-1}$, hence $\Phi_{v_-(a)}(\Psi) = g_{v_-(a)}$, and Equation (2.35) yields

$$\Sigma_L(\sigma^{-1}(g)) = g_{v_-(a)}(g_{v_-(a)}{}^{-1}g_{v_+(a)})g_{v_+(a)}{}^{-1} = e, \tag{2.36}$$

which proves that $\sigma^{-1}(g) \in D$ and $C \subset \sigma(D)$.

We have, therefore, shown that $\sigma(D) = C$, and completed the proof of the theorem. □

2.32. Remark. Theorem 2.30 shows that the articulated system $\mathcal{M} = \mathcal{A}(\Gamma, \rho, \gamma)$ is modeled by $(\mathcal{K}(D, Q), \sigma)$. This model, referred to as the *joint-space model* of the system, can be an alternative to the model based on the system $\mathcal{K}(C, SE(3)^M)$. (This second model can be called the *link-space model* of the system). The joint-space model is especially useful when most of the spaces Q_a are of dimension one, since then the dimension of Q may be significantly lower than $\dim SE(3)^M = 6M$. When the system $\mathcal{M}$ is described by $\mathcal{K}(D, Q)$, the elements of Q_a can be thought of as points of these manifolds rather than displacements in $SE(3)$. When this is the case, these elements will be referred to as *joint variables* (or joint parameters) and the notations used will be $q_a \in Q_a$ and $q = (q_1, \ldots, q_N) \in Q$.

2.7. Serial Chains

In this section, we address equivalent substitution of serial chains.

In a serial kinematic chain, the graph has a simple linear structure. Without loss of generality, we can assume that the arcs, $A = \{1, \ldots, M\}$, and vertices, $V = \{0, \ldots, M\}$, of the graph, (V, A, ν), of a serial kinematic chain are labeled in such a way that

$$\nu(a) = (\nu_-(a), \nu_+(a)) = (a - 1, a).$$

Such a graph will be denoted by $S(M)$.

In an articulated system, $\mathcal{S} = \mathcal{A}(S(M), \rho, \gamma)$, with a serial kinematic chain we will assume, without loss of generality, that the reference frames attached to the rigid bodies associated with the vertices are chosen in such a way that $\gamma_+(a) = e$ for all $a < M$.

For every vertex, $v = k$, the path from 0 to v is $P(v) = P(k) = (1, \ldots, k)$. Moreover, since all paths in $S(M)$ are composed of arcs with consecutive numbers, $P = (a + 1, \ldots, a + k)$, the maps Λ_P and Σ_P are given by:

$$\Lambda_P(h) = h_{a+1} \cdots h_{a+k},$$
$$\Sigma_P(f) = \Lambda_P(F^{-1}(f)) = \gamma_-(a + 1)f_{a+1} \cdot \cdots \gamma_-(a + k)h_{a+k}, \tag{2.37}$$
$$\Sigma_{P(k)}(f) = \gamma_-(1)f_1 \gamma_-(2)f_2 \cdot \cdots \gamma_-(k)h_k\gamma_+(k).$$

2.33. Definitions. *Let $\mathcal{S} = \mathcal{A}(S(M), \rho, \gamma)$ be a serial-chain articulated system and let $Q = Q_1 \times \cdots \times Q_M$ be the joint space of $\mathcal{S}$.*

(1) The mapping $\kappa : Q \to SE(3)$, $\kappa = \Sigma_{P(k)}\big|_Q$, is referred to as the **output map** *of the serial chain $\mathcal{S}$. The set $\kappa(Q)$ is the* **output space** *of $\mathcal{S}$. A kinematic pair, $\mathcal{P}$, is referred to as the* **substitute pair** *for $\mathcal{S}$ if $\mathcal{P} = \mathcal{K}(\{e\} \times \kappa(Q), SE(3)^2)$.*

(2) The system $\mathcal{S}$ is said to be a **substitute** *(system) of another serial-chain articulated system, $\mathcal{S}'$, (and vice versa) when the two systems have the same output spaces.*

(3) $\mathcal{S}$ is said to be a **diffeomorphic substitute** *of $\mathcal{S}'$ (and vice versa) when $\mathcal{S}$ is a substitute* and *it is diffeomorphic to $\mathcal{S}'$.*

2.34. Remarks.

(1) When two serial-chain systems are substitutes, the feasible locations of the rigid body associated with the last vertex of each of the chains are the same. Furthermore, if $\mathcal{S}$ is a subsystem in a larger articulated system, $\mathcal{M}$, then the substitution of $\mathcal{S}$ with $\mathcal{S}'$ would have no effect on the feasible position and orientation of any of the bodies in the system which are not part of $\mathcal{S}$. However, the new system obtained as a result of the substitution, $\mathcal{M}'$, will not be diffeomorphic to $\mathcal{M}$ unless $\mathcal{S}$ and $\mathcal{S}'$ are diffeomorphic substitutes.

(2) Of particular interest is the substitution of a kinematic pair by a serial-chain system. If $\mathcal{S}$ is a *diffeomorphic* substitute of a kinematic pair, $\mathcal{P}$, then the system, $\mathcal{M}'$, obtained by the replacement of $\mathcal{P}$ with $\mathcal{S}$ in a larger system, $\mathcal{M}$, can be considered equivalent to $\mathcal{M}$. The system $\mathcal{S}$ is a diffeomorphic substitute of a pair only if the map κ is bijective.

Many articulated systems of practical or theoretical importance have all their joints among the Reuleaux pairs. The image space of the joint-type distribution of such a system consists of the joint types shown in Table 2.1. It is, therefore, important to know whether some of the Reuleaux pairs of higher dimensions can be diffeomorphically substituted by a serial chain of Reuleaux pairs of dimension one. In fact, it can be seen that for pairs with joint types C and F there exist diffeomorphic substitutes with joint of types R and P. The following two propositions follow directly from the definition of the Releaux-pairs joint types in Example 2.24 and Definition 2.33.

2.35. Proposition. *Let $\mathcal{P}$ be a kinematic pair with joint $\mathcal{J} \in$ F. Then, there exists a serial-chain articulated system $\mathcal{S} = \mathcal{A}(S(3), \rho, \gamma)$, where $\rho(a) \in \{\mathrm{R}, \mathrm{P}\}$, such that $\mathcal{S}$ is a diffeomorphic substitute of $\mathcal{P}$.*

2.36. Proposition. *Let $\mathcal{P}$ be a kinematic pair with joint $\mathcal{J} \in$ C. Then, there exists a serial-chain articulated system $\mathcal{S} = \mathcal{A}(S(2), \rho, \gamma)$, where $\rho(a) \in \{\mathrm{R}, \mathrm{P}, \mathrm{H}\}$, such that $\mathcal{S}$ is a diffeomorphic substitute of $\mathcal{P}$.*

A serial-chain system with three revolute joints, whose axes intersect in one point, is a substitute of a pair with a spherical joint. However, this is not a diffeomorphic substitute.

2.37. Proposition. *Let $\mathcal{P}$ be a kinematic pair with joint $\mathcal{J} \in$ S. Then, there can be no serial-chain articulated system, $\mathcal{S} = \mathcal{A}(S(M), \rho, \gamma)$, where $\rho(a) \in \{\mathrm{R}, \mathrm{P}, \mathrm{H}\}$, such that $\mathcal{S}$ is a diffeomorphic substitute of $\mathcal{P}$.*

Proof. Such a system cannot have a joint of type P or H, since then the joint space would not be compact. On the other hand, if all joints in $\mathcal{S}$ are revolute, the joint space is homeomorphic to a torus, T^M, and therefore cannot be homeomorphic to $SO(3)$. □

It can be proven that for every M, any smooth map $\phi : T^M \to SO(3)$ has singularities (Gotlieb 1986). Therefore, the statement of Proposition 2.37 can be made even stronger. Namely, for the spherical pair, there is no substitute serial chain with a nonsingular output map.

2.8. Mobility

In this section, we define mobility, a concept which is widely used in mechanism theory, but is given only an essentially intuitive definition. Herein, we define mobility for arbitrary kinematic systems and make some observations valid for those kinematric systems, such as articulated systems, whose configuration spaces can be described as algebraic sets.

2.38. Definitions. *Let* $\mathcal{M} = \mathcal{K}(C, X)$ *be an abstract kinematic system with* $\dim X = n$.

(1) $x \in C$ *is a* **regular configuration of local mobility** *e,* $x \in C^{(e)}$*, if there is an open neighbourhood, U, of x in X such that:*

  (i) $U \cap C$ *is a smooth submanifold of X.*

  (ii) $\dim U \cap C = e$.

The set of all such configurations is denoted by $C^{(e)}$.

(2) $\mathcal{M}$ *has* **(global) mobility** $\mu(\mathcal{M})$, *if:*

  (i) There is a number, e, such that there exist regular configurations of mobility e, i.e. $C^{(e)} \neq \emptyset$.

  (ii) $\mu(\mathcal{M})$ *is the largest such number,* $\mu(\mathcal{M}) = \max\{e \mid C^{(e)} \neq \emptyset\}$.

(3) Let $\mathrm{Reg}\,\mathcal{M} = C^{(\mu(M))}$. *The elements of* $\mathrm{Reg}\,\mathcal{M}$ *are referred to as* **regular** *configurations of* $\mathcal{M}$. *The complement of* $\mathrm{Reg}\,\mathcal{M}$ *is denoted by* $\mathrm{Nrg}\,\mathcal{M}$.

When the configuration space C is a smooth manifold, all configurations are regular and the mobility of the system is equal to $\dim C$. There are kinematic systems for which the configuration space is immediately recognized as a smooth manifold. For example, an open-chain articulated system has a smooth configuration space provided that the configuration spaces of the individual joints, Q_a, are smooth (cf. Lemma 2.31).

For many other systems, including closed-chain articulated systems, the configuration space, C, is described as the vanishing set of a system of constraint equations. Then, C can be thought of as an algebraic set in some affine space. Indeed, Theorem 2.30 implies that the configuration space of an articulated system can always be defined as an algebraic set, provided that the joint configuration spaces, Q_a, are algebraic sets. To see this, we must recall that by virtue of Proposition 2.14, $SE(3)$ can be identified with an algebraic set in the affine space E^9. As an algebraic set, C is not guaranteed to be a smooth manifold, as is demonstrated by the system discussed in Example 2.12.

2.39. Proposition. *Let V be an algebraic set in E^n and let $\mathcal{M} = \mathcal{K}(V, E^n)$. If $\boldsymbol{x}$ is a nonsingular point of dimension e in C, then $\boldsymbol{x}$ is a regular configuration of local mobility e in $\mathcal{M}$.*

Proof. We recall that, according to Definition 1.20, $\boldsymbol{x} \in V$ is a nonsingular point of dimension e if, and only if, there exist $M - e$ polynomials, $p_1, \ldots, p_{M-e}$, in $\mathcal{J}(V)$ such that:

(i) Near $\boldsymbol{x}$, V is the vanishing set of the polynomials, i.e., for some neighbourhood U, we have $U \cap V = U \cap \{y \mid p_i(y) = 0,\ i = 1, \ldots, M - e\}$.

(ii) The polynomials have linearly independent gradients at $\boldsymbol{x}$.

According to the Implicit Function Theorem (Porteous 1981) a subset $V \subset E^n$ is a smooth manifold of dimension e near $\boldsymbol{x}$ if the conditions (i) and (ii) are satisfied for $M - e$ smooth functions p_i. Therefore, $\boldsymbol{x}$ is a regular configuration of mobility e. □

The converse is not true, as the following example indicates.

2.40. Example. Consider the system $\mathcal{K}(V, E^2)$, where V is the vanishing set of the equation $p(x, y) = y^3 + 2x^2y - x^4 = 0$. This is a cubic curve, which has a singularity at the point (0, 0). Indeed, the gradient of every polynomial in $\mathcal{J}(V)$ is zero at (0, 0). Yet, the curve is a smooth submanifold of E^2, and therefore all configurations of a mechanical system with configuration space V are regular.

2.41. Proposition. *Let V be an algebraic set in E^n and let $\mathcal{M} = \mathcal{K}(V, E^n)$. Then, the mobility of the kinematic system, $\mathcal{M}$, is equal to the dimension of its configuration space, i.e., $\mu(\mathcal{M}) = \dim V$.*

Proof. It is known that for any algebraic set, V, Nsg $V \neq \varnothing$ (Proposition 1.24(1)). Hence, there are points in V which are nonsingular of dimension $e = \dim V$. By Proposition 2.39, this implies that there exist regular configurations of mobility $e = \dim V$. Therefore, $\mu(\mathcal{M}) \geq \dim V$.

Let us assume that $\mu(\mathcal{M}) > \dim V$. Then, there exists an $x \in V$ and a neighbourhood, U, such that $U \cap V$ is a smooth manifold of dimension $d > \dim V$. This implies that $x \notin$ Nsg V and therefore $x \in$ Sing V. Moreover, the same is true for all points in $U \cap V$ and therefore $U \cap$ Sing $V = U \cap V$ is a smooth manifold of dimension d.

It is known that, for any V, Sing V is either the empty set or an algebraic set of dimension strictly smaller than dim V (Proposition 1.24(4)). Therefore, we can proceed by induction and prove that there exists a zero-dimensional algebraic set which contains $U \cap V$ in its singularity set. This is impossible, since the singularity set of a zero-dimensional algebraic set must be empty. □

2.42. Corollary. *Let V be an algebraic set in E^n and let $\mathcal{M} = \mathcal{K}(V, E^n)$. Then, the nonsingular points of the configuration space, V, are regular configurations of the kinematic system, $\mathcal{M}$, i.e.,* Nsg $V \subset$ Reg $\mathcal{M}$.

Proof. If $x \in$ Nsg V, then, by Proposition 2.39, x is a regular configuration of mobility $e = \dim V$. According to Proposition 2.41, $\dim V = \mu(\mathcal{M})$, and therefore x is a regular configuration of mobility $\mu(\mathcal{M})$, i. e., $x \in$ Reg $\mathcal{M}$. □

In particular, Corollary 2.42 shows that for articulated systems with lower pairs all nonsingular points of the configuration space are regular configurations.

2.9. Mechanism

In this section we define the term mechanism. An articulated system, $\mathcal{M}$, is referred to as a *mechanism*, when it is used as an input-output device for the transformation of motion. Two kinematic subsystems of $\mathcal{M}$ are specified: an input system, where the motions can be prescribed; and an output system, in which desirable motions must be obtained by choosing the motion in the input system.

Let $\mathcal{M} = \mathcal{A}(\Gamma, \rho, \gamma)$ be an articulated system with smooth joints (i.e., the sets Q_a are smooth manifolds). As we showed with Theorem 2.30, the articulated system, $\mathcal{M}$, can be modelled by two diffeomorphic kinematic systems: $\mathcal{K}(D, Q)$, the system of the feasible motions in joint space; or $\mathcal{K}(C, SE(3)^M)$, the space of the feasible link motions.

A subspace of the joint space, $I \subset Q$, is chosen as the *input space*. It is assumed that the input space has the structure $I = \prod_{a \in A_I} Q_a$, where A_I is a collection of arcs in the graph Γ. The joints that correspond to the arcs in A_I are referred to as *input joints* or *active joints* and it is assumed that their joint parameters can be actively controlled. The remaining joints are referred to as *passive*. Thus, the N-tuple $\boldsymbol{q}$ has two subsets: the active joint parameters $\boldsymbol{q}^a$; and the passive joint parameters $\boldsymbol{q}^p$. Since it has been assumed that the joint-configuration spaces are smooth manifolds, the output space, I, is a smooth submanifold as well. We denote $\dim I = n_I$. It is usual to assume that $n_I \geq \mu(\mathcal{M})$. The choice of the input space defines an *input projection*, $\pi_I : Q \to I$, which maps each configuration, $\boldsymbol{q}$, into the point $\boldsymbol{q}^a \in I$. The restriction of this map to the configuration space, D, is denoted by f_I and referred to as the *input map* of the mechanism. The kinematic system $\mathcal{K}(f_I(D), I)$ can be viewed as a subsystem of $\mathcal{K}(D, Q)$. The motions in this subsystem, the *input system* of the mechanism, are being actively selected and can be viewed as the control functions of the system $\mathcal{K}(D, Q)$.

The *output space*, O, is a chosen subspace of $SE(3)^M$, i.e of the space of possible locations of the links of the articulated system. For simplicity, we assume that O is a Cartesian factor of $SE(3)^M$. Since $SE(3)^M = \mathbf{R}^{3M} \times SO(3)^M$, O is of the form $\mathbf{R}^m \times SO(3)^n$. Thus, O is a smooth manifold and we denote $\dim O = n_O$. It is assumed that $n_O \le \mu(\mathcal{M})$.

In most practical applications, we have $O \subset SE(3)$, i.e., O is a subset of the copy of $SE(3)$ which corresponds to a chosen link associated with some vertex, v_O, of Γ. (In this case, O can be either the whole space, $SE(3)$, of displacements of the v_O-th link, or a proper submanifold of $SE(3)$ with dimension n_O.) The link v_O is referred to as the *output link* or the *end-effector* of the mechanism.

The choice of the output space, O, as a Cartesian factor of the link space defines an *output projection*, $\pi_O : SE(3)^M \to O$. For instance, when $O = SE(3)$ the output projection is $\pi_O(g_1, \ldots, g_M) = g_{v_O}$. The restriction of the map π_O to C will be denoted by g_O. The map $f_O : D \to O, f_O = g_O \circ \sigma|_D$ is referred to as the *output map* of the mechanism. These maps can be illustrated by the following diagram:

$$\begin{array}{ccccccccc} I & \overset{\pi_I}{\leftarrow} & Q & & SE(3)^M & \overset{\pi_O}{\rightarrow} & O \\ \cup & & \cup & & \cup & & \cup \\ \pi_I(D) & \overset{f_I}{\leftarrow} & D & \overset{\sigma|_D}{\rightarrow} & C & \overset{g_O}{\rightarrow} & \pi_O(C) \end{array} \tag{2.38}$$

We note that O need not be defined as a Cartesian factor of the link space. It would be sufficient to require that O is a submanifold of $SE(3)^M$ for which there exists a smooth map $\pi_O : SE(3)^M \to O$ such that $\operatorname{Im} \pi_O = O$ and $\pi_O|_O = \mathrm{id}_O$.

Similarly to the input system, we can view the kinematic system $\mathcal{K}(g_O(C), O) = \mathcal{K}(f_O(D), O)$ as a subsystem of $\mathcal{K}(C, SE(3)^n)$, and refer to it as the *output system* of the mechanism. The goal during the operation of the mechanism is is to achieve a desirable motion of the output system.

We can now summarize our definition of a mechanism:

2.43. Definitions. *Let $\mathcal{M}$ be an articulated system with smooth joints and let $\mathcal{K}(D, Q)$, and $\mathcal{K}(C, SE(3)^n)$ be the joint-space and link-space representations of $\mathcal{M}$. Let the submanifolds I and O be defined as the images of Q and $SE(3)^n$, respectively, under two chosen smooth surjective projections, π_I and π_O, as described above. Let f_I and f_O be the maps induced on D by these projections.*

(1) *The triple of kinematic systems $\mathfrak{M} = (\mathcal{M}, \mathcal{M}_I, \mathcal{M}_O)$, where $\mathcal{M}_I = \mathcal{K}(f_I(D), O)$ and $\mathcal{M}_O = \mathcal{K}(f_O(D), O)$, is referred to as a* **mechanism** *with articulated system $\mathcal{M}$, input system $\mathcal{M}_I$ and output system $\mathcal{M}_O$.*

(2) *The space D is referred to as the* **configuration space** *of the mechanism. The space I (respectively O) is said to be the* **input** *(respectively* **output***)* **space** *of* $\mathfrak{M}$, *while f_I (respectively f_O) is the* **input** *(respectively* **output***)* **map** *of* $\mathfrak{M}$.

(3) *The number $\mu = \mu(\mathcal{M}) = \dim D$ is referred to as the* **mobility** *of* $\mathfrak{M}$. *The mechanism is said to be* **non-redundant** *when $n_I = \mu = n_O$, where $n_I = \dim I$ and $n_O = \dim O$. When $n_I > \mu$, $\mathfrak{M}$ is said to be* **dynamically redundant** *(or an actuator redundancy is said to be present); if $\mu > n_O$ the mechanism is* **kinematically redundant** *(configuration-space redundancy is present).*

2.44. Definition. *Let $\mathfrak{M}$ be the mechanism defined in Definition 2.43.*

(1) *A configuration, $q \in D$, is said to be a* **nonsingular configuration of** $\mathfrak{M}$, *if both of the following conditions are satisfied:*

(i) $q \in \mathrm{Reg}\, \mathcal{M}$.

(ii) *Assuming that (i) is correct, let $U \subset Q$ be the neighbourhood of q such that $V = U \cap D$ is a smooth submanifold of Q of dimension n. Then, the restrictions of the maps f_I and f_O on V, i.e., the smooth*

mappings $f_I|_V : V \to I$ and $f_O|_V : V \to O$ must have Jacobian matrices of maximum rank.

Otherwise, q is said to be a **singular configuration** *(or a* **singularity***) of the mechanism.*

(2) *If $q \in$ Nrg $\mathcal{M}$, i.e., if condition (i) is violated, then q is a* **configuration-space singularity** *of* $\mathfrak{M}$.

(3) *If $q \in$ Reg $\mathcal{M}$ but the map $f_I|_V$ (respectively $f_O|_V$) is singular at q then the configuration is referred to as an* **input** *(respectively,* **output***)* **singularity** *of* $\mathfrak{M}$.

2.45. Remarks.

(1) We note that, according to Definition 2.43(1), the term "mechanism" is not synonymous to "articulated system". There exist many different mechanisms having the same articulated system, and in principle they may have completely different singularities.

(2) The definitions in the present section were formulated to apply to articulated systems with smooth joints, since these are the usual subject of mechanism theory. It can be noted, however, that Definitions 2.43 and 2.44 can be generalized for abstract kinematic systems. Thus an abstract "mechanism" is given by a system $\mathcal{K}(C, X)$, two submanifolds, I and O, of X and two smooth mappings, f_I and f_O, defined on some open set containing C. Since $\mu(\mathcal{M})$ was defined for arbitrary systems, singularity and redundancy can also be defined for abstract mechanisms.

2.10. Summary

In the present chapter, we have derived the basic notions and facts of mechanism theory, using as starting points the properties of the Euclidian group of isometries of the real affine space, introduced in Section 2.2, and the concept of an abstract kinematic system, defined in Section 2.3. Section 2.4 addresses the rigid body as a system of particles and provides a description of the configuration space of this system (Proposition 2.14). In Section 2.5, systems of rigid bodies are introduced, including precise definitions of a kinematic joint and a kinematic pair as kinematic systems. Articulated systems are the focus of Section 2.6, where we show that every articulated system has two diffeomorphic models, the joint-space and link-space representations. Section 2.7 discusses equivalent substitutions of serial chains and introduces the concept of diffeomorphic substitution. A novel definition of mobility of kinematic systems (and articulated systems in particular) is the focus of Section 2.8. Finally, Section 2.9 describes mechanisms and their input and output maps, and defines mechanism singularity, which is the central topic of the thesis.

CHAPTER 3

INSTANTANEOUS SINGULARITY ANALYSIS OF NON-REDUNDANT MECHANISMS

3.1. Introduction

In this chapter, mechanism singularity is analyzed from the viewpoint of instantaneous kinematics. The velocity kinematics is modelled using tangent spaces and Jacobian maps. The model is then applied for the classification of singularities.

The approach is applicable to the singularity analysis of non-redundant mechanisms with arbitrary kinematic chains and an equal number of inputs and outputs. The main features of this approach are as follows:

(i) The starting point of the singularity analysis is a system of linear equations (the *velocity equation*) including explicitly the passive-joint velocities. Such a system of equations can be obtained for any mechanism and therefore can be used for the practical identification of singularities.

(ii) A general definition of singularity of non-redundant mechanisms is utilized. A configuration is defined as singular when the kinematics of the mechanism is indeterminate with respect to either the input or the output velocities.

(iii) Singularities are classified on the basis of the physical (kinematic) phenomena that occur in such configurations, rather than on the sole basis of the mathematical concept of degenerating Jacobians.

The velocity equation is introduced in Section 3.2 and the definition of singularity from Chapter 2 is given a new infinitesimal interpretation in Section 3.3. Six types of singular configurations are defined in Section 3.4 and illustrated with the help of a 6-dof mechanism in Section 3.5. The motion-space interpretation of kinematic singularity, introduced in Section 3.6, is used to obtain a comprehensive singularity classification in Section 3.7.

3.2. Infinitesimal Model of Mechanism Kinematics

In Chapter 2, we showed that a mechanism can be viewed as a device targeted for the transformation of motions in the input system into motions in the output system. This approach, which emphasizes the local and global properties of the systems, provides insight into the position kinematics of mechanisms.

Instantaneous kinematics, on the other hand, regards the mechanism as a device for the transformation of instantaneous motion, i.e., for the control of the output velocity via the input velocities.

The global kinematic model of a mechanism, $\mathfrak{M}$, which we developed in Chapter 2, is given by the configuration space, D, defined as a subset of the joint space manifold, Q; the input space I; the output space, O; as well as the input and output maps, f_I and f_O. These two maps determine the relationship between the input and output parameters, and therefore describe the position kinematics of the mechanism.

The instantaneous kinematic model of $\mathfrak{M}$, at a fixed configuration $\boldsymbol{q} \in D$, is obtained by replacing the spaces in the global model by their tangent spaces and the maps by their Jacobians. The tangent spaces $T_{\boldsymbol{q}}Q$, $T_{\boldsymbol{q}}I$ and $T_{\boldsymbol{q}}O$ are well defined for any $\boldsymbol{q}$ since Q, I and O are smooth manifolds. The configuration space, D, however, may not be a smooth manifold near $\boldsymbol{q}$. Then, $T_{\boldsymbol{q}}D$ does not exist. If D is the vanishing set of a system of equations, we can replace $T_{\boldsymbol{q}}D$ with the null space of the Jacobian of this system of equations.

In this chapter, we consider a mechanism, $\mathfrak{M}$, with N 1-dof lower-pair joints. (There is no loss of generality since for any mechanism there is an instantaneous substitute mechanism with lower pairs (Hunt 1978)). As we pointed out in Chapter 2, the configuration space, D, of such a mechanism is a real algebraic set. Therefore, the number n, defined as the smallest possible dimension of the Zariski-tangent space at a point of D (cf. Definition 1.21, Notations 1.22), exists. We assume that the mechanism is non-redundant and $\mu = n_I = n_O = n$.

We adopt the following notations: A tangent vector of Q, i.e., an element of T_qQ, will be denoted by Ω. An *output vector* (an element of T_qO) and an *input vector* (element of T_qI) will be denoted, respectively, by $\boldsymbol{T}$ (the output twist), and $\boldsymbol{\Omega}^a$ (the active-joint velocities). The symbol $\boldsymbol{\Omega}^p$ will be used for the vector of passive-joint velocities. Also, hereafter, the tangent spaces T_qO and T_qI, will be denoted by $\mathcal{O}$ and $\mathcal{I}$, while $\mathcal{P}$ will be the space of all the vectors $\boldsymbol{\Omega}^p$. The dimensions of the vector spaces $\mathcal{I}$, $\mathcal{P}$ and $\mathcal{O}$ (and of the vectors $\boldsymbol{\Omega}^a$, $\boldsymbol{\Omega}^p$ and $\boldsymbol{T}$) are n, $N - n$, and n, respectively. We define a combined $(N + n)$-dimensional velocity vector, $\boldsymbol{m} = (\boldsymbol{T}, \boldsymbol{\Omega}) = (\boldsymbol{T}, \boldsymbol{\Omega}^a, \boldsymbol{\Omega}^p)$.

The definition of the output space in Section 2.10 implies that the differential output in any configuration is an explicit linear function of the joint velocities:

$$\boldsymbol{T} = \boldsymbol{A}(\boldsymbol{q})\boldsymbol{\Omega}. \tag{3.1}$$

In fact, the matrix A is the Jacobian of the smooth map $p \circ \pi_O$ defined in Section 2.10. If each of the output velocities is a component of the twist of the output link with respect to the fixed link, then Equation (3.1) is obtained by expressing $\boldsymbol{T}$ as a sum of joint twists (Davies, 1981). Equation (3.1) will be referred to as the *output equation* of the mechanism.

For any closed loop, the sum of the joint twists of all the kinematic pairs in the loop is zero. Hence, each loop imposes 6 linear equations for the joint velocities. A set of joint velocities will be feasible if and only if it satisfies these equations for all loops in the chain.

However, the system of all loop equations is equivalent to a system of equations obtained from a set of c independent loops. Therefore, by specifying c independent loops and writing the twist equations for each of them, the following system of $6c$ equations is obtained as a necessary and sufficient condition for the feasibility of $\boldsymbol{\Omega}$:

$$C(\boldsymbol{q})\Omega = 0, \tag{3.2}$$

where $C(\boldsymbol{q})$ is a $6c \times N$ matrix. The corank of $C(\boldsymbol{q})$, which we denote by n_q (i.e., rank $C(\boldsymbol{q}) = N - n_q$), is referred to as the *instantaneous mobility* at $\boldsymbol{q}$. By definition, the Zariski tangent space of D at $\boldsymbol{q}$ contains vectors normal to the gradients at $\boldsymbol{q}$ of all functions vanishing on D (cf. Definition 1.22). Therefore $T_q^{\mathrm{Zar}}D$ contains the kernel of $C(\boldsymbol{q})$ and thus we have $n \leq n_q$, for all $\boldsymbol{q}$. This implies that for any fixed $\boldsymbol{q}$, the vector Equation (3.2) can be transformed into an equivalent system of $N - n$ equations, which we denote:

$$D(\boldsymbol{q})\Omega = 0, \tag{3.3}$$

where $D(\boldsymbol{q})$ is a $(N-n) \times N$ matrix. (A discussion of twist equations like (3.2) and (3.3) for multi-loop chains can be found in (Baker 1980) and (Davies 1981), including a derivation of (3.3) as a mechanical analogy of Kirchhoff's circulation law).

Combining the $N - n$ equations of (3.3) with the n equations of (3.1), we obtain N linear equations which fully determine the instantaneous kinematics of the mechanism. The definition of the matrix $L(\boldsymbol{q})$ as:

$$L(\boldsymbol{q}) = \begin{bmatrix} I_{n \times n} & A(\boldsymbol{q}) \\ 0_{(N-n) \times n} & D(\boldsymbol{q}) \end{bmatrix}, \tag{3.4}$$

completes the proof of the following theorem:

3.1. Theorem. *For any given configuration,* $\boldsymbol{q}$*, an* $N \times (N+n)$ *matrix,* $L(\boldsymbol{q})$*, can be found, such that a velocity vector,* $\boldsymbol{m}$*, is a feasible motion vector of the mechanism if, and only if,*

$$L(\boldsymbol{q})\boldsymbol{m} = 0. \tag{3.5}$$

Equation (3.5) will be referred to as the **velocity equation** *of the mechanism for the configuration* q.

We remark that the rank of $L(q)$ is greater than the rank of $D(q)$ (or $C(q)$) by exactly n, i.e., $\operatorname{rank} L(q) = N + n - n_q$.

3.2. ***Example.*** Let us consider the velocity equation of the four-bar linkage shown in Figure 3.1.

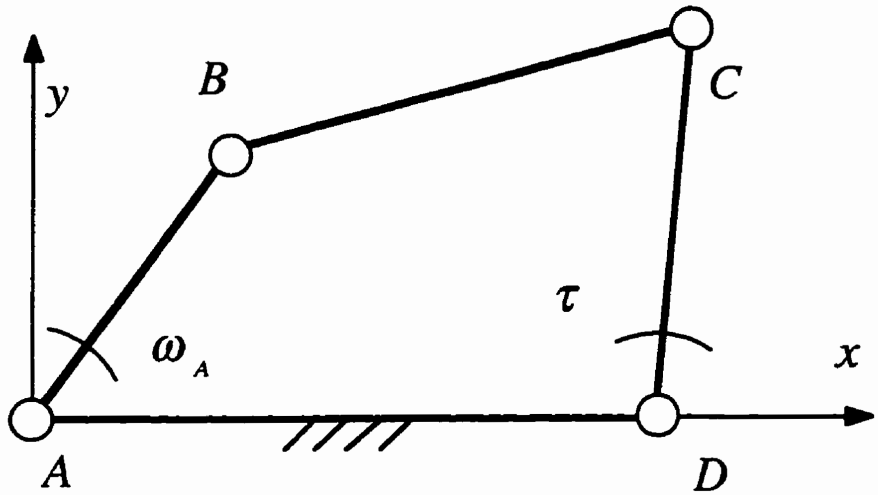

Figure 3.1. A four-bar mechanism.

There is only one loop and $c = 1$. The loop equation is:

$$\omega_A \mathbf{S}_A + \omega_B \mathbf{S}_B + \omega_C \mathbf{S}_C + \omega_D \mathbf{S}_D = 0, \tag{3.6}$$

where ω_P, $\mathbf{S}_P$ $(P = A, B, C, D)$ are the joint velocities and the joint screws, respectively. Only the planar components of the joint screws are nonzero. For the 6-dimensional space of twists we use the standard basis composed of the three rotations and three translations about the coordinate axes of a Cartesian reference frame. If we set the Cartesian reference frame with two of its axes lying in the plane of the mechanism, only three of the coordinates of the joint screws will be nonzero. Thus, whenever a planar linkage is considered, we shall assume that the joint screws are three-dimensional vectors. For this mechanism, we have $\mathbf{S}_P = (1,\ y_P,\ -x_P)$, where x_P, y_P are the coordinates of point P, $P = A, B, C, D$.

In this example, and everywhere else in this dissertation, when a four-bar linkage *ABCD* is considered, it will be assumed, unless the opposite is specified explicitly, that *AB* is the input link while *CD* is the output link and also that the joint velocity at A, ω_A, is the input, while the angular velocity of *CD*, τ, is the output. The output equation is:

$$\tau = -\omega_D . \tag{3.7}$$

Therefore, the velocity equation is:

$$\begin{bmatrix} -1 & 0 & 0 & 0 & -1 \\ 0 & \mathbf{S}_A & \mathbf{S}_B & \mathbf{S}_C & \mathbf{S}_D \end{bmatrix} \begin{bmatrix} \tau \\ \omega_A \\ \omega_B \\ \omega_C \\ \omega_D \end{bmatrix} = 0. \tag{3.8}$$

If point A is the origin and the x-axis is along *AD*, Equation (3.8) can be written as:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & y_B & y_C & 0 \\ 0 & 0 & -x_B & -x_C & -x_D \end{bmatrix} \begin{bmatrix} \tau \\ \omega_A \\ \omega_B \\ \omega_C \\ \omega_D \end{bmatrix} = 0. \tag{3.9}$$

3.3. ***Example.*** Let us consider the serial-chain 3-dof manipulator shown in Figure 3.2. There are no loops (and no passive joints) and the velocity equation is equivalent to the output equation:

$$\mathbf{T} = \omega_A \mathbf{S}_A + \omega_B \mathbf{S}_B + \omega_C \mathbf{S}_C . \tag{3.10}$$

In the twist Equation (3.10), only three of the six components are nonzero. As in Example 3.2, we can treat the screw vectors in (3.10) as three-dimensional. The matrix L can be written as:

$$L = \begin{bmatrix} -I_{3\times3} & \mathbf{S}_A & \mathbf{S}_B & \mathbf{S}_C \end{bmatrix}. \tag{3.11}$$

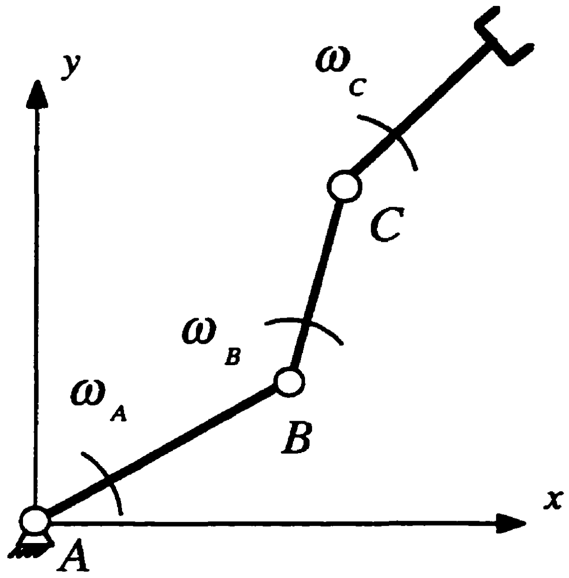

Figure 3.2. A 3-dof planar manipulator.

If point A is the origin, Equation (3.11) can be rewritten as:

$$\begin{bmatrix} -1 & 0 & 0 & 1 & 1 & 1 \\ 0 & -1 & 0 & 0 & y_B & y_C \\ 0 & 0 & -1 & 0 & -x_B & -x_C \end{bmatrix} \begin{bmatrix} \omega \\ v_x \\ v_y \\ \omega_A \\ \omega_B \\ \omega_C \end{bmatrix} = 0. \tag{3.12}$$

3.4. Example. Let us consider the 3-dof planar parallel manipulator shown in Figure 3.3. For each one of the three serial subchains connecting the base and the end-effector, we can express the twist of the end-effector as the sum of the joint twists:

$$\mathbf{T} = \sum_{i=0}^{2} \omega_i^A \mathbf{S}_i^A = \sum_{i=0}^{2} \omega_i^B \mathbf{S}_i^B = \sum_{i=0}^{2} \omega_i^C \mathbf{S}_i^C. \tag{3.13}$$

The first equality in (3.13) can be regarded as the output equation of the manipulator, and the second and the third as the loop equations of the loops $A_0A_2B_2B_0$ and $B_0B_2C_2C_0$. Therefore, the nine scalar equations in (3.13) are the velocity equations for this linkage. Rewriting the nine scalar equations yields the velocity equation in the form of (3.6) as:

$$\begin{bmatrix} -I_{3\times3} & \mathbf{S}_1^A & 0 & 0 & \mathbf{S}_0^A & 0 & 0 & \mathbf{S}_2^A & 0 & 0 \\ -I_{3\times3} & 0 & \mathbf{S}_1^B & 0 & 0 & \mathbf{S}_0^B & 0 & 0 & \mathbf{S}_2^B & 0 \\ -I_{3\times3} & 0 & 0 & \mathbf{S}_1^C & 0 & 0 & \mathbf{S}_0^C & 0 & 0 & \mathbf{S}_2^C \end{bmatrix} \begin{bmatrix} T \\ \Omega^a \\ \Omega^p \end{bmatrix} = 0, \quad (3.14)$$

where $T = [\omega, v_x, v_y]^T$, $\Omega^a = [\omega_1^A, \omega_1^B, \omega_1^C]^T$, and $\Omega^p = [\omega_0^A, \omega_0^B, \omega_0^C, \omega_2^A, \omega_2^B, \omega_2^C]^T$.

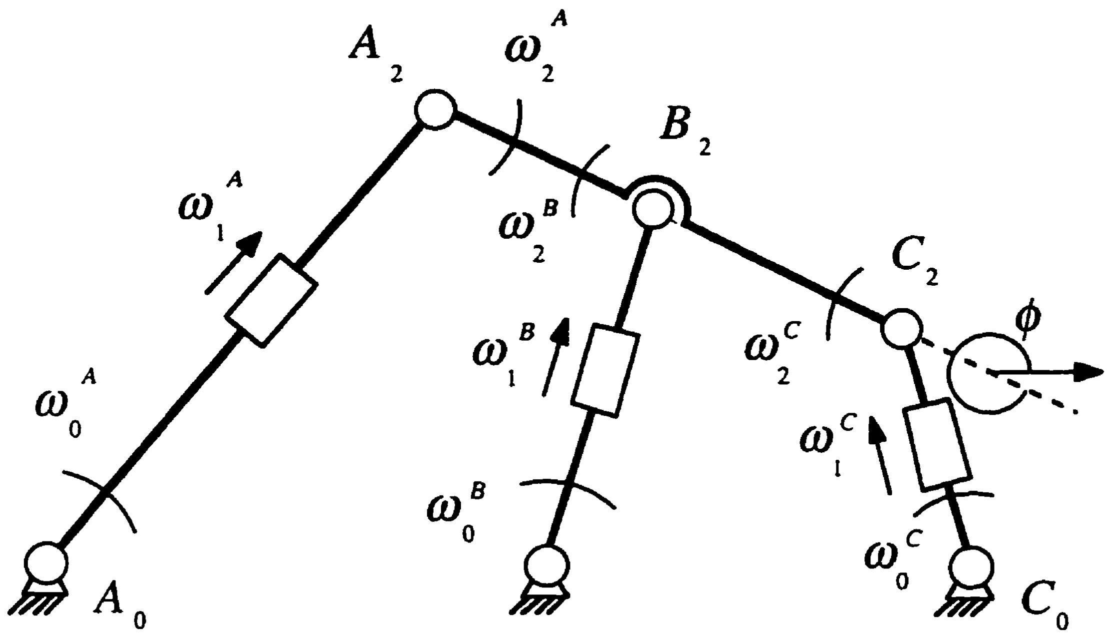

Figure 3.3. A planar parallel manipulator.

3.3. Instantaneous Definition of Singularity for Non-Redundant Mechanisms

The instantaneous-kinematics analysis of an input-output device addresses two main problems:

(i) The *forward* instantaneous kinematics problem (FIKP): where for a given configuration $\boldsymbol{q}$, the instantaneous motion of the mechanism is determined when the input $\boldsymbol{\Omega}^a$ is given; and,

(ii) The *inverse* instantaneous kinematics problem (IIKP): where for a given configuration $\boldsymbol{q}$, the instantaneous motion of the mechanism is determined when the output T is given.

For a non-redundant serial-chain robotic manipulator (i.e., a non-redundant mechanism with a serial-chain articulated system), it is well known that singularity occurs when the Jacobian is not invertible and the inverse instantaneous kinematics is indeterminate. Analogously, for non-redundant mechanisms, the singularity definition from Chapter 2 implies that singularity occurs whenever the instantaneous kinematics becomes indeterminate. Thus, a configuration is nonsingular, when *both* the forward instantaneous kinematic problem (FIKP) and the inverse instantaneous kinematic problem (IIKP) have unique solutions for any input or output.

3.5. ***Definition.*** *Let* $\boldsymbol{q}$ *be a feasible configuration of the mechanism.*

(1) *It is said that the* ***FIKP*** *is* **solvable** *at* $\boldsymbol{q}$, *if there exist matrices* $\boldsymbol{J}_{\mathrm{F}}$ *and* $\boldsymbol{P}_{\mathrm{F}}$ *of dimensions* $n \times n$ *and* $(N-n) \times n$ *respectively, such that the velocity equation is equivalent to the system:*

$$\begin{aligned} \boldsymbol{T} &= \boldsymbol{J}_{\mathrm{F}}\boldsymbol{\Omega}^a, \\ \boldsymbol{\Omega}^p &= \boldsymbol{P}_{\mathrm{F}}\boldsymbol{\Omega}^a. \end{aligned} \tag{3.14}$$

(2) *It is said that the* ***IIKP*** *is* **solvable** *at* $\boldsymbol{q}$, *if there exist matrices* $\boldsymbol{J}_{\mathrm{I}}$ *and* $\boldsymbol{P}_{\mathrm{I}}$ *of dimensions* $n \times n$ *and* $(N-n) \times n$ *respectively, such that the velocity equation is equivalent to the system:*

$$\begin{aligned} \boldsymbol{\Omega}^a &= \boldsymbol{J}_{\mathrm{I}}\boldsymbol{T}, \\ \boldsymbol{\Omega}^p &= \boldsymbol{P}_{\mathrm{I}}\boldsymbol{T}. \end{aligned} \tag{3.15}$$

(3) *If both FIKP and IIKP are solvable, the configuration is said to be* **nonsingular**, *otherwise it is a* **singular** *configuration.*

In a nonsingular configuration, both Jacobians J_F and J_I will be nonsingular and $J_I = J_F^{-1}$. However, it should be noted that, according to this definition, the existence of an invertible J_F, an $n \times n$ matrix, such that $T = J_F \Omega^a$ for any feasible pair (T, Ω^a), is not a sufficient condition for declaring that the configuration is nonsingular, unless the existence of the matrices P_F and P_I has also been established.

The formulation of singularity in terms of the velocity equation, given by Definition 3.5 allows the recognition of six substantially different types of singularities.

3.4. Definition of Singularity Types

Herein, six types of singular configurations are defined and illustrated by examples.

*3.6. **Definition.** A configuration is a singularity of* **redundant input** *(RI) type, if there exist a nonzero input, $\Omega^a \neq 0$, and a vector of passive-joint velocities, Ω^p, which satisfy the velocity equation for a zero-output, $T = 0$, i.e.,*

$$L \begin{bmatrix} 0 \\ \Omega^a \\ \Omega^p \end{bmatrix} = 0. \tag{3.16}$$

*3.7. **Proposition.** (Correctness of Definition 3.6)*

All RI-type configurations are singular.

Proof. We note that if q satisfies Definition 3.6 then it can not satisfy Definition 3.5 (2). Indeed, by Definition 3.6 it follows that the triple $T = 0$, $\Omega^a \neq 0$, Ω^p represents a feasible instantaneous motion at q. However, this motion clearly violates the first equation in (3.15) and therefore Equation (3.15) is not equivalent to the velocity equation of the mechanism (since there are feasible motions for which the Equation fails). Therefore, the IIKP is not solvable and $\boldsymbol{q}$ is a singularity. □

3.8. ***Example.*** The RI singularity type is illustrated by a four-bar linkage, Figure 3.4. In the configuration shown, the output link *CD* cannot move, since the velocity of point *C* must be zero. The instantaneous input, ω_A, however, can have any value. Therefore, Equation (3.16) holds, and an RI-type singularity exists, where the IIKP is insolvable.

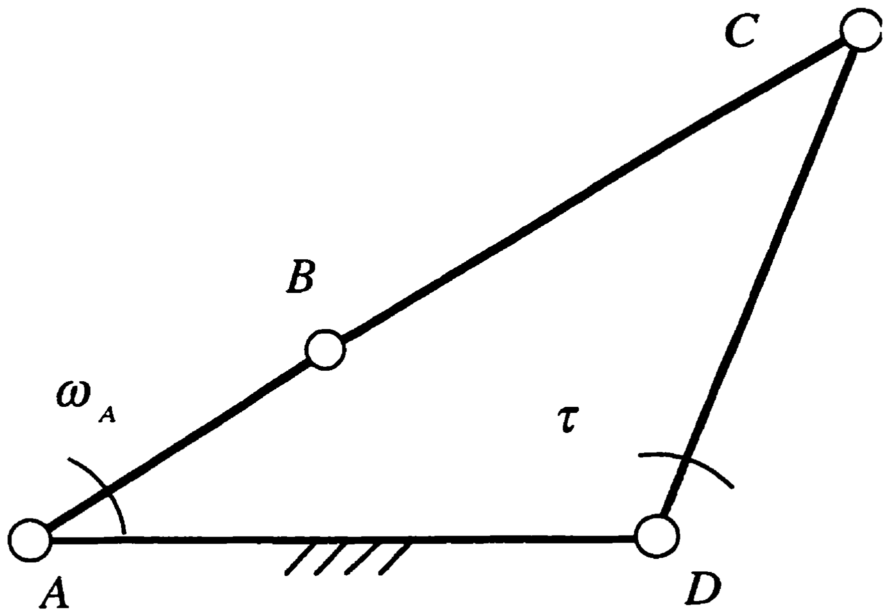

Figure 3.4. A four-bar mechanism in an RI- and IO-type singular configuration.

3.9. ***Definition.*** *A configuration is a singularity of* **redundant output** *(RO) type, if there exist a nonzero output,* $T \neq 0$*, and a vector of passive-joint velocities,* Ω^p*, which satisfy the velocity equation for a zero-input,* $\Omega^a = 0$*:*

$$L \begin{bmatrix} T \\ 0 \\ \Omega^p \end{bmatrix} = 0. \tag{3.17}$$

3.10. ***Proposition.*** *(Correctness of Definition 3.9)*

All RO-type configurations are singular.

Proof. If $\boldsymbol{q}$ satisfies Definition 3.9, then it can not satisfy Definition 3.5 (1). Indeed, by Definition 3.9 it follows that the triple $T \neq 0$, $\Omega^a = 0$, Ω^p represents a feasible instantaneous motion at $\boldsymbol{q}$. However, this motion clearly violates the first equation in (3.14) and therefore Equation (3.14) cannot be equivalent to the velocity equation of the mechanism. Hence, the FIKP is not solvable and $\boldsymbol{q}$ is a singularity. □

3.11. Example. Let us consider the four-bar linkage configuration shown in Figure 3.5. In the configuration shown, the input link *AB* is locked, while the instantaneous output, τ, can have any value. Thus, Equation (3.17) holds, and an RO-type singularity exists, where the FIKP is insolvable.

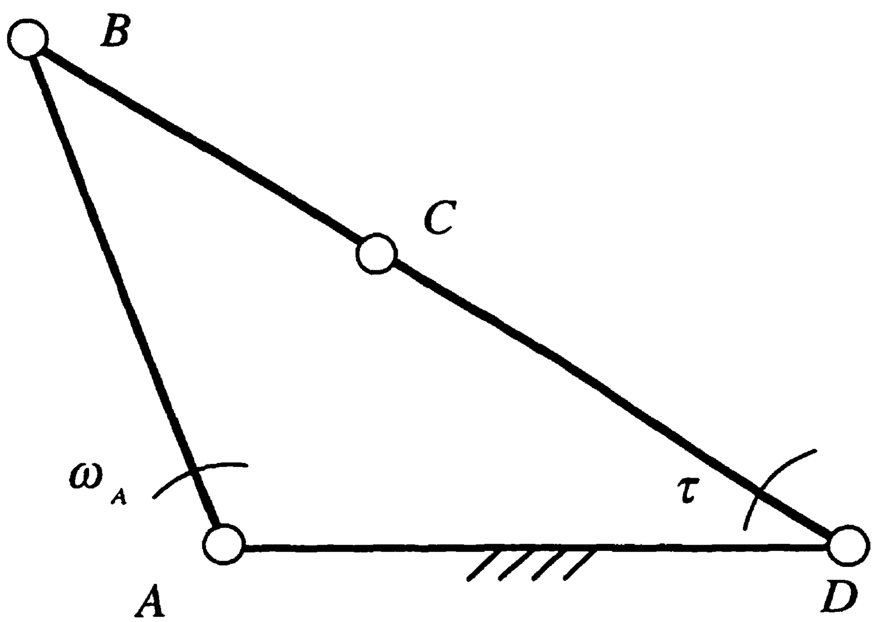

Figure 3.5. A four-bar mechanism in an RO- and II-type singular configuration.

3.12. Definition. *A configuration is a singularity of* **impossible input** *(II) type, if there exists a vector* Ω^a *for which the velocity equation cannot be satisfied for any combination of* $\boldsymbol{T}$ *and* Ω^p.

3.13. Proposition. *(Correctness of Definition 3.12)*

All II-type configurations are singular.

Proof. If $\boldsymbol{q}$ satisfies Definition 3.12, then the existence of an impossible input vector implies that the velocity equation cannot be written in the form of Equation (3.14) (since (3.14) allows for arbitrary values of $\boldsymbol{T}$). Thus, $\boldsymbol{q}$ violates the condition of Definition 3.5 (1) and is, therefore, a singularity. We note that an II-type singularity implies an insolvable FIKP. ☐

3.14. Example. The configuration in Figure 3.5 is an II-type singularity (in addition to being an RO-type singularity, as discussed in Example 3.11), since any nonzero input is impossible.

3.15. Definition. *A configuration is a singularity of* **impossible output** *(IO) type, if there exists a vector* $\boldsymbol{T}$ *for which the velocity equation cannot be satisfied for any combination of* Ω^a *and* Ω^p.

3.16. Proposition. *(Correctness of Definition 3.15)*

All IO-type configurations are singular.

Proof. Similarly to the proof of Proposition 3.13, it can be seen that the existence of an impossible output vector implies that the velocity equation cannot be written in the form of Equation (3.15) and hence an IO-type singularity implies an insolvable IIKP. ☐

3.17. Example. The configuration in Figure 3.4 is an IO-type singularity (as well as an RI-type singularity, as we showed in Example 3.8), since any nonzero output is impossible.

3.18. Definition. *A configuration is a singularity of* **increased instantaneous mobility** *(IIM) type, if* rank $\boldsymbol{L} < N$.

3.19. Proposition. *(Correctness of Definition 3.18)*

All IIM-type configurations are singular.

Proof. In an IIM-type singularity both the FIKP and the IIKP are insolvable. Indeed, when the velocity equation is in either of the forms (3.14) or (3.15), the matrix $L(q)$ contains unit matrix of dimension N as a submatrix and rank $L(q) = N$. □

Since rank $L = N + n - n_q$, an IIM-type singularity is, in fact, an uncertainty configuration (Hunt 1978), where the instantaneous mobility is greater than the full-cycle mobility $(n < n_q)$.

3.20. Example. Let us consider the four-bar mechanism shown in its "flattened" configuration in Figure 3.6, where it obtains a transitory mobility of 2, thus, having an IIM-type singularity. (It can be noted that this configuration also belongs to the singularity types RI and RO.)

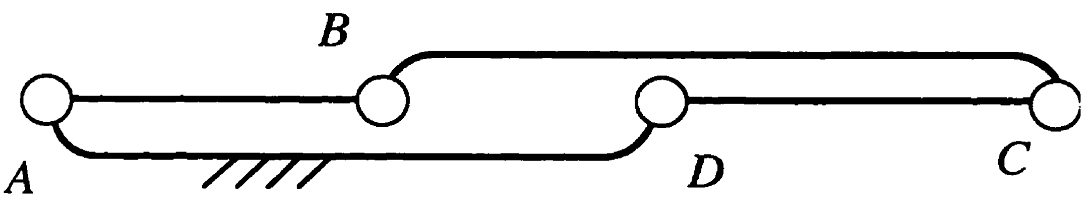

Figure 3.6. A four-bar mechanism in an IIM-, RI- and RO-type singular configuration.

3.21. Definition. *A configuration is a singularity of* **redundant passive motion** *(RPM) type, if there exists a nonzero passive-joint-velocity vector,* $\Omega^P \neq 0$, *which satisfies the velocity equation for a zero input and a zero output, i.e.,*

$$L \begin{bmatrix} 0 \\ 0 \\ \Omega^P \end{bmatrix} = 0. \qquad (3.18)$$

3.22. Proposition. *(Correctness of Definition 3.21)*

All RPM-type configurations are singular.

Proof. In an RPM-type configuration both the FIKP and the IIKP are insolvable. Indeed, if (3.18) is valid, neither of the matrices P_F and P_I can exist. □

3.23. Example. Let us consider the 1-dof slider, shown in Figure 3.7a.

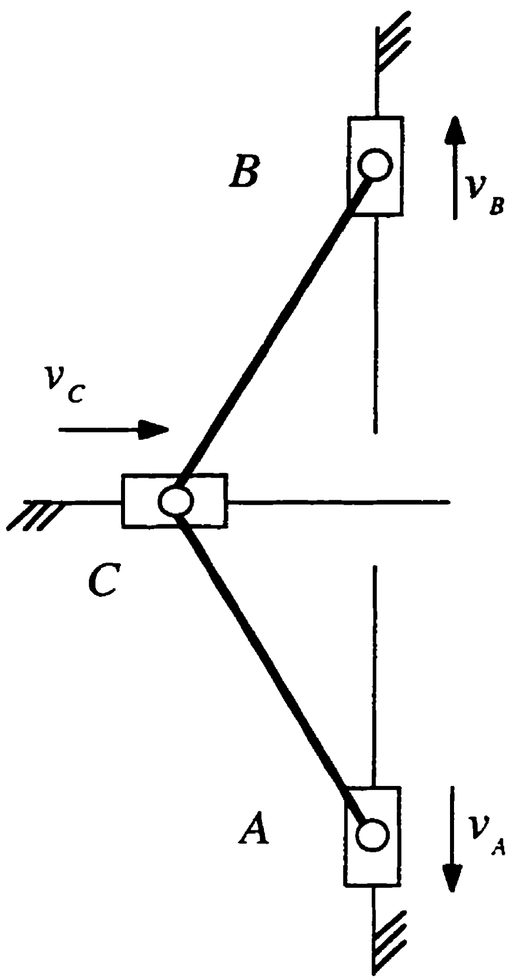

Figure 3.7a.
A 1-dof slider.

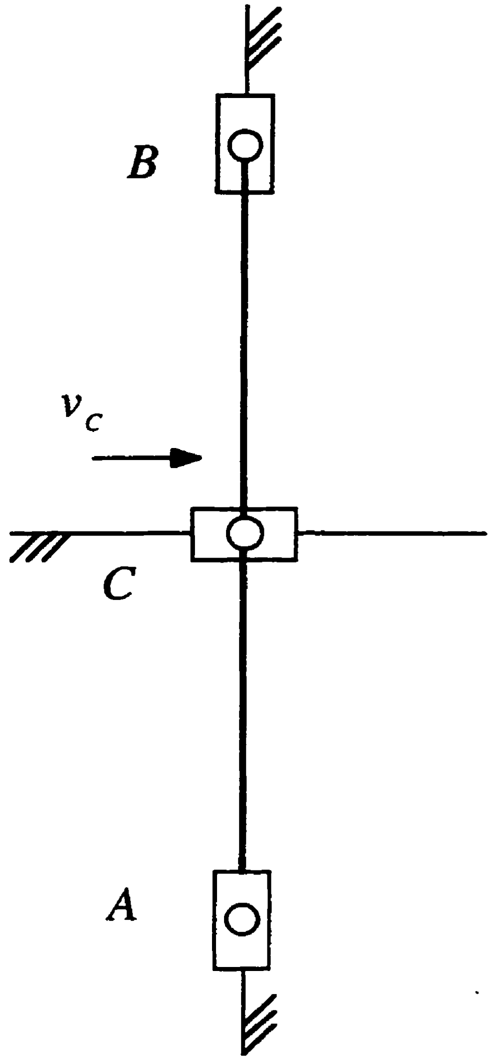

Figure 3.7b.
An RPM-type, (an II-type and an IO-type) singularity.

The velocity of point A is the input, the velocity of B is the output, and the velocity of C is a passive-joint rate. In the configuration shown in Figure 3.7b, both points A and B must have zero velocity, while the velocity of point C can be nonzero. Therefore, motion of the mechanism is possible while both the input and the output are zero, and thus an RPM-type singularity is present.

For all the configurations of the slider, the following equation linking the instantaneous input and output holds:

$$y_A v_A = y_B v_B, \tag{3.19}$$

where y_P and v_P are the coordinate and velocity of point P ($P = A, B$). Equation (3.19) can be obtained by differentiating the position-kinematics input-output equation, $y_A{}^2 = y_B{}^2$. If one solely uses such an input-output relation for the identification of singularities, the singularity in Figure 3.7b cannot be detected, since in this configuration Equation (3.19) does not degenerate. Thus, this configuration is not a singularity from a "traditional" point of view. Yet, this is not only an RPM-type configuration, but also an II- and IO-type singularity – any nonzero input or output is impossible.

When $\boldsymbol{q}$ belongs to a certain singularity type, this will be often denoted by $\boldsymbol{q} \in \{\text{type}\}$ (e.g., $\boldsymbol{q} \in \{\text{RI}\}$). The RI-, RO- and RPM-types will be referred to as R-types, and the others as I-types.

3.24. Remarks

(1) Each of the six singularity-type definitions describes an important change in the kinematic properties of the mechanism that occurs in a singular configuration of that type. When the mechanism is in an RO- or IO- (RI- or II-) type configuration the output (input) is indeterminate or restricted. In an IIM-type configuration the instantaneous motion of the mechanism is indeterminate with respect to any set of n velocities. In an RPM-type singularity, the passive motion of part of the mechanism is indeterminate, which may create problems such as interference with other links and obstacles. It is, therefore, desirable to know whether or not a given configuration belongs to each of these types, and a comprehensive singularity classification should clarify this.

(2) The fact that the same configuration was used to illustrate the RI and the IO type (Figure 3.4) or the RO and the II type (Figure 3.5) does not mean that one of these singularity types implies the other. (Such a wrong impression may be affirmed by the observation that the standard serial-manipulator singularity belongs to the IO and RI types, while the classical parallel-manipulator singularity is of the RO and II types.) On the contrary, in Figure 3.6 we have a configuration that belongs to both the RI and RO types, but is neither an IO nor an II singularity, while Figure 3.7b shows an IO- and II-type configuration which is neither an RI- nor an RO-type singularity. The novelty of the approach to kinematic singularity introduced in this thesis, consists partly in the recognition that IO and RI (II and RO) are separate phenomena which may or may not coincide.

(3) The defined singularity types are not non-intersecting, as the examples in this section have shown, and therefore do not form a classification of the set of all singular configurations. In fact, it can be shown that any singular configuration belongs to at least two types and is simultaneously an R-type singularity and an I-type singularity. This fact is proven later with Proposition 3.28 in Section 3.7.1. The result is obtained on the way to the stronger Theorem 3.30 (in Section 3.7.2), which fully characterizes the intersections of the singularity types and yields a refined and comprehensive classification of all possible singular configurations for all mechanisms.

3.5. Example

In this section a three-branch 6-dof parallel manipulator, shown in Figure 3.8, will be considered to further illustrate the singularity types introduced in Section 3.4. The mechanism has an RRRS joint distribution in each of the three legs (branches). Only the second and third rotary joints in each leg are actuated. This architecture is essentially equivalent to the one used by Collins and Long (1994) for their design of a hand controller for teleoperation.

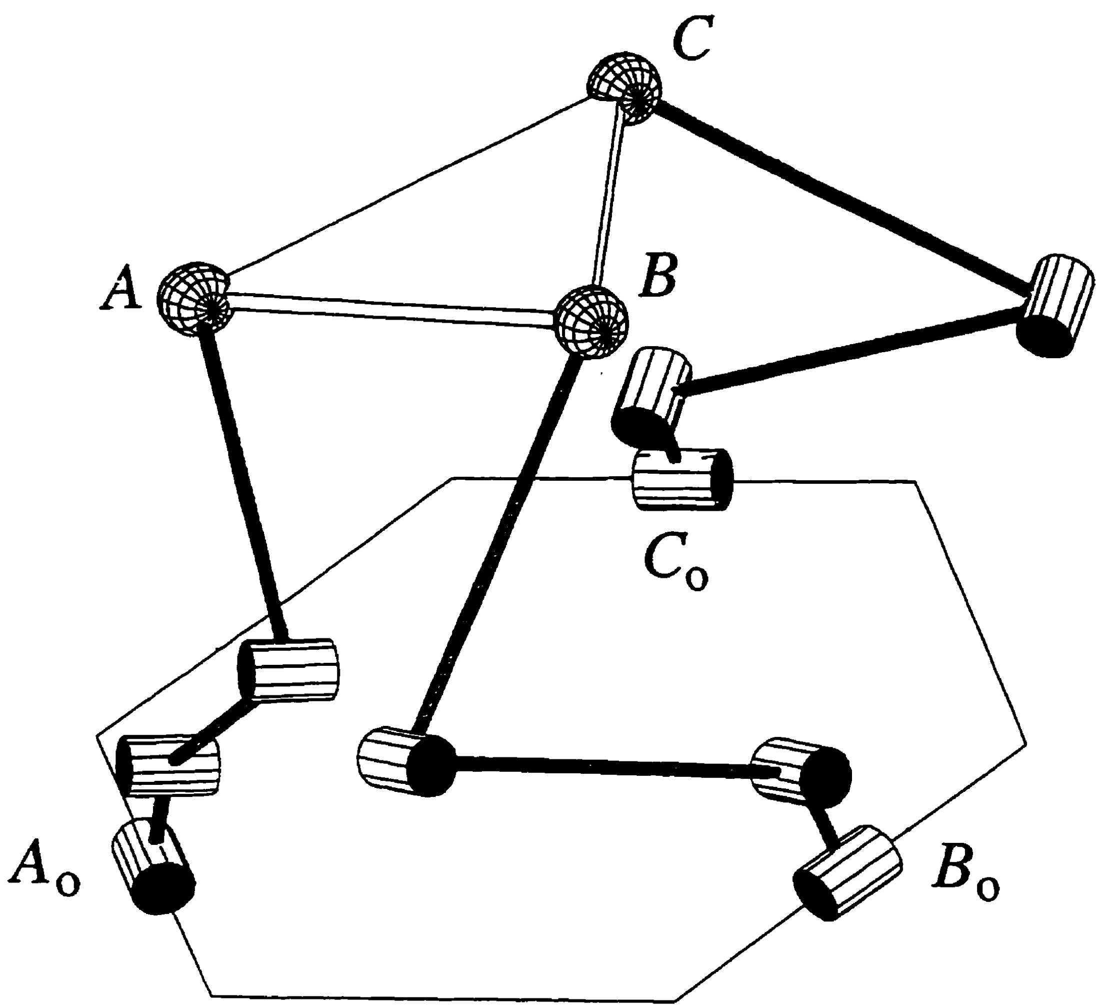

Figure 3.8. A 6-dof parallel manipulator.

The velocity equation, obtained using the method outlined in Section 3.2, is:

$$\begin{bmatrix} I_6 & -J_a^A & O & O & -J_p^A & O & O \\ O & J_a^A & -J_a^B & O & J_p^A & -J_p^B & O \\ O & O & J_a^B & -J_a^C & O & J_p^B & -J_p^C \end{bmatrix} \begin{bmatrix} T \\ \Omega^a \\ \Omega^p \end{bmatrix} = O \qquad (3.20)$$

where, for all P ($P = A, B, C$), $J_a^P = [\mathbf{S}_2^P, \mathbf{S}_3^P]$ is a 6×2 matrix which has as columns the active joint screws in the serial sub-chain, and $J_p^P = [\mathbf{S}_1^P, \mathbf{S}_4^P, \mathbf{S}_5^P, \mathbf{S}_6^P]$ is a 6×4 matrix composed of the passive screws in the sub-chain. The output is the twist of the moving platform, $T = \mathbf{T}$, the input, $\Omega^a = [\omega_2^A, \omega_3^A, \omega_2^B, \omega_3^B, \omega_2^C, \omega_3^C]^T$, is composed of the six active joint velocities, and the passive velocities are: $\Omega^p = [\omega_1^A, \omega_4^A, \omega_5^A, \omega_6^A, \omega_1^B, \ldots, \omega_6^C]^T$. (The spherical joints are modelled by three linearly-independent rotations through their centers). The first six scalar equations in (3.20) are the output equation (3.1) for this chain, while the remaining 12 equations are given by two loop-closure twist equations.

The velocity equation (3.20) can be shown to be equivalent to the system of equations: $\mathbf{T} = \sum_{i=1}^{6} \mathbf{S}_i^P \omega_i^P$, $P = A, B, C$, which is frequently used to describe the velocity kinematics of parallel-chain manipulators.

The definitions from Section 3.4 are used to identify the different types of singularities that can occur for the mechanism:

(i) ***RI-type singularity***

By substituting $T = 0$ in (3.20) and rearranging the columns of the velocity-equation matrix, it can be shown that for an RI-type singularity to be present, at least one of the serial sub-chains must be singular, (i.e., the six joint screws in the sub-chain must be linearly dependent). Although this is a necessary condition for the occurrence of an RI-type singularity, it is not a sufficient condition. For example, in the configuration shown in Figure 3.9, the mechanism does not have an RI-type singularity, although the B sub-chain is singular. (In Figure 3.9, the center of the spherical joint, B, lies on the axis of the passive rotary joint.) Indeed, one can see that, if in Figure 3.9 the end-effector were fixed, the input velocities could not be different from zero.

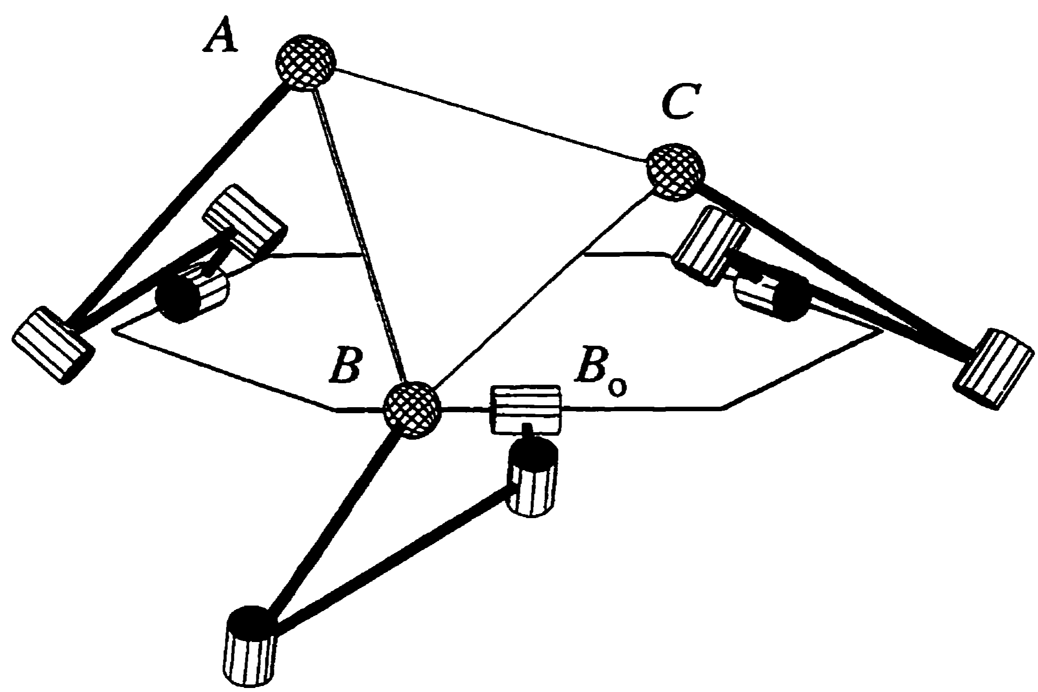

Figure 3.9. An RPM-, IO, and II-type singularity.

RI-type singularity would occur only when the sub-chain singularity is not due solely to a linear dependence of the passive-joint screws, more precisely, when the vanishing linear combination of joint screws includes active-joint screws (with nonzero coefficients). In this example, the active screws in a sub-chain are always linearly dependent. Therefore, an RI-type singularity occurs, if and only if for some P the column spaces of J_a^P and J_p^P have a nonzero intersection. For instance, if the joint angle at the third joint of one of the branches were 0° (or 180°), the input would be indeterminate (for a given output) and an RI-type singularity would be present.

(ii) RO-type singularity

The substitution of $\Omega^a = 0$ in (3.20) shows that an RO-type singularity occurs if and only if the column spaces of the three matrices J_p^P have a common nonzero screw. For example, the configuration shown in Figure 3.10 is an RO-type singularity since the sub-chain. (It should be noted that in Figure 3.10, the axis of the first joint of sub-chain B lies in the plane of the moving platform). Thus, a rotation of the moving platform about AC is possible even when all six inputs are locked.

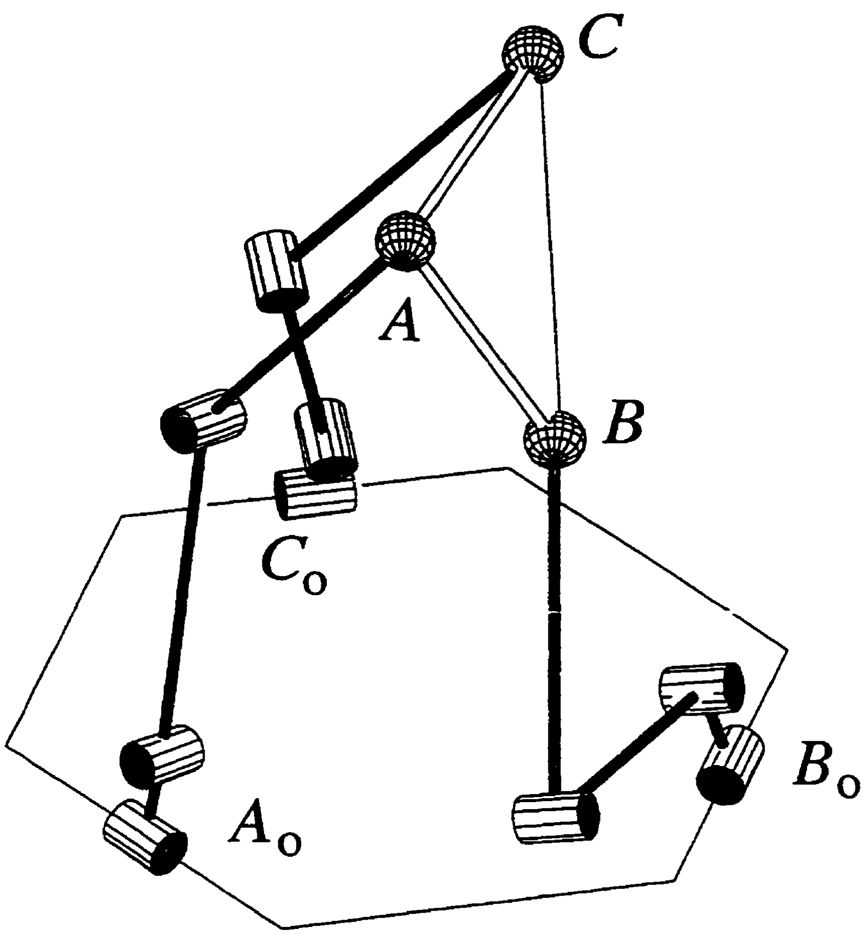

Figure 3.10. An RO-type (and II-type) singularity.

When all three matrices J_p^P are of full rank, the above-derived condition for RO-type singularity is equivalent to the linear dependence of the six reciprocal screws corresponding to each input. This formulation has been used in the literature (Kumar 1990) for the singularity analysis of parallel manipulators, and allows the detection of the RO-type singularities of the discussed mechanism as was done in (Collins and

Long, 1994). However, one can note that the configuration shown in Figure 3.9 is not an RO-type singularity, although six reciprocal screws intersect the line *AC* (just as in Figure 3.10) and are linearly dependent. (In Figure 3.9, the axes of the active joints in subchain *B* are perpendicular to the plane *ABC*.)

(iii) IO-type singularity

From Equation (3.20) and the definition of IO-type singularity, it can be deduced that an IO-type singularity occurs, if and only if at least one of the 6-dof serial sub-chains is singular. The configuration with a third joint angle of 0° (or 180°), discussed above in *(i)*, as well as the singularity shown in Figure 3.9 belong to this type.

(iv) II-type singularity

Figures 3.9 and 3.10 are examples of II-type singularities. For example, consider Figure 3.10 and assume that the input velocities in branches *A* and *C* are zero. Then, there exists a combination of the two inputs in sub-chain *B* which corresponds to no feasible motion of the mechanism. Indeed, if the second and third joints in subchains *A* and *C* are locked, the direction of the velocities of points *A* and *C* are fixed (perpendicular to the first joint in the sub-chain). Therefore, the direction of the *projection* onto the plane *ABC* of the velocity of point *B* is also fixed. Thus, the two input velocities in sub-chain *B* cannot be chosen arbitrarily and certain combinations of the two input velocities are *impossible*. Hence, an II-type singularity is present.

(v) IIM-type singularity

For an IIM-type singularity, the whole matrix of the velocity equation must be rank-deficient. This is equivalent to the singularity of the 12×18 matrix,

$$\begin{bmatrix} \boldsymbol{J}^A & -\boldsymbol{J}^B & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{J}^B & -\boldsymbol{J}^C \end{bmatrix},$$

where J^P are the sub-chain Jacobians. A necessary and sufficient condition for this phenomenon is the existence, for each of two of the three serial subchains, of a nonzero screw reciprocal to all joint screws in the subchain, such that a linear combination of the two screws is reciprocal to all joint screws in the *third* subchain. This condition is satisfied if a nonzero screw is reciprocal to all joint screws in two subchains. For example, IIM is present if subchains A and B are in the base plane and all their joints centres except A_0 and B_0 are collinear. Then the movable hexagon (with vertices A, B and the centres of joints $\mathbf{S}_2^P$ and $\mathbf{S}_3^P$) formed in the base plane by the two subchanes is "flattened". This would be possible only if the link lengths were specially proportioned.

(vi) RPM-type singularity

After considering Equation (3.20) for $\boldsymbol{T} = \Omega^a = 0$, it can be noted that an RPM-type singularity occurs if and only if at least one of the matrices J_p^P is singular, i.e. when the passive-joint screws in a serial subchain are linearly dependent. The configuration in Figure 3.9 is an RPM-type singularity. Even if both the input and the output in this configuration were zero, part of the mechanism (subchain B) could still move (rotation about the line B_0B is possible).

3.6. Motion-Space Interpretation of Kinematic Singularity

In this section the definitions of singularity and the singularity types are interpreted by the properties of the space of solutions of the velocity equation (the null-space of $\boldsymbol{L}$) referred to as the *motion space*. This linear-algebraic interpretation reveals the symmetric interdependence of the singularity types.

The spaces O, I and $\mathcal{P}$ (defined in Section 3.2) can be viewed as spanning an $(N+n_O)$-dimensional space $\mathcal{V} = O \oplus I \oplus \mathcal{P}$. $\mathcal{V}$ is the tangent space of $Q \times O$ at $(\boldsymbol{q}, f_O(\boldsymbol{q}))$. The elements of $\mathcal{V}$ are velocity vectors of the form $\boldsymbol{m} = (\boldsymbol{T}, \boldsymbol{\Omega}) = (\boldsymbol{T}, \boldsymbol{\Omega}^a, \boldsymbol{\Omega}^p)$. The *feasible* velocity vectors form a subspace of $\mathcal{V}$, $\mathcal{M}_q$, the ***motion space*** at $\boldsymbol{q}$. $\mathcal{M}_q$ is the space of solutions of the velocity equation, and its dimension is equal to the instantaneous mobility $\boldsymbol{n_q}$. All instantaneous kinematics properties are determined by the orientation of the $\mathcal{M}_q$ in $\mathcal{V}$.

Consider the maps $\boldsymbol{p}_I : \mathcal{M}_q \rightarrow I$, and $\boldsymbol{p}_O : \mathcal{M}_q \rightarrow O$, defined as the restrictions on the motion space $\mathcal{M}_q$ of the projections which map $\mathcal{V}$ onto I and O. They map any motion vector into the vector of its input or output, respectively. The ranks of $\boldsymbol{p}_I$ and $\boldsymbol{p}_O$, (the dimensions of their image spaces) will be denoted by r_I and r_O. Note that the maps $\boldsymbol{p}_I$ and $\boldsymbol{p}_O$ (and their ranks) are dependent on the configuration $\boldsymbol{q}$.

The singularity definition can be now reformulated in terms of the properties of $\boldsymbol{p}_I$ and $\boldsymbol{p}_O$. The FIKP is equivalent to the problem of finding the inverse of the map $\boldsymbol{p}_I$, while solving the IIKP is equivalent to finding the inverse map of $\boldsymbol{p}_O$. Therefore, the following proposition is true:

3.25. Proposition

*(i) The **FIKP** is* **solvable** *for a configuration **q**, if and only if $\boldsymbol{p}_I$ is a one-to-one mapping of $\mathcal{M}_q$ onto I, i.e.,*

$$\dim\ \mathcal{M}_q = n_q = r_I = n = \dim\ I.$$

*(ii) The **IIKP** is* **solvable** *for a configuration **q**, if and only if $\boldsymbol{p}_O$ is a one-to-one mapping of $\mathcal{M}_q$ onto O, i.e.,*

$$\dim\ \mathcal{M}_q = n_q = r_O = n = \dim\ O.$$

The six singularity types are redefined below in terms of the projection maps.

3.26. Proposition

(i) $\quad \boldsymbol{q} \in \{\mathrm{RI}\} \Leftrightarrow \mathrm{Ker}\ \boldsymbol{p}_O - \mathrm{Ker}\ \boldsymbol{p}_I \neq \varnothing,$

(ii) $\quad \boldsymbol{q} \in \{\mathrm{RO}\} \Leftrightarrow \mathrm{Ker}\ \boldsymbol{p}_I - \mathrm{Ker}\ \boldsymbol{p}_O \neq \varnothing,$

(iii) $\quad \boldsymbol{q} \in \{\mathrm{II}\} \Leftrightarrow I - \mathrm{Im}\ \boldsymbol{p}_I \neq \varnothing,$

(iv) $q \in \{IO\} \Leftrightarrow O - \text{Im}\, p_O \neq \varnothing$,

(v) $q \in \{IIM\} \Leftrightarrow \dim \mathcal{M}_q > \dim I$,.

(vi) $q \in \{RPM\} \Leftrightarrow \text{Ker}\, p_I \cap \text{Ker}\, p_O \neq 0$.

The proof follows directly from the definitions of p_I, p_O and the singularity types.

The next proposition states the restrictions imposed on r_I and r_O when q belongs to different singularity types:

3.27. Proposition

(i) $q \in \{II\} \Leftrightarrow r_I < n$,

(ii) $q \in \{IO\} \Leftrightarrow r_O < n$,

(iii) $q \in \{IIM\} \Leftrightarrow n < n_q$,

(iv) $q \in \{RI\} \Rightarrow r_O < n_q$,

(v) $q \in \{RO\} \Rightarrow r_I < n_q$,

(vi) $q \in \{RPM\} \Rightarrow r_O < n_q$ and $r_I < n_q$,

(vii) $r_O < n_q \Rightarrow q \in \{RI\}$ or $q \in \{RPM\}$,

(viii) $r_I < n_q \Rightarrow q \in \{RO\}$ or $q \in \{RPM\}$.

Proof.

(i)-(vi) Follow directly from the definitions of the singularity types.

(vii) If $r_O < n_q$, there are nonzero motion vectors projected onto zero by p_O (i.e., $\text{Ker}\, p_O \neq O$). If such a vector M in $\text{Ker}\, p_O$ belongs also to $\text{Ker}\, p_I$, then an RPM-type singularity is present due to (vi). Otherwise, an RI-type singularity is implied by (iii).

(viii) The proof is analogous to the proof of (vii). □

In this section we relate the definitions of singularity and singularity types to the various velocity spaces associated with a configuration q. The sets $\mathcal{V}$, O, and I are the spaces of the *potential* motions, output motions and input motions, respectively. Their subspaces, $\mathcal{M}_q$,

$\text{Im}\,\boldsymbol{p}_O$ and $\text{Im}\,\boldsymbol{p}_I$ consist of the *feasible* motions, output motions and input motions. The subspaces $\text{Ker}\,\boldsymbol{p}_O$ and $\text{Ker}\,\boldsymbol{p}_I$ of $\mathcal{M}_q$ are, in fact, the spaces of the zero-output and zero-input motions, respectively. Singularity occurs when at least one of these kernels is greater than zero. The difference of the two kernels determines an RI- or RO-type singularity, while their nonzero intersection leads to an RPM-type singularity. The IIM-type is present when the existence of the kernels is due to the higher dimension of $\mathcal{M}_q$. When a nonzero kernel is due to the singularity of the maps $\boldsymbol{p}_O$ and $\boldsymbol{p}_I$, the configuration is IO- or II-type.

3.7. Classification of Singularities

3.7.1. Singularity-type combinations

For any configuration, singular or nonsingular, $r_I \leq n \leq n_q$ and $r_O \leq n \leq n_q$. A configuration is nonsingular, only if $r_I = n = n_q$ and $r_O = n = n_q$. The cases in which these equalities do not hold are analyzed below:

Case 1. $\quad n < n_q$

This is an IIM-type singularity. It can be noted that in this case $r_I < n_q$ and $r_O < n_q$. Therefore, as implied by Proposition 3.27, (vii) and (viii), an RPM-type singularity or a singularity belonging to both the RI- and the RO-type must be present as well.

Case 2. $\quad r_I < n = n_q$ and $r_O = n = n_q$

This case is a combination of the II- and RO- singularity types. Indeed, $r_I < n$ implies an II-type singularity according to Proposition 3.27, (i). According to Proposition 3.27, (viii), either an RO- or an RPM-type singularity is present. But, if the configuration were an RPM-type singularity, according to Proposition 3.27, (vi), r_O would be smaller than n_q.

Case 3. $r_I = n = n_q$ and $r_O < n = n_q$

This case is symmetrical to Case 2 and is a combination of the IO- and RI-type singularities. The reasoning is the same as above.

Case 4. $r_I < n = n_q$ and $r_O < n = n_q$

This case is a combination of the II and IO types together with either an RPM type or any combination of at least two different R types. The II-type and the IO-type singularities are implied by Proposition 3.27, (i) and (ii), while (vii) and (viii) show that in this case there should be either an RPM-type singularity or a singularity of at least two different R-types.

The above discussion of the four cases provides the proof for the following proposition:

3.28. Proposition. *Let q be a singular configuration. Then,*

(1) *q belongs to at least one of the types RO, RI, and RPM.*

(2) *q belongs to at least one of the types IO, II, and IIM.*

Indeed, each individual singularity belongs to exactly one of Cases 1 to 4, and for each case, it was shown that the configuration must be of at least one I-type and one R-type.

3.7.2. Enumeration of all possible combinations

Below, the velocity-space formulation of the singularity problem is applied to find *all* feasible combinations of the six singularity types for the general case of an arbitrary kinematic chain. First, in the following proposition the rules for the simultaneous occurrence of the singularity types are stated.

3.29. Proposition

(i) $q \in \{\mathrm{RI}\} \Rightarrow q \in \{\mathrm{IO}\}$ or $q \in \{\mathrm{IIM}\}$,

(ii) $q \in \{\mathrm{RO}\} \Rightarrow q \in \{\mathrm{II}\}$ or $q \in \{\mathrm{IIM}\}$,

(iii) $q \in \{\mathrm{II}\} \Rightarrow q \in \{\mathrm{RO}\}$ or $q \in \{\mathrm{RPM}\}$,

(iv) $q \in \{\mathrm{IO}\} \Rightarrow q \in \{\mathrm{RI}\}$ or $q \in \{\mathrm{RPM}\}$,

(v) $q \in \{\mathrm{RPM}\} \Rightarrow (q \in \{\mathrm{II}\}$ and $q \in \{\mathrm{IO}\})$ or $q \in \{\mathrm{IIM}\}$,

(vi) $q \in \{\mathrm{IIM}\} \Rightarrow (q \in \{\mathrm{RI}\}$ and $q \in \{\mathrm{RO}\})$ or $q \in \{\mathrm{RPM}\}$,

(vii) $q \in \{\mathrm{II}\} \Rightarrow q \in \{\mathrm{IO}\}$ or $q \in \{\mathrm{RO}\}$,

(viii) $q \in \{\mathrm{IO}\} \Rightarrow q \in \{\mathrm{II}\}$ or $q \in \{\mathrm{RI}\}$,

(ix) $q \in \{\mathrm{RI}\} \Rightarrow q \in \{\mathrm{IO}\}$ or $q \in \{\mathrm{RO}\}$,

(x) $q \in \{\mathrm{RO}\} \Rightarrow q \in \{\mathrm{II}\}$ or $q \in \{\mathrm{RI}\}$.

Proof.

(i) Proposition 3.27, (iv), implies $r_O < n_q$. Therefore, since $r_O \leq n \leq n_q$, either $r_O < n$, which is equivalent to an IO-type singularity according to Proposition 3.27, (ii), or $n < n_q$, which is the condition for an IIM-type singularity.

(ii) Similar to (i).

(iii) From Proposition 3.27, (i) and (vii).

(iv) From Proposition 3.27, (ii) and (viii).

(v) From Proposition 3.27, (vi), (i) and (ii).

(vi) From Proposition 3.27, (vi), (i) and (ii).

(vii) An II-type singularity implies that $\boldsymbol{p}_I$ is not of maximum rank (Proposition 3.27, (i)), and therefore: $\mathrm{Ker}\,\boldsymbol{p}_I \neq \boldsymbol{O}$. Let us consider the image of $\mathrm{Ker}\,\boldsymbol{p}_I$ under $\boldsymbol{p}_O$, $\boldsymbol{p}_O(\mathrm{Ker}\,\boldsymbol{p}_I)$. Then, if $\boldsymbol{p}_O(\mathrm{Ker}\,\boldsymbol{p}_I) \neq \boldsymbol{O}$, an RI is present. If $\boldsymbol{p}_O(\mathrm{Ker}\,\boldsymbol{p}_I) = \boldsymbol{O}$, then $\mathrm{Ker}\,\boldsymbol{p}_O \supset \mathrm{Ker}\,\boldsymbol{p}_I$, and hence $r_I \geq r_O$. Since $\boldsymbol{p}_I$ is rank-deficient, r_O is also smaller than n, and therefore an II-type singularity is present.

(viii) Analogous to the proof of (vii).

(ix) Assume there is no IO-type singularity. Then, $\boldsymbol{p}_O$ is of maximum rank and the motion space, $\mathcal{M}$, can be decomposed as $\mathcal{M} = \mathrm{Ker}\,\boldsymbol{p}_O \oplus \mathcal{M}_O$, where $\mathcal{M}_O$ is a subspace of $\mathcal{M}$ with $\dim \mathcal{M}_O = \dim O = r_O$. Let us consider $\boldsymbol{p}_I(\mathcal{M}_O)$. If $\boldsymbol{p}_I(\mathcal{M}_O) \neq I$, an RO-type singularity is present.

We assume $p_I(\mathcal{M}_O) = I$. Then, any input vector is an image under p_I of a motion vector with nonzero output (since $\text{Ker}\, p_O \cap \mathcal{M}_O = O$). On the other hand, since an RI-type singularity is present, there are motion vectors with nonzero input and zero output. Thus, there exist two different outputs (one is nonzero and the other is equal to zero) which are feasible with one and the same input. If we subtract the motion vectors corresponding to these two different outputs, a motion vector with nonzero output and zero input is obtained, which implies an RO-type singularity.

(x) Analogous to the proof of (ix). □

3.30. Theorem. *Let* S *be a combination of singularity types. There exists a non-redundant mechanism with a configuration,* q, *such that* $q \in$ S, *if and only if S is marked with* "Y" *in Table 3.1.*

	IO	II	IO *and* II	IIM	IO *and* IIM	II *and* IIM	IO *and* II *and* IIM
RI	Y						
RO		Y					
RI *and* RO			Y	Y	Y	Y	Y
RPM			Y	Y			Y
RI *and* RPM			Y		Y		Y
RO *and* RPM			Y			Y	Y
RI *and* RO *and* RPM			Y	Y	Y	Y	Y

Table 3.1. Possible combinations of singularity types.

Proof. To prove the theorem, we need to establish that: (i) all combinations not marked with "Y" in the table can never occur; and (ii) there exist mechanisms and configurations with the marked singularity-type combinations.

(i) There are six singularity types and therefore there are $2^6 = 64$ combinations (one of them is the nonsingular combination). From Propositions 3.28 we conclude that it is sufficient to consider the ones that include at least one I-type and one R-type. These combinations are represented by the 49 cells of Table 3.1. The cell in the *i*-th row and *j*-th column of the table corresponds to a combination of all singularity types listed to the left of the *i*-th row and on the top of the *j*-th column.

We must show that the combinations corresponding to blank cells of the table are impossible. This is proven with the help of Proposition 3.29 as illustrated by Table 3.2.

	IO	II	IO *and* II	IIM	IO *and* IIM	II *and* IIM	IO *and* II *and* IIM
RI	**Y**	*(iii)*	*(iii)*	*(vi)*	*(vi)*	*(iii)*	*(iii)*
RO	*(ii)*	**Y**	*(iv)*	*(vi)*	*(iv)*	*(vi)*	*(iv)*
RI *and* RO	*(ii)*	*(i)*	**Y**	**Y**	**Y**	**Y**	**Y**
RPM	*(v)*	*(v)*	**Y**	**Y**	*(viii)*	*(vii)*	**Y**
RI *and* RPM	*(v)*	*(i)*	**Y**	*(ix)*	**Y**	*(ix)*	**Y**
RO *and* RPM	*(ii)*	*(v)*	**Y**	*(x)*	*(viii)*	**Y**	**Y**
RI *and* RO *and* RPM	*(ii)*	*(i)*	**Y**	**Y**	**Y**	**Y**	**Y**

Table 3.2. Impossible combinations of singularity types for non-redundant mechanisms.

Each of the 28 empty cells represents a combination of singularity types which, if it occurred in some configuration, would violate (at least) one statement in Proposition 3.29. Table 3.2 illustrates which statement each blank-cell combination violates.

(ii) We need to give an example for each of the 21 combinations, corresponding to "Y" cells. Four of these combinations were already illustrated in this section:

(RI, IO)	Figure 3.4 in Examples 3.8 and 3.17;
(RO, II)	Figure 3.5 in Examples 3.11 and 3.15;
(RI, RO, IIM)	Figure 3.6 in Example 3.20;
(RPM, IO, II)	Figure 3.7b in Example 3.23, and Figure 3.9 in Section 3.5.

Twelve additional combinations occur in different examples in Chapters 4 and 5 of the thesis:

(RI, RO, IO, II)	Figure 4.5 in Example 4.7, this combination also occurs for the mechanism in Figure 5.7 as discussed in Section 5.6.3;
(RI, RO, IO, IIM)	Figure 4.7 in Example 4.25, and Figure 5.10 in Section 5.6.3;
(RI, RPM, IO, II)	A variation of Figure 5.8 as discussed in Section 5.6.3;
(RO, RPM, IO, II)	Figure 4.6 in Example 4.11, and a variation of Figure 5.12 as discussed in Section 5.6.3;
(RI, RPM, IO, IIM)	A variation of Figure 5.9 as discussed in Section 5.6.3;
(RI, RO, IO, II, IIM)	Figure 4.8 in Example 4.30;
(RI, RO, RPM, IO, II)	A variation of Figure 5.11 as discussed in Section 5.6.3;
(RPM, IO, II, IIM)	Figure 5.9 in Section 5.6.3;
(RI, RPM, IO, II, IIM)	Figure 5.8 in Section 5.6.3;
(RI, RO, RPM, IO, IIM)	Figure 4.9 in Example 4.31;
(RO, RPM, IO, II, IIM)	Figure 5.12 in Section 5.6.3;
(RI, RO, RPM, IO, II, IIM)	Figure 5.11 in Section 5.6.3.

The remaining five combinations, namely (RPM, IIM), (RI, RO, RPM, IIM), (RI, RO, II, IIM), (RI, RO, RPM, II, IIM) and (RO, RPM, II, IIM) are illustrated with the five examples below.

3.31. Example. Let us consider a four-bar mechanism such that $AB = AD$ and $BC = CD$. Furthermore, let the active joint be at B (rather than the customary, A), while the output velocity is the usual (the angular velocity of link CD). In the configuration shown in Figure 3.11, the points A and C coincide. It can be seen that this is an RPM-type singularity. Indeed, when both the input and output are set to zero, $\omega_B = \tau = 0$, the mechanism retains mobility: a rotation of links AB and CB about point $A = C$ is possible. On the other hand, this singular configuration belongs to neither of the types RI, RO, IO and II, since $\omega_B = -\tau$ can have any value. The configuration is therefore an example for an (RPM, IIM) singularity type combination.

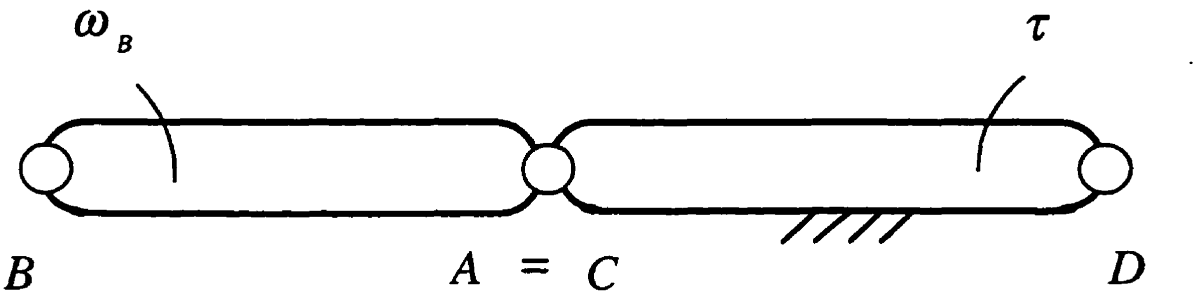

Figure 3.11. A four-bar mechanism in an RPM- and IIM-type singularity.

3.32. Example. We consider a six-bar mechanism shown in Figure 3.12. The input is the joint velocity at A and the output is the angular velocity of link EF.

Assuming that the link lengths are appropriately chosen, the mechanism in Figure 3.12 can be positioned in the configuration shown in Figure 3.13.

By fixing, respectively, $\tau = 0$, $\omega_A = 0$, and $\omega_A = \tau = 0$, it can be seen that the configuration belongs to types RI, RO, and RPM. It is also clear that this is not an IO-type or an II-type configuration (since neither the input nor the output need to be zero). Therefore, the singularity in Figure 3.13 is a representative of the singularity-type combination (RI, RO, RPM, IIM).

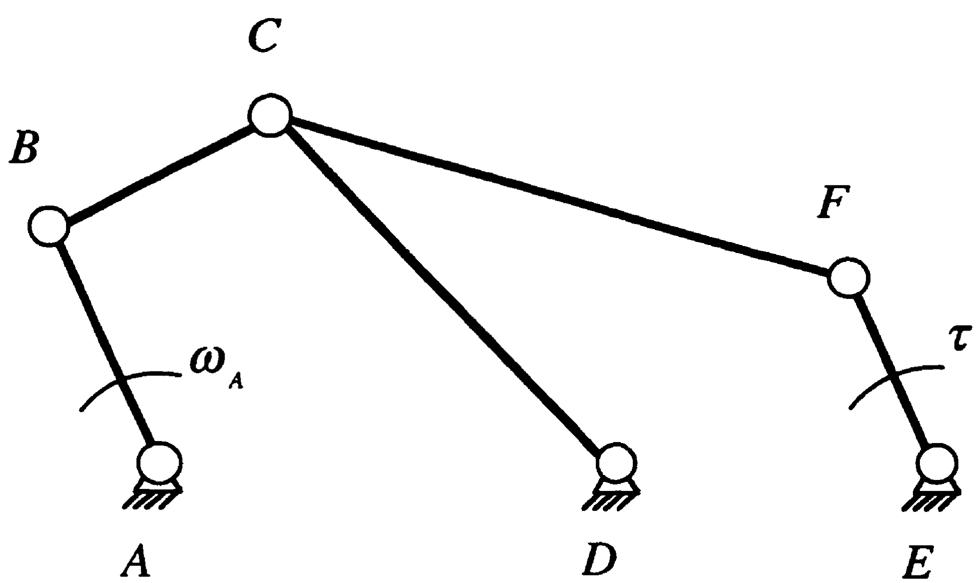

Figure 3.12. A six-bar mechanism.

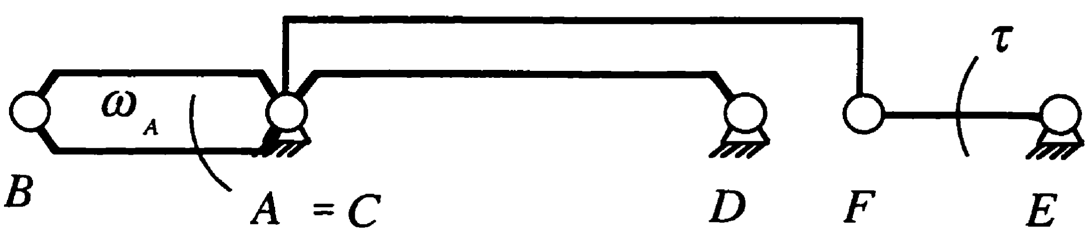

Figure 3.13. A configuration of singularity types RI, RO, RPM and IIM.

*3.33. **Example.*** The mechanism shown in Figure 3.14 has 3 dof. The active joints are A_1, B_1 and C_0. The output is the motion of link ABC. The configuration shown belongs to types RI and RO but it is not an RPM-type singularity. On the other hand, this is an II-type configuration (since the input velocities at A_1 and B_1 must always be equal) and not an IO-type singularity (since the output link can have an arbitrary instantaneous motion). Therefore, Figure 3.14 proves the existence of singularities belonging to the combination of types (RI, RO, II, IIM).

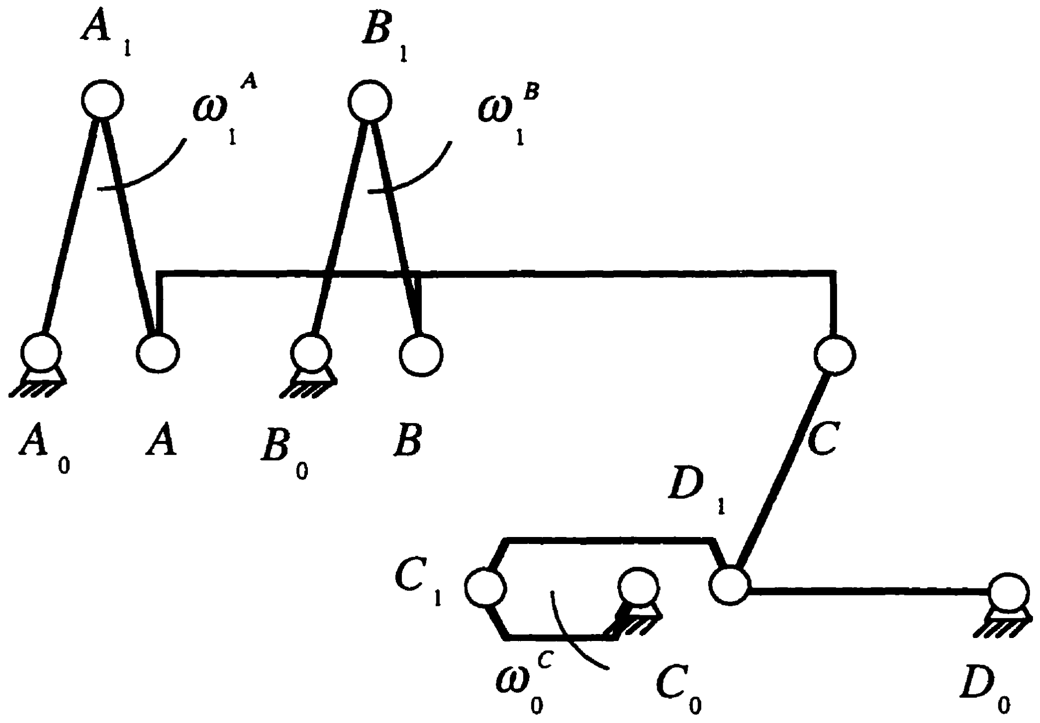

Figure 3.14. A configuration of singularity types RI, RO, II and IIM.

3.34. Example. The mechanism and the configuration in Figure 3.15 are very similar to the ones presented in Figure 3.14 (and discussed in the previous Example 3.33) except for two changes: the third input joint is C_1 rather than C_0; and the points C_0 and D_1 coincide. Just like the configuration in Figure 3.14, the present example belongs to the singularity types RO, II and IIM but not IO. In addition, an RPM-type singularity is present, since the point C_1 can have a nonzero velocity even when the output link is fixed and the inputs are equal to zero. However, unlike Figure 3.14 the present configuration is not an RI-type singularity, since when the output link is fixed all inputs, including the joint velocity at C_1

must be zero. Thus, Figure 3.15 presents a configuration, which is an RO-, RPM-, II-, IIM-type singularity.

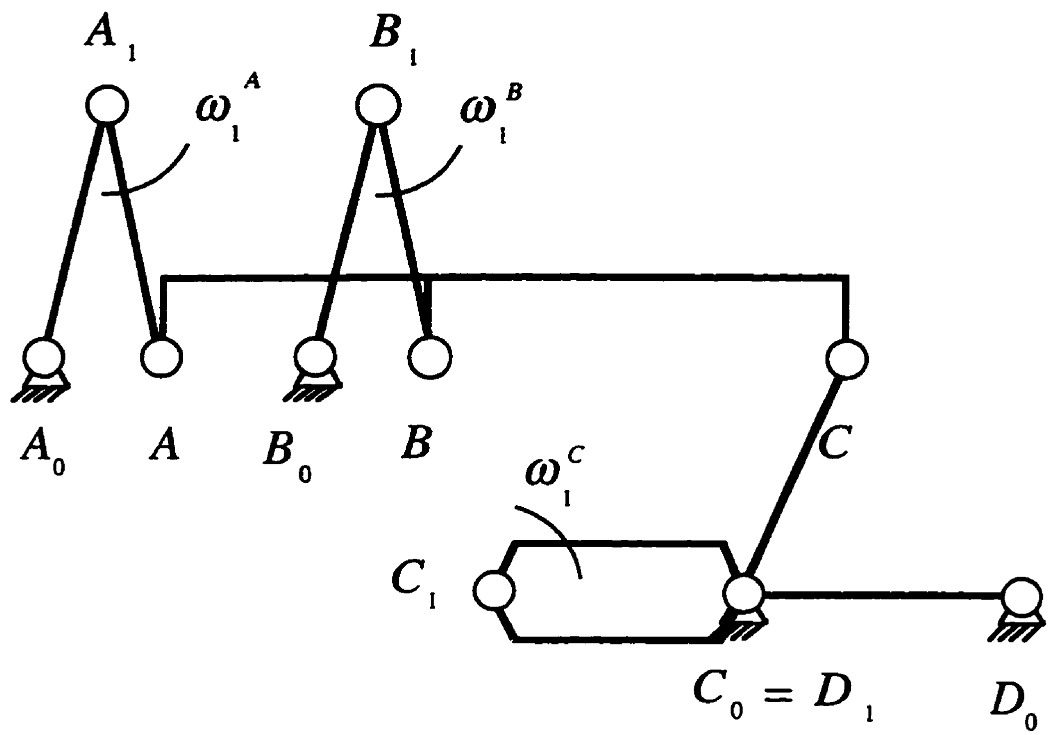

Figure 3.15. A configuration of singularity types RI, RO, II and IIM.

3.35. Example. The mechanism shown in Figure 3.16 is similar to the ones in Figures 3.14 and 3.15. However, here the four-bar subchain $C_0C_1D_1D_0$ is replaced with a six-bar subchain $C_0C_1D_0D_1E_0E_1$, which is similar to the one shown in Figure 3.13. As in both Examples 3.33 and 3.34, it is established that the configuration belongs to types RO, II and that it is not an IO singularity. Assuming that the output link ABC is fixed, the study of the six-bar subchain reveals in a way analogous to Example 3.32 that RI- and RPM-type singularities are present. Therefore, we have a configuration which is a representative of the combination (RI, RO, RPM, II, IIM).

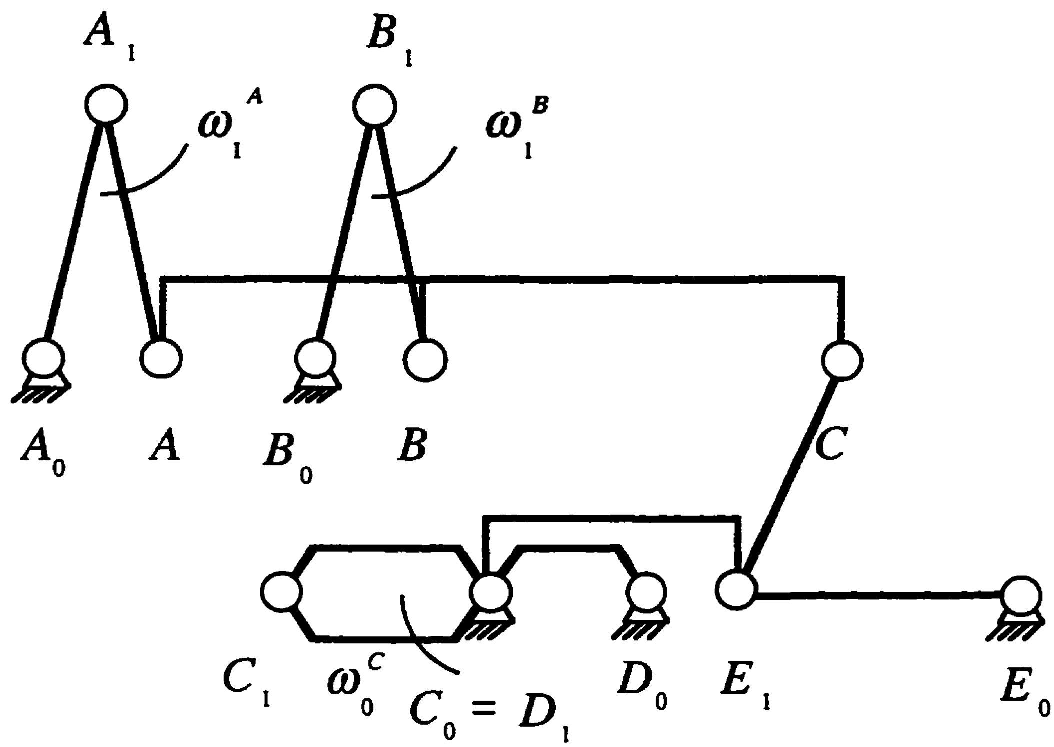

Figure 3.16. A configuration of singularity types RI, RO, II and IIM.

This completes the proof of Theorem 3.30. □

3.36. Remark. In Remark 3.24(3) it was noted that the introduction of the six singularity types does not immediately provide a rigorous classification of the singular configurations of non-redundant mechanisms, since each singularity belongs to more than one type. A proper classification of some set is a representation of the set as a union of non-intersecting classes. Theorem 3.30 proves that the set of all singularities of all non-redundant mechanisms consists of 21 non-intersecting non-empty subsets, each being the set of singularities that belong to the combination of singularity types corresponding to one of the non-blank cells of Table 3.1. Thus, Table 3.1 presents a comprehensive classification of the singularities of a general non-redundant mechanism with 21 non-intersecting classes.

3.8. Summary

In this chapter, a general framework for the singularity analysis of non-redundant mechanisms was developed. On the basis of the velocity equation, derived as a necessary and sufficient condition for the feasibility of the instantaneous motion of a mechanism, a new general definition of singularity was proposed. A configuration is regarded as singular, when either the forward or the inverse kinematics problem does not have a general solution. Six types of singularities, reflecting different possibilities for the occurrence of indeterminacy of the instantaneous kinematics, were defined. On the basis of a motion-space interpretation of these definitions, the relationship between the singularity types was revealed and a comprehensive and refined classification was developed. The presented approach can be used as a starting point for the singularity analysis of specific mechanisms, since the velocity equation can be obtained for any given mechanism as an explicit function of the joint screws.

CHAPTER 4

HYBRID-CHAIN MANIPULATORS

4.1. Introduction

In this chapter, the concepts introduced in Section 3 are applied to a narrower set of mechanisms, namely a class of parallel-like manipulators, herein referred to as hybrid-chain manipulators (HCMs). As a result, new mathematical tools for the instantaneous kinematics and singularity analysis of HCMs are obtained.

The HCMs are formally defined in Section 4.2. They have a parallel-like topology of the kinematic chain, which is similar to the one found in walking machines and multi-fingered grippers. The velocity equations for such mechanisms is presented in the same section.

In Section 4.3, the passive-joint velocities are eliminated from the velocity equation, in such a way that the resulting input-output equation is a necessary and sufficient condition for feasible input and output. A new screw-theory based formulation of the instantaneous kinematics for this class of mechanisms is obtained. Unlike existing solutions for parallel manipulators, the derived (instantaneous) input-output equation is a necessary and sufficient condition for the feasibility of the manipulator's motions.

The formulation of singularity for non-redundant mechanisms, given in Chapter 3, is applied to HCMs. In Section 4.4, conditions for each of the six singularity types are derived. In Section 4.5, a comprehensive classification of all singularities of all HCMs is given.

4.2. Hybrid-Chain Manipulators

A typical HCM is a non-redundant mechanism (i.e., $n_I = \mu = n_O$) with mobility $\mu = n \leq 6$, which consists of a base link, an end-effector, and k serial subchains connecting the base and the end-effector. Each serial chain consists of joints with total dimension of n, e.g., n 1-dof joints. Only n of the kn joints are actively controlled. These active joints are distributed in an arbitrary way amongst the subchains. We denote the number of active joints in the j-th subchain by n_j, $n_j \leq n$. The classical serial and parallel manipulators can be regarded as special cases of HCMs. The former has only 1 subchain ($k = 1$) and all n joints are active, while the latter has n subchains ($k = n$) and 1 active joint in each.

The joint space, Q, of a HCM is of dimension nk. Due to the specific symmetric structure of the kinematic chain the system of loop equations defining the configuration space D (Theorem 2.30 and Equation (2.21)) are equivalent to a system of the form:

$$h_1^j(q_1^j)h_2^j(q_2^j) \cdots h_n^j(q_n^j) = g(\boldsymbol{q}), \qquad j = 1, \ldots, k \tag{4.1}$$

where the subscript denotes the number of the joint and the superscript is the number of the subchain. The right-hand side $g(\boldsymbol{q})$ is the displacement of the end-effector and is the same for each of the Equations (4.1). Thus, the configuration space, D, of the HCM is the subset of Q composed of all $\boldsymbol{q}$, which satisfy (4.1).

The input space, I, of the HCM is the n-dimensional Cartesian product of the configuration spaces of the active joints (as defined in Section 2.6). The output space is defined as the smallest Lie subgroup of $SE(3)$ containing all possible displacements. There exist HCMs with output spaces all possible Lie sub-groups of $SE(3)$, however three types have the greatest practical importance. These are manipulators with O diffeomorphic to $\mathbf{R}^2$, $SE(2)$ or $SO(3)$. It is assumed that each serial sub-chain consists of n joints which all belong to the same n-dimensional Lie subgroup of $SE(3)$ and do not belong to any smaller subgroup. This ensures that the mobility of the mechanism is equal to n and that the dimension of the output space is n, i.e., the mechanism is non-redundant.

Since in this chapter, we are interested mainly in instantaneous analysis, we shall represent the joints by their joint screws $\mathbf{S}_i^j$, where j is the index of the subchain, while i indicates the joint in the chain. The subscripts of the n_j active (actuated) joints in the j-th subchain form a set that is denoted by A_j ($A_j = \{i \in \{1, \ldots, n\} \mid \mathbf{S}_i^j$ is active$\}$).

Using the notation first introduced in Section 3.2, the (instantaneous) input is the n-dimensional vector (column matrix) Ω^a consisting of the active joint rates. In the 6-dof case ($n = 6$) the output will be the twist $\mathbf{T} = (\omega, \mathbf{v})$ representing the instantaneous motion of the end-effector. When $n < 6$ (e.g., a planar or spherical mechanism), the output will be an n-dimensional vector including only part of the components of $\mathbf{T}$. The n-dimensional column matrix of the instantaneous outputs will be denoted by T. In this case (i.e., $n < 6$), suppose that for any configuration, all the joint screws (and therefore the output twist) belong to a common n-dimensional subspace, $\mathcal{S}$, of the 6-dimensional vector space of twists, $\mathcal{T}$. Also, suppose that $\mathcal{S}$ allows a "standard" basis, i.e., that a family of Cartesian reference frames in the 3-dimensional Euclidean space exists, such that the basis vectors of $\mathcal{S}$ can be chosen only among the three rotations and three translations about the coordinate axes. This condition is satisfied for all the screw systems (i.e., the subspaces of $\mathcal{T}$), which guarantee full-cycle mobility as listed in (Hunt 1978), p. 378. These subspaces are in fact the Lie algebras of the Lie subgroups of $SE(3)$. Then, if we use only such reference frames, the same $6 - n$ coordinates of the joint twists and the output twist will be zero at any configuration. For example, in the case of 3-dof planar manipulators, $\mathcal{S}$ is the screw system of planar motion, which can be spanned by a rotation and two translations, and thus all the twists involved will have only three nonzero coordinates.

The twist of the output link can be expressed as a linear combination of the joint twists in each of the n subchains:

$$\mathbf{T} = \sum_{i=1}^{n} \omega_i^j \mathbf{S}_i^j, \qquad j = 1, \ldots, k \tag{4.2}$$

where ω_i^j is the joint velocity of the joint along $\mathbf{S}_i^j$. If only n screw-coordinates are nonzero, (4.2) is a set of nk linear equations relating the output twist $\mathbf{T}$, the column matrix of the input velocities $\boldsymbol{\Omega}^a$, and the passive joint rates $\boldsymbol{\Omega}^p$. It is satisfied for any feasible instantaneous motion ($\boldsymbol{T}, \boldsymbol{\Omega}^a, \boldsymbol{\Omega}^p$) of the HCM. On the other hand, if $\boldsymbol{T}$, $\boldsymbol{\Omega}^a$, and $\boldsymbol{\Omega}^p$ satisfy (4.2), they represent a feasible motion of the manipulator. Therefore, these nk equations in (4.2) are equivalent to the *velocity equation* of the HCM according to the definition in Section 3.2.

The results presented in this chapter are valid for any mechanism whose instantaneous kinematics is described by n of the rows of a twist equation of the type of (4.2), even if the mechanism's architecture does not correspond to the exact description of HCMs above. Thus, an HCM can be redefined as follows:

4.1. Definition. *A mechanism with an n-dimensional configuration space, C, is referred to as a* **Hybrid Chain Manipulator**, *when there exist n-dimensional screw subspace,* $\mathcal{S}$, $\mathcal{S} \subset \mathcal{T}$, *and nk maps* $\mathbf{S}_i^j : C \rightarrow \mathcal{S}$, *such that for every configuration* $\boldsymbol{q}$ *in C, the screws* $\mathbf{S}_i^j(\boldsymbol{q})$ *satisfy Equation* (4.2), *if and only if all the velocities* ω_i^j *are feasible for the mechanism in this configuration.*

In other words, HCMs are mechanisms, whose instantaneous kinematics is entirely described by (4.2).

Equation (4.2) can be modified into an equivalent expression that will match the form $\boldsymbol{Lm} = 0$ of the velocity equation defined in Chapter 3, Equation (3.5),where $\boldsymbol{L}$ is a

$kn \times (k+1)n$ matrix and $\boldsymbol{m} = (\boldsymbol{T}, \boldsymbol{\Omega}) = (\boldsymbol{T}, \boldsymbol{\Omega}^a, \boldsymbol{\Omega}^p)$ Let $\boldsymbol{J}_j$ be the Jacobian of the j-th serial subchain, i.e., a matrix of columns $\mathbf{S}_i^j$, $i = 1, \ldots, n$. Let $\boldsymbol{J}_j^a$ be the matrix composed of the active-joint screws only, i.e., of columns $\mathbf{S}_i^j$, $i \in A_j$. Let $\boldsymbol{J}_j^p$ be the matrix composed of the passive-joint screws only, i.e., of columns $\mathbf{S}_i^j$, $i \notin A_j$. By neglecting the zero rows, one can consider these three matrices as $n \times n$, $n \times n_j$ and $n \times (n - n_j)$-dimensional, respectively. Then, (4.2) can be rewritten as,

$$\boldsymbol{T} = \boldsymbol{J}_j \boldsymbol{\Omega}_j = \boldsymbol{J}_j^a \boldsymbol{\Omega}_j^a + \boldsymbol{J}_j^p \boldsymbol{\Omega}_j^p \quad j = 1, \ldots, k \tag{4.3}$$

where $\boldsymbol{\Omega}_j$, $\boldsymbol{\Omega}_j^a$ and $\boldsymbol{\Omega}_j^p$ are vectors (column matrices) composed of all, the active-, and the passive-joint velocities respectively, in the j-th subchain. By rearranging (4.3), we obtain the velocity equation for a general HCM:

$$\begin{bmatrix} -\boldsymbol{I}_n & \boldsymbol{J}_1^a & 0 & \cdots & 0 & \boldsymbol{J}_1^p & 0 & \cdots & 0 \\ -\boldsymbol{I}_n & 0 & \boldsymbol{J}_2^a & & \vdots & 0 & \boldsymbol{J}_2^p & & \vdots \\ \vdots & \vdots & & \ddots & 0 & \vdots & & \ddots & 0 \\ -\boldsymbol{I}_n & 0 & \cdots & 0 & \boldsymbol{J}_k^a & 0 & \cdots & 0 & \boldsymbol{J}_k^p \end{bmatrix} \begin{bmatrix} \boldsymbol{T} \\ \boldsymbol{\Omega}^a \\ \boldsymbol{\Omega}^p \end{bmatrix} = 0. \tag{4.4}$$

In Equation (4.4) above, $\boldsymbol{I}_n$ is the $n \times n$ unit matrix.

4.2. ***Example.*** Consider the 6-dof platform manipulator shown in Figure 4.1. In this case, $n = 6$, $\mathcal{S} = \mathcal{T}$, the output is the twist of the platform ABC, $\boldsymbol{T} = \mathbf{T}$, and $\mathbf{S}_i^p$ are the joint screws of the mechanism. The spherical joints are modelled by three linearly-independent rotations through the center of the joint. The sets of active joints in each subchain are $A_A = \{1, 2, 3\}$, $A_B = \{2, 3\}$ and $A_C = \{3\}$. The velocity equation, in either of the forms (4.2), (4.3) or (4.4), is $6 \times 3 = 18$ dimensional. The mechanical design and kinematic analysis of this manipulator architecture was reported in (Zlatanov et al. 1992).

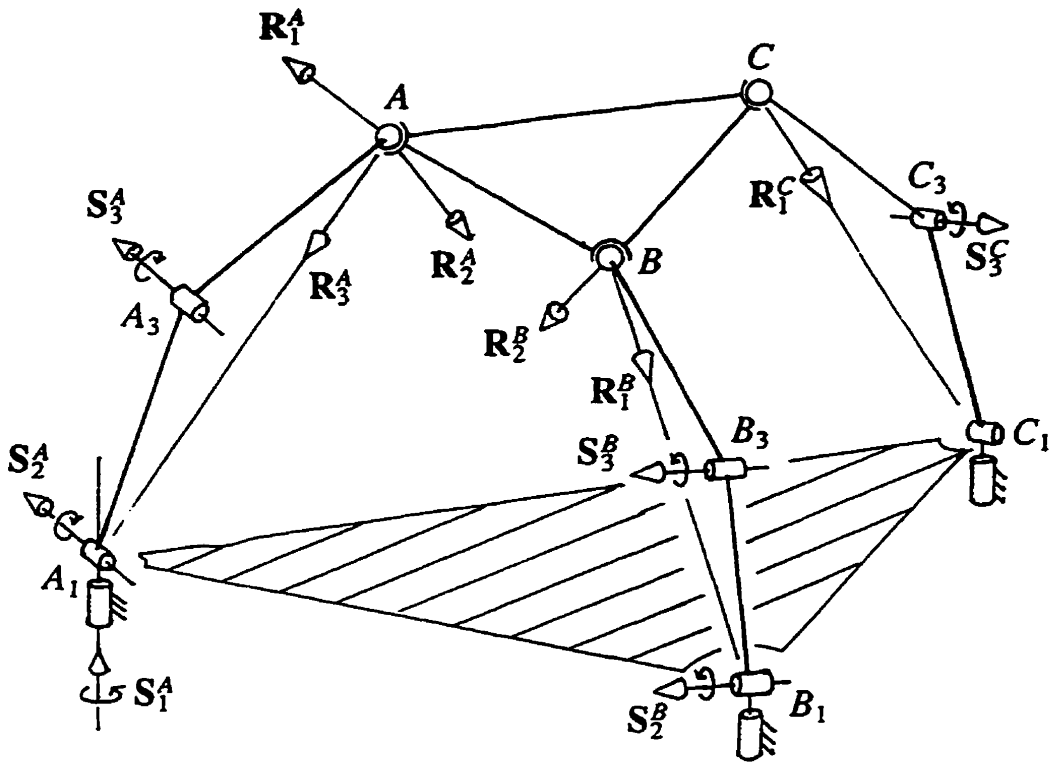

Figure 4.1. A 6-dof hybrid-chain manipulator.

4.3. Example. For the manipulator shown in Figure 4.2, $n = 3$, $\mathcal{S}$ is the screw system of planar motion, $\boldsymbol{T} = [\omega, v_x, v_y]^{\mathrm{T}}$, where v_x and v_y are the planar velocity components of point C, $\Omega^a = [\omega_2^A, \omega_2^B, \omega_2^C]^{\mathrm{T}}$, and $\Omega^P = [\omega_1^A, \omega_3^A, \omega_1^B, \omega_3^B, \omega_1^C, \omega_3^C]^{\mathrm{T}}$. $\mathbf{S}_i^P$, $i = 1, \ldots, n$, $P = A, B, C$, are the joint screws of the mechanism. In a screw basis of $\mathcal{T}$, corresponding to a Cartesian frame with two coordinate axes (x and y) in the plane of the mechanism, only three of the equations of (4.2) are nonzero ($\mathbf{T}$ and all $\mathbf{S}_i^P$ being in $\mathcal{S}$), therefore (4.3) and (4.4) are $3 \times 3 = 9$ dimensional. Equation (4.4) is:

$$\begin{bmatrix} I_3 & S_2^A & 0 & 0 & S_1^A & S_3^A & 0 & 0 & 0 & 0 \\ I_3 & 0 & S_2^B & 0 & 0 & 0 & S_1^B & S_3^B & 0 & 0 \\ I_3 & 0 & 0 & S_2^C & 0 & 0 & 0 & 0 & S_1^C & S_3^C \end{bmatrix} \begin{bmatrix} T \\ \Omega^a \\ \Omega^P \end{bmatrix} = 0, \quad (4.5)$$

where S_i^P are the 3-dimensional vectors composed of the three nonzero coordinates of the joint screws, e.g., $S_2^A = [0, \cos q_1^A, \sin q_1^A]^{\mathrm{T}}$, $S_1^A = [1, y_1^A, -x_1^A]^{\mathrm{T}}$.

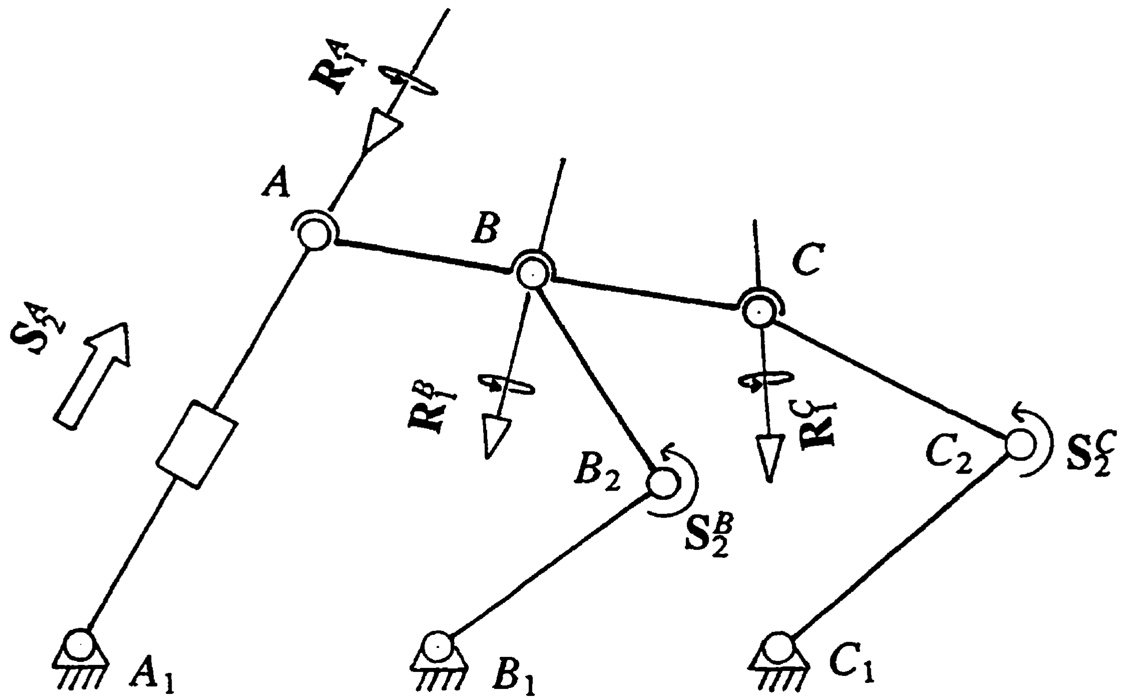

Figure 4.2. A 3-dof planar hybrid-chain manipulator.

4.4. ***Example.*** The five-bar linkage shown in Figure 4.3 can be considered as an HCM, if the "output link" is the point C and two of the four joint angles that are not at C are actively controlled. Then, the output is $\boldsymbol{T} = [v_x, v_y]^T$, where v_x and v_y are the velocity components of point C, and therefore $\mathcal{S}$ is the system of planar translations. Since the actual joint screws of the linkage are all rotational, and thus, they do not belong to $\mathcal{S}$, we need to redefine the joint screws. They will be considered only in coordinate systems with an origin coinciding with C and their rotational coordinates will be ignored. Thus, the new "joint screws" are translations equal to the moment with respect to C of the actual joint rotations. In this way, the instantaneous kinematics of the linkage is described by two of the equations in (4.2) and the mechanism can be treated as an HCM. If the actuated joints are these at the base, the 4-dimensional velocity equation will be:

$$\begin{bmatrix} I_2 & S_1^A & 0 & S_2^A & 0 \\ I_2 & 0 & S_1^B & 0 & S_2^A \end{bmatrix} \begin{bmatrix} \boldsymbol{T} \\ \Omega^a \\ \Omega^p \end{bmatrix} = 0, \tag{4.6}$$

where $\Omega^a = [\omega_1^A, \omega_1^B]^T$, $\Omega^p = [\omega_2^A, \omega_2^B]^T$, and $S_i^P = [y_i^P, -x_i^P]^T$. This linkage was considered as a manipulator and analyzed in (Asada and Youcef-Toumi 1984) and (Kumar 1990).

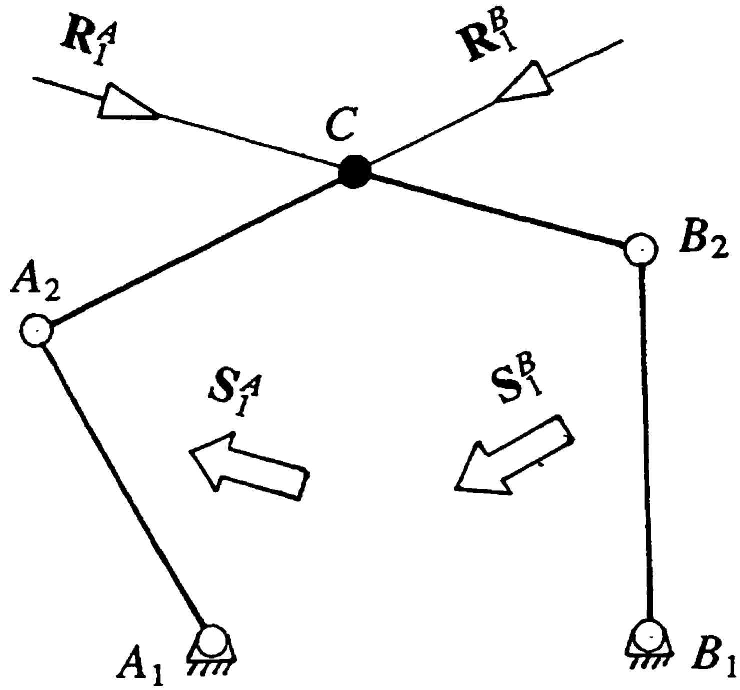

Figure 4.3. A five-bar linkage considered as a hybrid-chain manipulator.

4.3. The Input-Output Equation

Let us denote the subspace of $\mathcal{T}$ spanned by the active joint screws in the j-th subchain by $\mathcal{A}_j$ ($\mathcal{A}_j = \mathrm{Span}\{ \mathbf{S}_i^j \mid i \in A_j\}$). The subspace spanned by the passive joint screws in the j-th subchain is denoted by $\mathcal{P}_j = \mathrm{Span}\{ \mathbf{S}_i^j \mid i \notin A_j\}$, while the subspace of all the joint screws in the subchain is referred to as $\mathcal{T}_j = \mathrm{Span}\{ \mathbf{S}_i^j \mid i = 1, \ldots, n \}$.

If all the $n - n_j$ passive screws in the subchain are linearly independent, then $\mathcal{P}_j$ is of dimension $n - n_j$, otherwise the dimension is smaller. Thus, in general,

$$\dim \mathcal{P}_j = (n - n_j) - d_j \,, \tag{4.7}$$

where $0 \le d_j \le n - n_j$.

If $\mathcal{G}$ is a subspace of $\mathcal{T}$, then its *(reciprocal) orthogonal complement* $\mathcal{G}^{\perp}$ is defined by, $\mathcal{G}^{\perp} = \{\mathbf{T} \in \mathcal{T} \mid \mathbf{T} \circ \mathbf{G} = 0 \ \forall \mathbf{G} \in \mathcal{G}\}$. Here, the symbol "$\circ$" denotes the so-called "reciprocal product" – an indefinite scalar product in $\mathcal{T}$, given by $\mathbf{A} \circ \mathbf{B} = \mathbf{A} \bullet \Pi \mathbf{B} = \mathbf{A}^{\mathrm{T}} \Pi \mathbf{B}$, where "$\bullet$" denotes the standard dot product, and Π is the symmetric matrix:

$$\Pi = \begin{bmatrix} 0 & I_3 \\ I_3 & 0 \end{bmatrix}.$$

I_3 is the 3×3 identity matrix. It is known that $\dim \mathcal{G}^{\perp} = 6 - \dim \mathcal{G}$. Therefore,

$$\dim \mathcal{P}_j^{\perp} = 6 - n + n_j + d_j. \tag{4.8}$$

For each j, one can choose a maximum collection of linearly independent twists in $\mathcal{P}_j^{\perp} - \mathcal{S}^{\perp}$ and denote them by $\mathbf{R}_l^j$, $l = 1, \ldots, n_j + d_j$. In other words, a basis of $\mathcal{P}_j^{\perp}$ can be chosen in such a way that $6 - n$ of the basis vectors are in $\mathcal{S}^{\perp}$ and the remaining $n_j + d_j$ basis vectors are the twists $\mathbf{R}_l^j$. When $n = 6$, $\mathbf{R}_l^j$ can be any basis of $\mathcal{P}_j^{\perp}$. Obviously, such a set $\{\mathbf{R}_l^j\}$ is not unique. It can be shown that the results presented in this chapter are invariant with respect to the choice of the twists $\mathbf{R}_l^j$.

We now take the reciprocal product of each $\mathbf{R}_l^j$ and Equation (4.2) for the j-th subchain,

$$\mathbf{R}_l^j \Pi \mathbf{T} = \sum_{i=1}^{n} \dot{q}_i^j \mathbf{R}_l^{j\mathrm{T}} \Pi \mathbf{S}_i^j \,. \tag{4.9}$$

Since the twists $\mathbf{R}_l^j$ belong to $\mathcal{P}_j^{\perp}$, their reciprocal products with the elements of $\mathcal{P}_j$ are zero. Therefore,

$$\mathbf{R}_l^j \Pi \mathbf{T} = \sum_{i \in A_j} \dot{q}_i^j \mathbf{R}_l^{j\mathrm{T}} \Pi \mathbf{S}_i^j \tag{4.10}$$

for $l = 1, \ldots, n_j + d_j$ and $j = 1, \ldots, k$. In a matrix form, (4.9) and (4.10) are written as:

$$R_j T = R_j J_j \Omega_j = R_j J_j^a \Omega_j^a \tag{4.11}$$

where $\boldsymbol{R}_j$ is a matrix of dimension $(n_j + d_j) \times n$ obtained from n of the columns of the matrix

$$\boldsymbol{R}_j = \begin{bmatrix} \mathbf{R}_1^{j\,\mathrm{T}} \Pi \\ \mathbf{R}_2^{j\,\mathrm{T}} \Pi \\ \vdots \\ \mathbf{R}_{n_j+d_j}^{j\;\;\mathrm{T}} \Pi \end{bmatrix}. \tag{4.12}$$

Only the columns that correspond to nonzero coordinates of the output twist, **T**, are considered.

Thus, $n + \sum_{j=1}^{k} d_j$ scalar equations are obtained,

$$\begin{bmatrix} \boldsymbol{R}_1 \\ \boldsymbol{R}_2 \\ \vdots \\ \boldsymbol{R}_k \end{bmatrix} \boldsymbol{T} = \begin{bmatrix} \boldsymbol{H}_1 & 0 & \cdots & 0 \\ 0 & \boldsymbol{H}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \boldsymbol{H}_k \end{bmatrix} \Omega^a. \tag{4.13}$$

$\boldsymbol{H}_j = \boldsymbol{R}_j J_j^a$ is a matrix of dimension $(n_j + d_j) \times n_j$ with elements

$$(H_j)_{lm} = \mathbf{R}_l^{j\mathrm{T}} \Pi \mathbf{S}_{i_m}^{j}, \quad l = 1, \ldots, n_j + d_j; \quad m = 1, \ldots, n_j$$

where i_m, $m = 1, \ldots, n_j$ are the elements of the index set A_j.

Denoting the $(n + \sum_{j=1}^{k} d_j) \times n_j$-dimensional matrices in (4.13) by $\boldsymbol{R}$ and $\boldsymbol{H}$, the following theorem can be stated:

4.5. Theorem. *Let the HCM be in a given configuration, $\boldsymbol{q}$, and let the matrices $\boldsymbol{R} = \boldsymbol{R}(\boldsymbol{q})$ and $\boldsymbol{H} = \boldsymbol{H}(\boldsymbol{q})$ be defined as above. Then an n-dimensional input vector, $\boldsymbol{T}$, (i.e., the corresponding twist, $\mathbf{T} \in \mathcal{S}$) and an n-dimensional vector, Ω^a, can be a feasible pair of output and input for the HCM in $\boldsymbol{q}$, if and only if:*

$$\boldsymbol{R}\boldsymbol{T} = \boldsymbol{H}\,\Omega^a. \tag{4.14}$$

Proof

Sufficiency. If $\boldsymbol{T}$ and Ω^a are feasible, there exist passive-joint velocities, Ω^p, for which the velocity Equation (4.2) is satisfied. By multiplying this equation with a basis of $\mathcal{P}_j^{\perp}$, as was described above, Equation (4.14) is obtained.

Necessity. Let $\boldsymbol{T}$ and Ω^a satisfy (4.14). We must show that there exists a passive-joint velocity vector, Ω^p, such that $\boldsymbol{T}$, Ω^a and Ω^p satisfy the velocity Equation (4.2). Let us fix j and consider the twist,

$$\mathbf{V}_j = \mathbf{T} - \sum_{i \in A_j} \dot{q}_i^j \mathbf{S}_i^j .$$

Equation (4.14) implies that the reciprocal product of $\mathbf{V}_j$ and the twist $\mathbf{R}_l^j$ is zero for all l, $l = 1, \ldots, n_j + d_j$. Since $\mathbf{V}_j$ is in $\mathcal{S}$, this is equivalent to $\mathbf{V}_j \in (\mathcal{P}_j^{\perp})^{\perp} = \mathcal{P}_j$. Therefore, $\mathbf{V}_j$ can be presented as a linear combination of the passive screws of the j-th subchain, i.e., there exist scalars $\dot{q}_i^j$, $i \notin A_j$, such that

$$\mathbf{V}_j = \mathbf{T} - \sum_{i \in A_j} \dot{q}_i^j \mathbf{S}_i^j = \sum_{i \notin A_j} \dot{q}_i^j \mathbf{S}_i^j ,$$

i.e., Equation (4.2) is satisfied for that j. Since the argument is valid for any j, the existence of Ω^p is established. □

4.6. Example. Consider the linkage described in Example 4.4, Figure 4.3. In this case, $\mathcal{S}^{\perp}$ is 4-dimensional and is spanned by the planar motions and a vertical translation (perpendicular to the linkage plane). $\mathcal{P}_P$ is a 1-system (1-dimensional screw subspace) along $[0, 0, 0, y_2^P, -x_2^P, 0]^{\mathrm{T}}$. The reciprocal screws, which are in $\mathcal{P}_P^{\perp}$ but not in $\mathcal{S}^{\perp}$, are $\mathbf{R}_1^P = [x_2^P, y_2^P, 0, 0, 0, 0]^{\mathrm{T}}$, i.e., rotations with axes along CP_2, $P = A, B$. The input output equation is:

$$\begin{bmatrix} x_2^A y_2^A \\ x_2^B y_2^B \end{bmatrix} \begin{bmatrix} v_x \\ v_y \end{bmatrix} = \begin{bmatrix} x_2^A y_1^A - y_2^A x_1^A & 0 \\ 0 & x_2^B y_1^B - y_2^B x_1^B \end{bmatrix} \begin{bmatrix} \omega_1^A \\ \omega_1^B \end{bmatrix} . \tag{4.15}$$

$\mathcal{T}_P$ could degenerate only if the points C and P_2 were to coincide, which is an impossible case. If, however, one of the active joints were at A_2 instead of A_1, it could be possible to have $C = A_1$ (Figure 4.4) and $\mathcal{T}_A$ would be zero. Then, there would be two linearly independent screws $\mathbf{R}_i^A$: $\mathbf{R}_1^A = (1, 0, 0, 0, 0, 0)$ and $\mathbf{R}_2^A = (0, 1, 0, 0, 0, 0)$, and Equation (4.14) would become:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ x_2^B & y_2^B \end{bmatrix} \begin{bmatrix} v_x \\ v_y \end{bmatrix} = \begin{bmatrix} y_2^A & 0 \\ -x_2^A & 0 \\ 0 & -y_2^B x_1^B \end{bmatrix} \begin{bmatrix} \omega_2^A \\ \omega_1^B \end{bmatrix}. \tag{4.16}$$

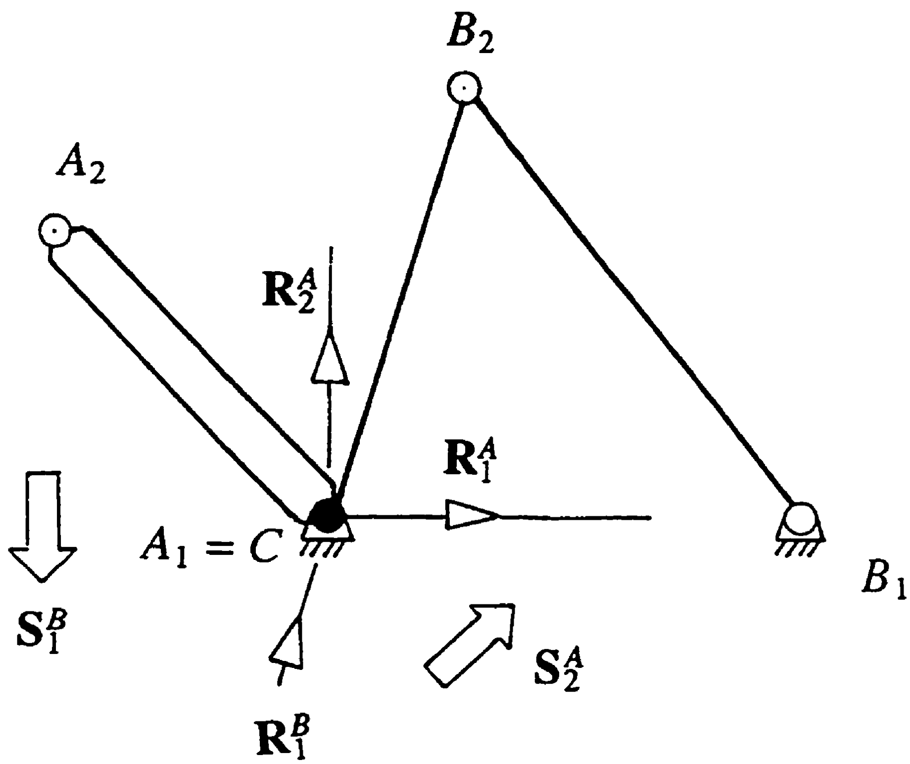

Figure 4.4. An RPM-, IO, and II-type singularity.

4.7. Example. For the mechanism of Example 4.3, Figure 4.2, $\mathcal{T}_P = \mathrm{Sp}(\mathbf{S}_1^P, \mathbf{S}_3^P)$ is a 2-system unless the two points P_1 and P coincide for some P. Assuming $P_1 \neq P$ for all P, the spaces $\mathcal{T}_P^\perp$ are 4 dimensional and spanned by the twists of planar motion and a rotation intersecting the axes of both $\mathbf{S}_1^P$ and $\mathbf{S}_3^P$. Since, in this case, $\mathcal{S}^\perp = \mathcal{S}$, the reciprocal screws can be chosen as the rotations with axes through the points P and P_1, i.e., $\mathbf{R}_1^P = (x_1^P - x^P, y_1^P - y^P, 0, 0, 0, m^P)$, where $m^P = |PP_1| \mathrm{dist}(C, PP_1)$ are the moments of these rotations with respect to C (the origin of the Cartesian frame is assumed at C). Then, the input-output equation can be obtained as:

$$\begin{bmatrix} m^A & x_1^A - x^A & y_1^A - y^A \\ m^B & x_1^B - x^B & y_1^B - y^B \\ 0 & x_1^C & y_1^C \end{bmatrix} \begin{bmatrix} \omega \\ v_x \\ v_y \end{bmatrix} = \begin{bmatrix} -|AA_1| & 0 & 0 \\ 0 & \mathbf{R}_1^B \circ \mathbf{S}_2^B & 0 \\ 0 & 0 & x_1^C y_2^C - y_1^C x_2^C \end{bmatrix} \begin{bmatrix} \omega_2^A \\ \omega_2^B \\ \omega_2^C \end{bmatrix}, \quad (4.17)$$

where $\mathbf{R}_1^B \circ \mathbf{S}_2^B = (x_1^B - x^B)(y_2^B - y^B) - (y_1^B - y^B)(x_2^B - x^B)$.

As in Example 4, if $P_1 = P$ for some P, the space $\mathcal{P}_P^\perp$ would be of a higher dimension (five) and there would be two linearly independent twits $\mathbf{R}_i^P$ for this P. Then the matrices in the input-output equation would be rectangular.

4.8. Example. For the platform-type manipulator in Example 4.2, Figure 4.1, $n = 6$, $\mathcal{S}^\perp = 0$ and the screws $\mathbf{R}_i^P$ must be chosen as a basis of $\mathcal{P}_P^\perp$. The unit vector parallel to the axis of $\mathbf{S}_i^P$ is denoted by $\mathbf{k}_i^P$, the unit vectors along PP_1 are denoted by by $\mathbf{p}_1$ (= $\mathbf{a}_1$, $\mathbf{b}_1$, $\mathbf{c}_1$), and finally the vectors along AB and AC by $\mathbf{b}$ and $\mathbf{c}$. For simplicity, we assume that $|P_1P_3| \neq |PP_3|$ for all P, and configurations in which $P_1 = P$ do not exist.

The subspace $\mathcal{P}_A^\perp$ is the screw system of all rotations with axes through A. It is convenient to choose as its basis the three rotations with axes parallel to $\mathbf{k}_2^A$, $\mathbf{a}_1 \times \mathbf{k}_2^A$, and $\mathbf{a}_1$. Then, if we assume that the origin of the Cartesian reference frame is at A,

$$\boldsymbol{R}_A = \begin{bmatrix} 0 & 0 & 0 & \mathbf{k}_2^{A\,\mathrm{T}} \\ 0 & 0 & 0 & [\mathbf{a}_1 \times \mathbf{k}_2^A]^{\mathrm{T}} \\ 0 & 0 & 0 & \mathbf{a}_1^{\mathrm{T}} \end{bmatrix}. \quad (4.18)$$

Also, for $\boldsymbol{H}_A = \boldsymbol{R}_A J_A^a$, we obtain:

$$H_A = \begin{bmatrix} \varepsilon m_1^A & 0 & 0 \\ 0 & m_2^A & (m_3^A)_2 \\ 0 & 0 & (m_3^A)_3 \end{bmatrix}, \tag{4.19}$$

where $m_i^P = |\mathbf{m}_P(\mathbf{S}_i^P)|$, $(m_i^P)_l$ is the projection of $\mathbf{m}_P(\mathbf{S}_i^P)$ on the axis of $\mathbf{R}_l^P$ and $\varepsilon = \pm 1$. $\mathbf{m}_Q(\mathbf{S}_i^P)$ is the moment of $\mathbf{S}_i^P$ with respect to point Q.

In the subchain B, there are four passive joint screws which span either a 4-dimensional or a 3-dimensional subspace, and $\mathcal{P}_B^{\perp}$ is either 2 or 3 dimensional. The passive screws are linearly dependent, only when BB_1 is vertical and the axis of $\mathbf{S}_1^B$ passes through P (the so-called "wrist-above-shoulder" configuration of the serial subchain). In this case, $\mathcal{P}_B^{\perp}$ is analogous to $\mathcal{P}_A^{\perp}$. When the passive screws are independent, $\mathcal{P}_B^{\perp}$ consists of all rotations through B lying in a plane perpendicular to $\mathbf{k}_2^B$. We choose the first basis vector parallel to $\mathbf{b}_1 \times \mathbf{k}_2^B$, the second basis vector parallel to $\mathbf{b}_1$, and the third (if necessary) parallel to $\mathbf{k}_2^B$. Then, the matrices R_B and H_B are obtained as,

$$R_B = \begin{bmatrix} [\mathbf{b}\times(\mathbf{b}_1\times\mathbf{k}_2^B)]^T & [\mathbf{b}_1\times\mathbf{k}_2^B]^T \\ [\mathbf{b}\times\mathbf{b}_1]^T & \mathbf{b}_1^T \\ \{\ [\mathbf{b}\times\mathbf{k}_2^B]^T & \mathbf{k}_2^{BT}\ \} \end{bmatrix} \tag{4.20}$$

$$H_B = \begin{bmatrix} m_2^B & (m_3^B)_1 \\ 0 & (m_3^B)_2 \\ \{\ 0 & 0\ \} \end{bmatrix}. \tag{4.21}$$

The last rows appear only if the subchain is in a wrist-above-shoulder configuration.

In a similar way, R_C and H_C are obtained as:

$$R_C = \begin{bmatrix} [\mathbf{c}\times\mathbf{c}_1]^T & \mathbf{c}_1^T \\ \{\ [\mathbf{c}\times\mathbf{k}_2^C]^T & \mathbf{k}_2^{CT}\ \} \end{bmatrix} \tag{4.22}$$

$$H_C = \begin{bmatrix} (m_3^C)_1 \\ \{\ 0\ \} \end{bmatrix} \tag{4.23}$$

where the rows enclosed in braces are necessary only when C is exactly above C_1.

From Equations (4.19) to (4.20) we obtain the input-output equation in the form of (4.14),

$$
\begin{bmatrix}
0 & \mathbf{k}_2^{A\mathrm{T}} \\
0 & [\mathbf{a}_1 \times \mathbf{k}_2^A]^{\mathrm{T}} \\
0 & \mathbf{a}_1^{\mathrm{T}} \\
[\mathbf{b} \times (\mathbf{b}_1 \times \mathbf{k}_2^B)]^{\mathrm{T}} & [\mathbf{b}_1 \times \mathbf{k}_2^B]^{\mathrm{T}} \\
[\mathbf{b} \times \mathbf{b}_1]^{\mathrm{T}} & \mathbf{b}_1^{\mathrm{T}} \\
[\mathbf{c} \times \mathbf{c}_1]^{\mathrm{T}} & \mathbf{c}_1^{\mathrm{T}} \\
\{\ [\mathbf{b} \times \mathbf{k}_2^B]^{\mathrm{T}} & \mathbf{k}_2^{B\mathrm{T}}\ \} \\
\{\ [\mathbf{c} \times \mathbf{k}_2^C]^{\mathrm{T}} & \mathbf{k}_2^{C\mathrm{T}}\ \}
\end{bmatrix}
\begin{bmatrix} \boldsymbol{\omega} \\ \mathbf{v}_A \end{bmatrix} =
$$

$$
\begin{bmatrix}
\varepsilon m_1^A & 0 & 0 & 0 & 0 & 0 \\
0 & m_2^A & (m_3^A)_2 & 0 & 0 & 0 \\
0 & 0 & (m_3^A)_3 & 0 & 0 & 0 \\
0 & 0 & 0 & m_2^B & (m_3^B)_1 & 0 \\
0 & 0 & 0 & 0 & (m_3^B)_2 & 0 \\
0 & 0 & 0 & 0 & 0 & (m_3^C)_1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix} \omega_1^A \\ \omega_2^A \\ \omega_3^A \\ \omega_2^B \\ \omega_3^B \\ \omega_3^C \end{bmatrix}. \tag{4.24}
$$

The matrices $\boldsymbol{R}$ and $\boldsymbol{H}$ in (4.14) are square, only when $d_j = 0$ for all j (i.e., when in all subchains $\mathcal{P}_j$ is of maximum dimension). In this case, if specific bases of $\mathcal{P}_j^{\perp}$ are chosen, the matrix $\boldsymbol{H}$ will be diagonal. To achieve this, each of the basis vectors is chosen to be not only reciprocal to $\mathcal{P}_j$, but also reciprocal to all active joint screws in the subchain but one. Based on this idea, an equation similar to (4.14), with a square matrix on the left-hand side and a diagonal matrix on the right-hand side, was first obtained in (Mohammed and Duffy 1984), although the standard dot product was used instead of the reciprocal product. Later, (Kumar 1990, Agrawal 1990) proposed the use of reciprocity for the solution of the instantaneous kinematics of HCM and the analysis of their singular configurations. In all these works, unlike in the present paper, the reciprocal screws did not have to be linearly independent for a fixed j. However, this type of an input-output equation is effective only when the joint screws of each subchain are linearly independent and therefore a large class of singular configurations cannot be explored. The reason for this limitation is that, unlike (4.14), such an equation is in general only a sufficient but *not* a *necessary* condition for the feasibility of the input and output.

4.4. Conditions for Singularity

In this section, the conditions for each of the six singularity types are given by six theorems. In all of them, we assume that the HCM is in a given configuration, $\boldsymbol{q}$.

4.9. ***Theorem.*** *The following are equivalent:*

(i) $\boldsymbol{q}$ *is a redundant-output (RO) type singularity.*

(ii) rank $\boldsymbol{R} < n$.

(iii) $\bigcap_{j=1}^{k} \mathcal{P}_j \neq 0$.

Proof.

(i) $\Rightarrow$ *(ii)*. Let $\mathbf{T} \neq 0$ and $\Omega^a = 0$ satisfy the velocity equation (for some Ω^p). Then, from Equation (4.11), $\boldsymbol{RT} = 0$ and therefore rank $\boldsymbol{R} < n$.

(ii) $\Rightarrow$ *(iii)*. If rank $\boldsymbol{R} < n$, then a twist $\mathbf{V} \in \mathcal{S}$, $\mathbf{V} \neq 0$, exists, such that $\boldsymbol{RV} = 0$. This means that $\mathbf{V}$ will be reciprocal to the $\mathbf{R}_l^j$ vectors for all l and all j. Therefore, $\mathbf{V}$ will be in the reciprocal complement of $\mathcal{P}_j^{\perp}$ for all j . But $(\mathcal{P}_j^{\perp})^{\perp} = \mathcal{P}_j$, and therefore,

$$0 \neq \mathbf{V} \in \bigcap_{j=1}^{k} \mathcal{P}_j .$$

(iii) ⇒ *(i)*. Let $\mathbf{V} \neq 0$ and $\mathbf{V} \in \mathcal{P}_j$, for all j. Then, there are scalars, λ_i^j, such that:

$$\mathbf{V} = \sum_{i \notin A_j} \lambda_i^j \mathbf{S}_i^j \quad j = 1, \ldots, k .$$

Let us define $\mathbf{T}$ as $\mathbf{T} = \mathbf{V}$ and Ω^p by $q_i^j = \lambda_i^j,\ i \notin A_j$. Then T, Ω^p and $\Omega^a = 0$ will satisfy the velocity equation (4.2), and a RO-type singularity is present. □

The theorem shows that the occurrence of RO-type singularity, which is often the only type of singularity addressed when parallel manipulators are analyzed, is entirely determined by the configuration of the passive joints and not by the active-joint screws.

4.10. Example. The configuration shown in Figure 4.5 is an RO-type singularity.

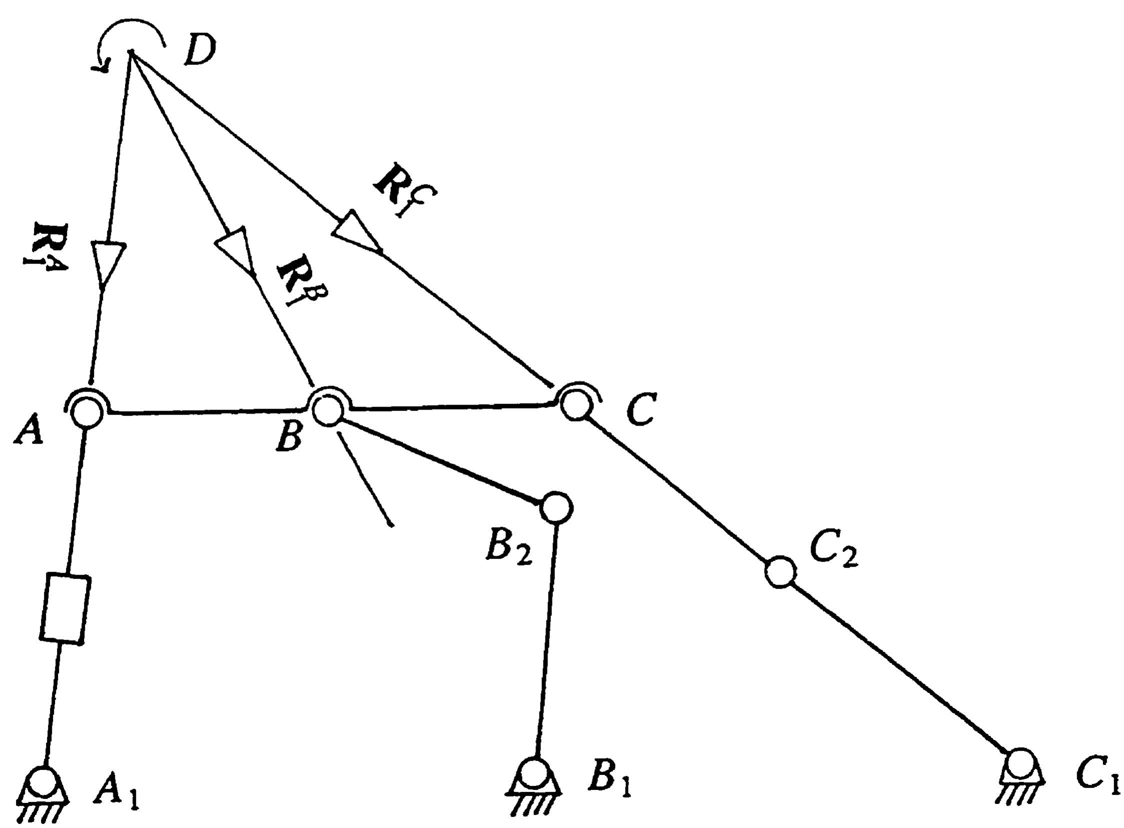

Figure 4.5. An RI-, RO-, IO, and II-type singularity.

The end-effector can rotate (instantaneously) about the point D, even if all the inputs, ω_2^P, (see Figure 4.2) are zero. On the other hand, the matrix $\boldsymbol{R}$ for this manipulator, as shown in the left side of Equation (4.17) is singular in this configuration, since it is composed of the nonzero coordinates of the three rotations $\mathbf{R}_1^P$, $P = A, B, C$, which are linearly dependent as they are co-planar and intersect in D.

4.11. Example. Figure 4.6 also shows an RO-type singularity.

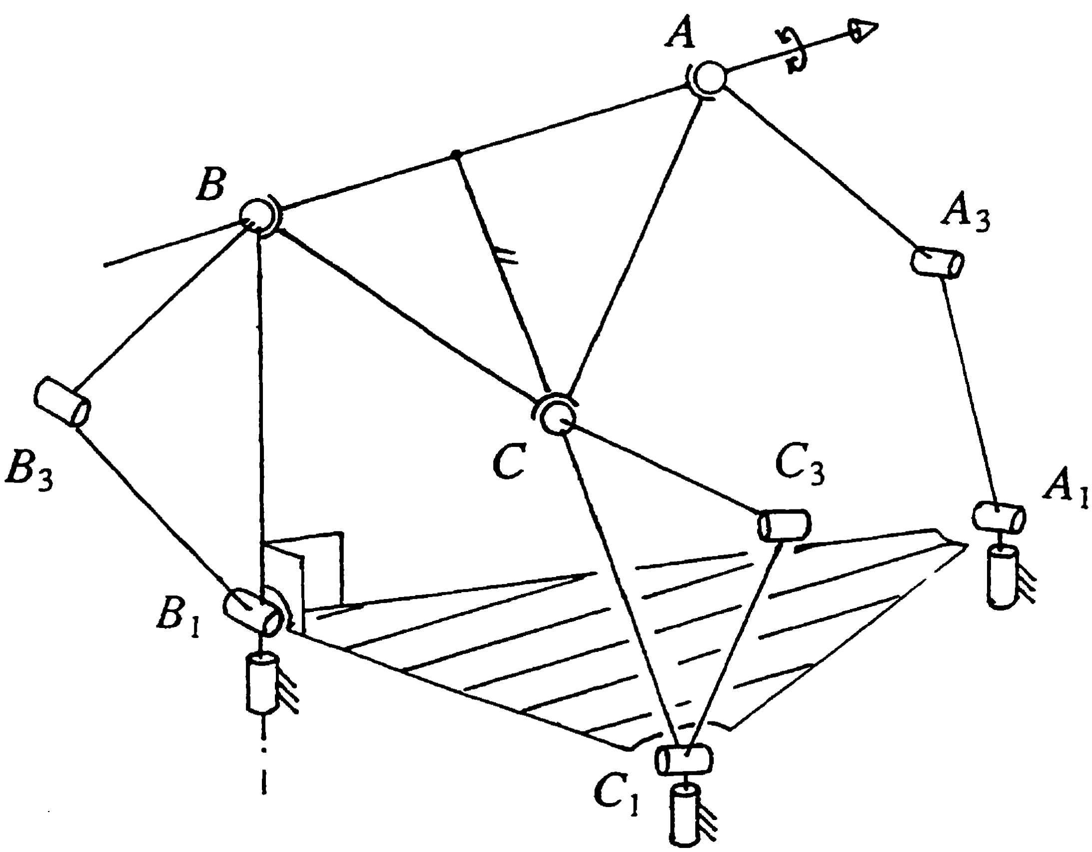

Figure 4.6. An RO-, RPM-, IO, and II-type singularity.

In the depicted configuration, the only singular serial subchain is B (wrist-above-shoulder) and also the point C_1 lies in the plane ABC. The redundant-output freedom is a rotation about the axis AB, which is feasible even if all the active-joint velocities are zero. As predicted by Theorem 4.9, the matrix $\boldsymbol{R}$ is singular (rank $\boldsymbol{R} < 6$) although it has seven rows. For any configuration of the manipulator, $\boldsymbol{R}$ (given in (4.24)) is rank deficient if and only if rank $\Delta < 3$, where Δ is the matrix,

$$\Delta = \begin{bmatrix} [\mathbf{b}\times(\mathbf{b}_1\times\mathbf{k}_2^B)]^\mathrm{T} \\ [\mathbf{b}\times\mathbf{b}_1]^\mathrm{T} \\ [\mathbf{c}\times\mathbf{c}_1]^\mathrm{T} \\ \left\{ [\mathbf{b}\times\mathbf{k}_2^B]^\mathrm{T} \right\} \\ \left\{ [\mathbf{c}\times\mathbf{k}_2^C]^\mathrm{T} \right\} \end{bmatrix}. \tag{4.25}$$

In the considered configuration, Δ is composed of only the first four rows of Equation (4.25). When the points C_1, A, B and C lie on one plane, the vector $\mathbf{c}\times\mathbf{c}_1$ is perpendicular to this plane and to $\mathbf{b}$ in particular. Thus, all row vectors in Δ are perpendicular to $\mathbf{b}$, and therefore are co-planar.

4.12. Example. It can be seen that the configuration shown in Figure 4.4 is *not* an RO-type singularity. When the input at A_2 is zero, the point C is fixed. Accordingly, the matrix $\boldsymbol{R}$, as given by (4.16) is of rank two.

4.13. Theorem. *The following are equivalent:*

(i) *q is a redundant-input (RI) singularity.*

(ii) rank $\boldsymbol{H} < n$.

(iii) *For some j,* rank $\boldsymbol{H}_j < n_j$.

(iv) *For some j, there exist scalars λ_i, $i \in A_j$, not all zero, such that,*

$$\textstyle\sum_{i\in A_j} \lambda_i\, \mathbf{S}_i^j \in \mathcal{P}_j\,.$$

(v) *For some j,* $\dim \mathcal{T}_j < n_j + \dim \mathcal{P}_j$.

(vi) *For some j, either $\mathcal{A}_j \cap \mathcal{P}_j \neq 0$ or* $\dim \mathcal{A}_j < n_j$.

Proof

(i) ⇔ *(ii)*. According to Definition 3.6 an RI-type singularity occurs, when a nonzero input and a zero output are simultaneously feasible. Theorem 4.5 implies that this is possible, if and only if $\boldsymbol{H\Omega}^a = 0$ for some $\boldsymbol{\Omega}^a \neq 0$, which is equivalent to rank $\boldsymbol{H} < n$.

(ii) ⇔ *(iii)*. Implied by the block-diagonal structure of $\boldsymbol{H}$.

(iii) ⇔ *(iv)*. rank $\boldsymbol{H}_j < n_j$, if and only if there exists a vector, $\lambda \neq 0$, such that $\boldsymbol{H}_j\lambda = \boldsymbol{R}_j J_j^a \lambda = 0$. According to the definition of $\boldsymbol{R}_j$, this is equivalent to $\mathbf{R}_l^j \circ J_j^a \lambda = 0$ for all j. In the last expression, $\boldsymbol{J}_j^a$ is interpreted as a matrix with 6-dimensional screws as columns. Since the $\mathbf{R}_l^j$ are (a part of) a basis of $\mathcal{P}_j^{\perp}$, this is equivalent to $\boldsymbol{J}_j^a\lambda \in \mathcal{P}_j$, and the statement of (iv).

(iv) ⇔ *(vi)*. $0 \neq J_j^a\lambda \in \mathcal{P}_j$ is equivalent to $\mathcal{A}_j \cap \mathcal{P}_j \neq 0$, while $0 = J_j^a\lambda$ is the necessary and sufficient condition for dim $\mathcal{A}_j < n_j$.

(v) ⇔ *(vi)*. Follows from: $\dim \mathcal{T}_j \leq \dim \mathcal{A}_j + \dim \mathcal{P}_j \leq n_j + \dim \mathcal{P}_j$. □

Theorem 4.13 shows that the HCM is in an RI-type singularity, when a serial subchain has a singularity "involving" active-joint screws.

4.14. Example. The configuration discussed in Example 4.10, Figure 4.5, is also an RI-type singularity. Indeed, if the end-effector were to be fixed, the input velocity at C_2 could be instantaneously nonzero. As predicted by Theorem 4.13, the $\boldsymbol{H}$ matrix in (4.17) is singular, since one of the diagonal elements, $l_{33} = \boldsymbol{H}_3$, is zero. Also, as suggested by (iv) of the Theorem, the joint screw at C_2 belongs to $\mathcal{P}_C$.

4.15. Example. On the contrary to the manipulator configuration in Example 4.14, the configuration in Figure 4.6 is *not* of RI-type. If the output platform *ABC* were to be fixed, the active-joint velocities would be zero. The matrix $\boldsymbol{H}$ is of full rank since none of the diagonal elements in the left-hand side of (4.24) is zero in this configuration.

4.16. Theorem. *The following are equivalent:*

(i) $\boldsymbol{q}$ *is a redundant-passive-motion (RPM) type singularity.*

(ii) *For some j,* $\dim \mathcal{P}_j < n - n_j$.

(iii) ***H*** *and* ***R*** *are not square.*

Proof

(i) ⇔ *(ii).* An RPM-type singularity is present, when a nonzero motion is possible for zero input and zero output. According to (4.3) this is true if and only if, for some j, there exists a vector, $\Omega_j^p \neq 0$, such that $J_j^p \Omega_j^p = 0$. That is equivalent to $\operatorname{rank} J_j^p < n - n_j$ and (ii).

(ii) ⇔ *(iii).* Follows from the definition of ***R*** and ***H***. □

4.17. Example. The configurations in Figures 4.4 and 4.6 are RPM-type singularities, however the one in Figure 4.5 is not. In Figure 4.6, the subchain singularity (i.e., the singularity of J_B) is due only to the linear dependence of the passive screws, while in Figure 4.5 the active screw in subchain C is "involved" in the linear dependence of the subchain screws.

4.18. Theorem. *The following are equivalent:*

(i) *q is an impossible-output (IO) type singularity.*

(ii) $\operatorname{Im} \boldsymbol{R} - \operatorname{Im} \boldsymbol{H} \neq \varnothing$.

(iii) *For some i,* $1 \leq i \leq n$, *no input vector,* Ω^a, *can satisfy the equation* $r_i = \boldsymbol{H}\Omega^a$, *where* r_i *is the i-th column of* ***R***.

(iv) *For some j,* J_j *is singular.*

Proof

(i) ⇔ *(ii).* A configuration is an IO-type singularity when there exists an n-dimensional vector $\boldsymbol{T}$, which cannot be a feasible output for any values of the joint velocities. According to Theorem 4.5 this is true if and only if there exists $\boldsymbol{T}$, such that $\boldsymbol{RT} \neq \boldsymbol{H}\Omega^a$ for any Ω^a. Thus, for some $\boldsymbol{T}$, $\boldsymbol{RT} \notin \operatorname{Im} \boldsymbol{H}$, or, equivalently $\boldsymbol{RT} \in \operatorname{Im} \boldsymbol{R} - \operatorname{Im} \boldsymbol{H}$.

(ii) $\Leftrightarrow$ *(iii)*. Since $\mathrm{Im}\,\boldsymbol{R} = \mathrm{Span}\{r_i \mid i = 1, \ldots, n\}$.

(i) $\Leftrightarrow$ *(iv)*. A twist, $\mathbf{T} \in \mathcal{S}$, is impossible as an output if and only if, for some j, $\mathbf{T} \notin \mathcal{T}_j$. However, $\mathcal{T}_j \neq \mathcal{S}$ only if $\dim \mathcal{T}_j = \mathrm{rank}\, J_j < n$. □

4.19. Example. All the singular configurations shown in this section are of the IO-type, since in all of them a serial subchain is singular. However, if in Figure 4.5 the angle at C_2 were not extended and the three lines still intersected in D, the configuration would have been an RO- and an II-type but *not* an IO-type singularity.

4.20. Theorem. *The following are equivalent:*

(i) *q is an impossible input (II) type singularity.*

(ii) $\mathrm{Im}\,\boldsymbol{H} - \mathrm{Im}\,\boldsymbol{R} \neq \varnothing$.

(iii) *For some i,* $1 \le i \le n$, *no output vector,* $\boldsymbol{T}$, *can satisfy the equation* $\boldsymbol{RT} = \boldsymbol{l}_i$, *where* $\boldsymbol{l}_i$ *is the i-th column of* $\boldsymbol{H}$.

Proof. The proof is similar to the proof of (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) in Theorem 4.18. □

4.21. Example. All configurations in Figures 4.4, 4.5 and 4.6 are II-type singularities. For example, in Figure 4.4 the input (1, 0, 0) is impossible, since the velocity of the point A must be perpendicular to AA_1. It can be checked from (4.17) that the equation $\boldsymbol{RT} = \boldsymbol{l}_1$ has no solution for $\boldsymbol{T}$.

4.22. Theorem. *The following are equivalent:*

(i) *q is an increased-instantaneous-mobility (IIM) type singularity.*

(ii) *There exist* l, $2 \le l \le n$, *twists,* $\mathbf{A}_{j_1}, \ldots, \mathbf{A}_{j_l}$, $\{j_1, \ldots, j_l\} \subset \{1, \ldots, k\}$, *such that:*

1) $\mathbf{A}_{j_s} \in \mathcal{T}_{j_s}^{\perp} - \mathcal{S}^{\perp}$,

2) $\sum_{s=1}^{l} \mathbf{A}_{j_s} = 0$.

(iii) $\dim \sum_{j=1}^{k} \mathcal{T}_j^{\perp} < \sum_{j=1}^{k} \dim \mathcal{T}_j^{\perp} - (k-1)(6-n).$

The proof is obtained with the help of the following two lemmas.

4.23. Lemma. *An IIM-type singularity is present if and only if there exist l, $l \geq 2$, n-dimensional nonzero vectors, $\Lambda_{j_1}, \ldots, \Lambda_{j_l}$, $\{j_1, \ldots, j_l\} \subset \{1, \ldots, k\}$, such that,*

$$\sum_{s=1}^{l} \Lambda_{j_s} = 0 \quad and \quad \Lambda_{j_s}^{\mathrm{T}} J_{j_s} = 0.$$

Proof. According to Definition 3.18 an IIM-type singularity is equivalent to a singular velocity-equation matrix L. This is so, if and only if there is nonzero nk-dimensional vector, Λ, $\Lambda = (\Lambda_1, \ldots, \Lambda_k)$ such that $\Lambda^{\mathrm{T}} L = 0$. The matrix product of Λ with the first n columns of L gives $\sum_{s=1}^{k} \Lambda_s = 0$. The product of Λ with the n columns of L which contain the columns of the Jacobian of the j-th subchain gives $L_j^{\mathrm{T}} J_j = 0$. Therefore, when L is singular at least two of the Λ_j vectors are nonzero and satisfy the conditions of the Lemma. Conversely, when such a set of at least two vectors exists, the vector Λ can be constructed by filling in zeros. □

4.24. Lemma. *Let $\mathcal{V}$ be a vector space and let $\mathcal{L}_j$, $j = 1, \ldots, k$, $\mathcal{L} \subset \bigcap_{j=1}^{k} \mathcal{L}_j$, be subspaces of $\mathcal{V}$. Then*

$$\dim \sum_{j=1}^{k} \mathcal{L}_j \leq \sum_{j=1}^{k} \dim \mathcal{L}_j - (k-1)\dim \mathcal{L}$$

and equality is present if and only if $\mathcal{L} = \bigcap_{j=1}^{k} \mathcal{L}_j = \left(\sum_{s=1}^{l-1} \mathcal{L}_{j_s}\right) \cap \mathcal{L}_{j_l}$ for any set of subscripts $\{j_1, \ldots, j_l\} \subset \{1, \ldots, k\}$.

Proof. By using the formula:

$$\dim (A + B) = \dim A + \dim B - \dim A \cap B,$$

we obtain:

$$\dim \sum_{j=1}^{k} \mathcal{L}_j = \sum_{j=1}^{k} \dim \mathcal{L}_j - \dim(\mathcal{L}_{j_1} \cap \mathcal{L}_{j_2}) - \dim\left[(\mathcal{L}_{j_1} + \mathcal{L}_{j_2}) \cap \mathcal{L}_{j_3}\right] -$$

$$\cdots - \dim\left[(\sum_{s=1}^{k-1} \mathcal{L}_{j_s}) \cap \mathcal{L}_{j_k}\right] =$$

$$\sum_{j=1}^{k} \dim \mathcal{L}_j - \sum_{l=2}^{k} \dim\left[(\sum_{s=1}^{l-1} \mathcal{L}_{j_s}) \cap \mathcal{L}_{j_l}\right] \leq$$

$$\sum_{j=1}^{k} \dim \mathcal{L}_j - (k-1)\dim \bigcap_{j=1}^{k} \mathcal{L}_j \leq$$

$$\sum_{j=1}^{k} \dim \mathcal{L}_j - (k-1)\dim \mathcal{L}$$

and we have equality only when $\mathcal{L} = \bigcap_{j=1}^{k} \mathcal{L}_j = \left(\sum_{s=1}^{l-1} \mathcal{L}_{j_s}\right) \cap \mathcal{L}_{j_l}$. □

Proof (of Theorem 4.22)

(i) ⇔ *(ii)*. This part is easily proven by Lemma 4.23. Indeed the vectors $\mathbf{A}_{j_1}, \ldots, \mathbf{A}_{j_l}$ are obtained by $\mathbf{A}_{j_s} = \Pi \Lambda_{j_s}$.

(i) ⇔ *(iii)*. Apply Lemma 4.24 for $\mathcal{L}_j = \mathcal{T}_j^{\perp}$. □

4.25. Example. Consider the "flattened" five-bar linkage shown in Figure 4.7.

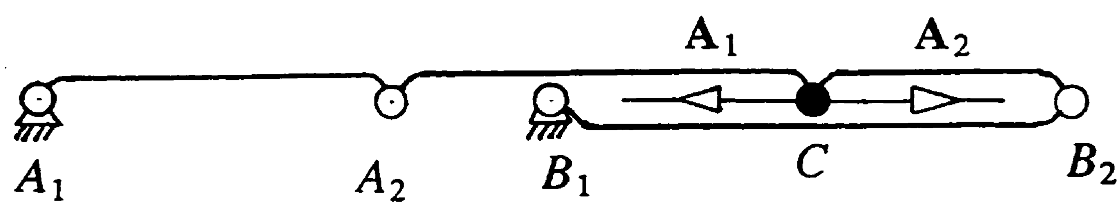

Figure 4.7. An RO-, RI-, IO- and IIM-type singularity.

The first and the third rows of the matrix of the velocity equation in Equation (4.6) are identical, since the x-coordinates of all joint screws are zero. Therefore, the matrix is not of full rank, which is the definition of an IIM-type singularity. On the other hand, the two rotations with axes CA_1 and CB_1 satisfy all the conditions of (ii) in Theorem 4.22. The condition from (iii), for this configuration, yields $5 < (5 + 5) - (2 - 1)(6 - 2) = 6$, which is correct.

4.5. Classification of Singularities

An individual singular configuration always belongs to more than one singularity type. For example, the configuration shown in Figure 4.4 is simultaneously an RPM-, an IO- and an II-type singularity. The configuration of the 6-dof manipulator shown in Figure 4.6, on the other hand, belongs to types RPM, RO, II and IO. If the active joints are at A_1 and B_1, the singularity shown in Figure 4.7 would be IIM-, IO-, RI- and RO-type.

Not all combinations of singularity types are feasible. In Chapter 3, different rules were derived for the simultaneous occurrence of the singularity types for the case of a general mechanism. All of these apply for HCMs. Some additional rules can be derived from the results of Section 4.4.

4.26. Proposition. *For an HCM, if a configuration is an RI-type singularity, then it is an IO-type singularity as well.*

Proof. This result follows directly from Theorem 4.13 (v) and Theorem 4.18 (iv). □

4.27. Proposition. *For an HCM, if a configuration is an RPM-type singularity, then it is an IO-type singularity as well.*

Proof. The proposition follows from Theorem 4.16 (ii) and Theorem 4.18 (iv). □

4.28. Proposition. *For an HCM, if a configuration is an IIM-type singularity, then it is an IO-type singularity as well.*

Proof. This is implied by Lemma 4.23 and Theorem 4.18 (iv). □

In Chapter 3, it was shown that all possible combinations of singularity types for a general mechanism are given by the 21 non-empty cells of Table 3.1 (these are both the cells marked by "Y" and the ones marked by "N" in Table 4.1 below). In this way, the set of all singular configurations of all non-redundant mechanisms can be divided into 21 non-intersecting classes. The following theorem establishes an analogous classification for the singular configurations of HCMs.

4.29. Theorem. *Let* S *be a combination of singularity types. There exists an HCM with a configuration,* q*, such that* $q \in$ S*, if and only if* S *is marked with "Y" in Table 4.1.*

	IO	II	IO *and* II	IIM	IO *and* IIM	II *and* IIM	IO *and* II *and* IIM
RI	**Y**						
RO		**Y**					
RI *and* RO			**Y**	**N**	**Y**	**N**	**Y**
RPM			**Y**	**N**			**Y**
RI *and* RPM			**Y**		**Y**		**Y**
RO *and* RPM			**Y**			**N**	**Y**
RI *and* RO *and* RPM			**Y**	**N**	**Y**	**N**	**Y**

Table 4.1. Possible combinations of singularity types for HCMs.

Proof. To prove this theorem, it is necessary to show that: (i) the combinations not marked "Y" are impossible; and (ii) that the ones marked "Y" are indeed possible.

(i) The blank cells in Table 4.1 correspond to singularity-type combinations, which are impossible for any mechanism. The 6 cells marked "N" are not possible for HCMs because they violate Proposition 4.28.

(ii) To show that the "Y" cells denote possible combinations it is sufficient to give one example for each of these combinations. In this chapter, we already considered four singular combinations, which illustrate four different singularity-type combinations:

(RPM, IO, II)	Figure 4.4 in Example 4.23;
(RI, RO, IO, II)	Figure 4.5 in Example 4.7;
(RI, RO, IO, IIM)	Figure 4.7 in Example 4.25;
(RO, RPM, IO, II)	Figure 4.6 in Example 4.11.

Although the two basic combinations, (RI, IO) and (RO, II), are not shown, they can be easily visualized by small modifications of Figure 4.5. If the only change in the figure were a slight shift of point C_2 so it no longer lies on the line defined by points C_0 and C_1, then the altered figure would represent a HCM in a singularity of class (RO, II). On the other hand, if the configuration shown in Figure 4.5 were changed by a small motion of the mechanism while keeping subchain C fixed (so that the three lines and PP_1 no longer intersect in one point), the result would be a singularity of class (RI, IO).

In Chapter 5, the singularity set of a 6-dof spatial HCM is considered in considerable detail. Seven additional singularity types are illustrated with figures and discussed in Section 5.6.3:

(RI, RPM, IO, II)	A variation of Figure 5.8 as discussed in Section 5.6.3;
(RI, RPM, IO, IIM)	A variation of Figure 5.9 as discussed in Section 5.6.3;
(RI, RO, RPM, IO, II)	A variation of Figure 5.11 as discussed in Section 5.6.3;
(RPM, IO, II, IIM)	Figure 5.9 in Section 5.6.3;
(RI, RPM, IO, II, IIM)	Figure 5.8 in Section 5.6.3;
(RO, RPM, IO, II, IIM)	Figure 5.12 in Section 5.6.3;
(RI, RO, RPM, IO, II, IIM)	Figure 5.11 in Section 5.6.3.

The remaining two combinations, namely (RI, RO, RPM, IO, IIM) and (RI, RO, IO, II, IIM), are proven to exist by the following two examples.

4.30. Example. Consider the configuration shown in Figure 4.8.

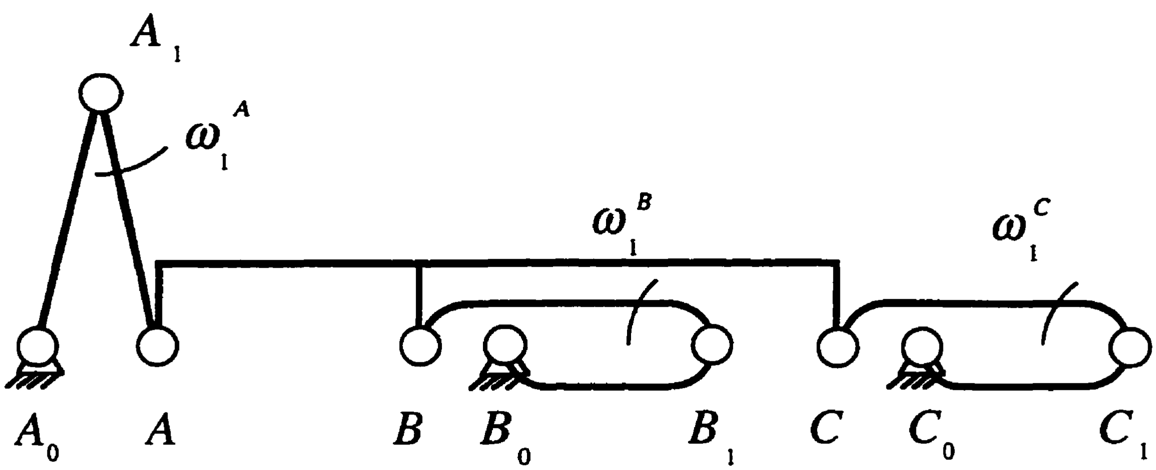

Figure 4.8. A planar HCM in a singular configuration of class (RI, RO, IO, II, IIM).

The mechanism shown is a planar HCM similar to the one in Figure 4.2, however, in the present example all joints are revolute. As in Figure 4.2, the active joints are the second joints in all subchains.

Since subchains B and C are singular, the configuration in Figure 4.8 is an IO-type singularity. Moreover, it is an II-type singularity as well, since the input velocity in joint A_1 must be zero. When the output link ABC is fixed and the input velocities are zero, there can be no instantaneous motion in the present configuration. Therefore, this is not an RPM-type singularity. (Also, it is clear that condition (iii) in Theorem 4.16 is not satisfied, since $\dim \mathcal{P}_j \geq 2$ for all three subchains.) Furthermore, condition (iii) in Theorem 4.22, applied for the configuration in Figure 4.8, yields: $4 < (3 + 4 + 4) - (3 - 1)(6 - 2) = 5$, and therefore the configuration is an IIM-type singularity. A configuration which belongs to type IIM but not RPM, must be an RI- and RO-type singularity. Thus, the present example establishes the existence of singularities of singularity type combination (RI, RO, IO, II, IIM).

4.31. Example. The mechanism shown in Figure 4.9 is similar to the one in Figure 4.8. The difference is in subchain A: in the present figure point A coincides with point A_0, and point A_1 is aligned with the other joint centres.

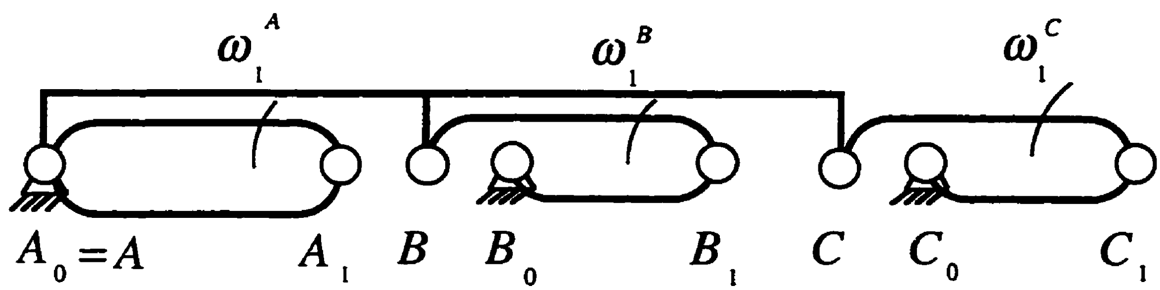

Figure 4.9. A planar HCM in a singular configuration of class (RI, RO, RPM, IO, IIM).

Due to the position of point A_1, unlike Figure 4.8 the configuration shown in the present figure is no longer an II-type singularity. (It can be checked that each one of the input joints can move while the other two are fixed.) On the other hand, this is an RPM-type singularity, since subchain A can rotate about point $A = A_0$ even when both the input and output are zero. Therefore, the configuration must be an IIM-type singularity as well (since it is of RPM-type but not II-type). Furthermore, it can be shown that this is an IO-, RI and RO-type singularity. Indeed, the IO-type singularity is due to the singular serial subchains, the RO-type singularity becomes apparent when we fix the input joints and observe that the output link can still rotate about point A, and, finally, the RI-type singularity is established by noting that the inputs at joints B_1 and C_1 need not be zero even when points B and C of the output link are fixed. In conclusion, Figure 4.9, provides an example of an HCM singularity of class (RI, RO, RPM, IO, IIM).

This completes the proof of Theorem 4.29. □

4.6. Summary

This chapter presented the analysis of the instantaneous kinematics of a class of mechanisms with several serial subchains arranged in parallel to connect the base with the end-effector. The velocity equation (as defined in Chapter 3) of such mechanisms, which completely describes the mechanism rate kinematics, was used as the starting point for the analysis. A method for the elimination of the passive-joint velocities from the velocity equation was described. This method is applicable for all HCMs and the resulting equation fully characterizes the input and output at any configuration, even at singularity. This equation was then applied to the singularity analysis of HCMs, which was performed in accordance with the general theory of kinematic singularity for non-redundant mechanisms developed in Chapter 3. For each of the six singularity types introduced there, the present chapter provides several criteria (necessary and sufficient conditions for their occurrence) for the case of HCMs. A refined and comprehensive classification of the singular configurations of HCMs is obtained by the enumeration of all 15 feasible combinations of singularity types.

CHAPTER 5

SINGULARITY IDENTIFICATION

5.1. Introduction

In the present chapter, the problem of singularity identification is addressed. The objective is to provide a method for the solution of the following problem: Given an arbitrary non-redundant mechanism with lower pairs, find all the singularities of the mechanism and determine their type. The end result of the solution process must be a description of the singularity set as a whole, as well as a division of this set into subsets belonging to exactly the same singularity types.

The proposed solution technique is based on the velocity-equation formulation of kinematic singularity, introduced in Chapter 3. The definitions of Sections 3.3 and 3.4 are used in Section 5.2 to derive the singularity criteria, i.e., necessary and sufficient conditions for the occurrence of singularities of different types. On the basis of these criteria, methods for computing the singularity set and revealing its division into singularity classes are proposed in Sections 5.3 and 5.4, respectively. The application of these methods to complex spatial mechanisms is discussed in Section 5.5 and illustrated in Section 5.6, where the singularity set of a 6-dof parallel manipulator is obtained and analyzed.

5.2. Conditions for Singularity

The singularity of a given configuration, $\boldsymbol{q}$, can be determined by examining the matrix $\boldsymbol{L}(\boldsymbol{q})$ of the velocity equation (introduced in Section 3.2, Equation (3.5)). Let $\boldsymbol{L}_I$, $\boldsymbol{L}_O$ and $\boldsymbol{L}_p$ be the submatrices of $\boldsymbol{L}$ obtained by removing the columns corresponding to the input, output, and both the input *and* output, respectively. Then, the following general singularity condition holds:

5.1. Theorem. *For any non-redundant mechanism, a configuration, $\boldsymbol{q}$, is nonsingular, if and only if both the matrices $\boldsymbol{L}_I$ and $\boldsymbol{L}_O$ are nonsingular at $\boldsymbol{q}$.*

Proof. Let $\boldsymbol{L}_T$ be the matrix formed by the columns of $\boldsymbol{L}$, which correspond only to the output velocities, and $\boldsymbol{L}_a$ be the matrix of the columns of the input velocities. Then, the velocity equation (3.5) can be rewritten as: $\boldsymbol{L}_T\boldsymbol{T} + \boldsymbol{L}_a\boldsymbol{\Omega}^a + \boldsymbol{L}_p\boldsymbol{\Omega}^p = 0$, or, in any of the following two forms:

$$\boldsymbol{L}_I\begin{bmatrix} \boldsymbol{T} \\ \boldsymbol{\Omega}^p \end{bmatrix} = -\boldsymbol{L}_a\boldsymbol{\Omega}^a, \tag{5.1}$$

and

$$\boldsymbol{L}_O\begin{bmatrix} \boldsymbol{\Omega}^a \\ \boldsymbol{\Omega}^p \end{bmatrix} = -\boldsymbol{L}_T\boldsymbol{T} \ . \tag{5.2}$$

From Equations (5.1) and (5.2), it is evident that all velocities can be expressed in terms of the output (input) velocities, if and only if $\boldsymbol{L}_I$ (respectively $\boldsymbol{L}_O$) is invertible. According to the definition of singularity in Section 3.3 this proves the theorem. □

The conditions for the occurrence of the different singularity types are described by the following proposition:

5.2. Proposition

(i) $\boldsymbol{q} \in \{\mathrm{RI}\} \Leftrightarrow \mathrm{rank}\ \boldsymbol{L}_O < \mathrm{rank}\ \boldsymbol{L}_p + n,$

(ii) $q \in \{\text{RO}\} \Leftrightarrow \text{rank}\, \boldsymbol{L}_I < \text{rank}\, \boldsymbol{L}_p + n$,

(iii) $q \in \{\text{RPM}\} \Leftrightarrow \text{rank}\, \boldsymbol{L}_p < N - n$,

(iv) $q \in \{\text{II}\} \Leftrightarrow \text{rank}\, \boldsymbol{L}_I < \text{rank}\, \boldsymbol{L}$,

(v) $q \in \{\text{IO}\} \Leftrightarrow \text{rank}\, \boldsymbol{L}_O < \text{rank}\, \boldsymbol{L}$,

(vi) $q \in \{\text{IIM}\} \Leftrightarrow \text{rank}\, \boldsymbol{L} < N$,

(vii) $q \in \{\text{RI}\}$ or $q \in \{\text{RPM}\} \Leftrightarrow q \in \{\text{IO}\}$ or $q \in \{\text{IIM}\} \Leftrightarrow \boldsymbol{L}_O$ *is singular,*

(viii) $q \in \{\text{RO}\}$ or $q \in \{\text{RPM}\} \Leftrightarrow q \in \{\text{II}\}$ or $q \in \{\text{IIM}\} \Leftrightarrow \boldsymbol{L}_I$ *is singular.*

Proof

(i) $q \in \{\text{RI}\}$ is equivalent to the existence of a $\Omega^a \neq 0$ such that Equation (5.2) is satisfied with a zero right-hand side.

Let d, $d \geq 0$, be defined by $\text{rank}\, \boldsymbol{L}_p = N - n - d$. Then, $\dim(\text{Ker}\, \boldsymbol{L}_O)$ is exactly d, if and only if the left-hand side of (5.2) can be zero only for a zero Ω^a. Therefore, an RI-type singularity is present only when $\dim(\text{Ker}\, \boldsymbol{L}_O) > d$. This proves (i), since $\text{rank}\, \boldsymbol{L}_O = N - \dim(\text{Ker}\, \boldsymbol{L}_O)$ and $\text{rank}\, \boldsymbol{L}_p + n = N - d$.

(ii) Analogous to (i).

(iii) Follows directly from the definition of the RPM-type.

(iv) Equation (5.1) implies that $q \in \{\text{II}\}$ is equivalent to the existence of a vector $\boldsymbol{v}$, which is in $\text{Im}\, \boldsymbol{L}_a$ but not in $\text{Im}\, \boldsymbol{L}_I$, i.e., equivalent to $\text{Im}\, \boldsymbol{L}_a - \text{Im}\, \boldsymbol{L}_I = \varnothing$. Since $\text{Im}\, \boldsymbol{L} = \text{Im}\, \boldsymbol{L}_a + \text{Im}\, \boldsymbol{L}_I$, this in turn is equivalent to $\text{Im}\, \boldsymbol{L} - \text{Im}\, \boldsymbol{L}_I = \varnothing$, i.e., $\text{rank}\, \boldsymbol{L}_I < \text{rank}\, \boldsymbol{L}$.

(v) Analogous to (iv).

(vi) Follows directly from the definition of the IIM-type.

(vii) For any configuration, it is true that

$$\text{rank}\, \boldsymbol{L}_O \leq \text{rank}\, \boldsymbol{L} \leq N.$$

The matrix $\boldsymbol{L}_O$ is singular, when $\text{rank}\, \boldsymbol{L}_O < N$. Therefore, $\boldsymbol{L}_O$ is singular, if and only if either

$$\text{rank}\, L_O < \text{rank}\, L \quad \textit{or} \qquad \text{rank}\, L < N.$$

Using (v) and (vi), we conclude that L_O is singular if and only if the configuration belongs to either IO or IIM.

On the other hand, it is always true that

$$\text{rank}\, L_O \leq n + \text{rank}\, L_p \leq N.$$

For L_O to be singular, rank $L_O < N$. Therefore, a necessary and sufficient condition for the singularity of L_O is that at least one of the following two inequalities holds:

$$\text{rank}\, L_O < n + \text{rank}\, L_p \quad \textit{or} \qquad \text{rank}\, L_p \leq N - n.$$

It follows from (i) and (iii) that L_O is singular, if and only if the configuration is either an RO- or an RPM-type singularity.

(viii) Analogous to the proof of (vii).

□

5.3. Remark. A mechanism configuration, $\boldsymbol{q}$, is an N-tuple of values of all joint parameters. As was pointed out in Chapter 2, in the case of closed-loop mechanisms not all such N-tuples correspond to feasible configurations. The configuration space is given by the solution set of a system of equations, $\Sigma_L(f) = e$, $L \in \mathcal{L}$ (Theorem 2.30). When a local coordinate system is chosen on the joint-space manifold, Q, Equations (2.3) become a system of nonlinear scalar equations, $\boldsymbol{l}(\boldsymbol{q}) = 0$. In the present chapter, when referring to the "loop equations" of the kinematic chain, we will have in mind the scalar equations.

When attempting to find the singularities of a given mechanism, it must be assured that the values obtained for $\boldsymbol{q}$ are compatible with the loop equations. If only parts of the configuration space need to be considered, additional inequality constraints on the joint parameters are imposed. The feasible set consistent with the joint constraints will be denoted by F. Thus, the *set of feasible configurations* is $\{\boldsymbol{q} \in F \mid \boldsymbol{l}(\boldsymbol{q}) = 0\}$.

5.3. Determination of the Singularity Set

When a feasible configuration, q, is given, the rank of the matrices L_I, L_O, L_p and L are computed and the type of singularity is determined by reviewing conditions (i) to (viii) listed in Section 5.2, Theorem 5.2. However, to obtain the singularities of a mechanism, without considering all feasible q, the conditions must be interpreted as systems of equations for q, and the singularity set and its subsets be obtained as solutions of these equations. This process is described below.

For singularity identification of closed-loop mechanisms, the matrices L_I and L_O play a role analogous to the one of the Jacobian in the case of a serial chain. The singularities of a non-redundant mechanism with known kinematic chain, link parameters and joint constraints, can be determined by solving the following two systems of nonlinear equations:

$$\begin{aligned} \det L_I(q) &= 0, \\ l(q) &= 0, \end{aligned} \tag{5.3}$$

and

$$\begin{aligned} \det L_O(q) &= 0, \\ l(q) &= 0, \end{aligned} \tag{5.4}$$

subject to the joint constraints F.

Therefore, the problem of singularity identification can be resolved by the execution of the following steps:

(1) Derive the loop equations, $l(q) = 0$, of the mechanism.

(2) Derive the velocity equation, $L(q)M = 0$, of the mechanism.

(3) Solve the system

$$\det L_I(q) = 0,$$
$$l(q) = 0,$$

subject to the joint constraints F.

(4) Solve the system

$$\det L_O(q) = 0,$$
$$l(q) = 0,$$

subject to the joint constraints F.

(5) Obtain the singularity set as the union of the sets obtained as solutions of the systems in Steps (3) and (4).

For a non-redundant mechanism each of the two subsets of the singularity set, obtained by Equations (5.3) and (5.4) (Steps 3 and 4), is the solution of a system of $(N - n + 1)$ equations. Therefore, the singularity set will be typically of dimension $(n - 1)$ or, equivalently, of co-dimension 1 in the n-dimensional configuration space of the mechanism. Thus, mechanisms with mobility of 1 usually have a finite number of isolated singularities, while for higher values of n the singularity set will have ∞^{n-1} points.

*5.4. **Example.*** As an example, the above procedure is applied to a four-bar linkage (shown in Figure 5.1a and 5.1b) with dimensions $AB = AD = DC = 1$, $BC = 2$, with no joint constraints. The input link is AD and the output link is BC.

The four-bar mechanism is parameterized by the coordinates of the points C and D. (When planar linkages are considered, it is often convenient to use Cartesian position coordinates rather than joint angles). The base reference frame is such that the coordinates of A and B are $(0, 0)$ and $(1, 0)$.

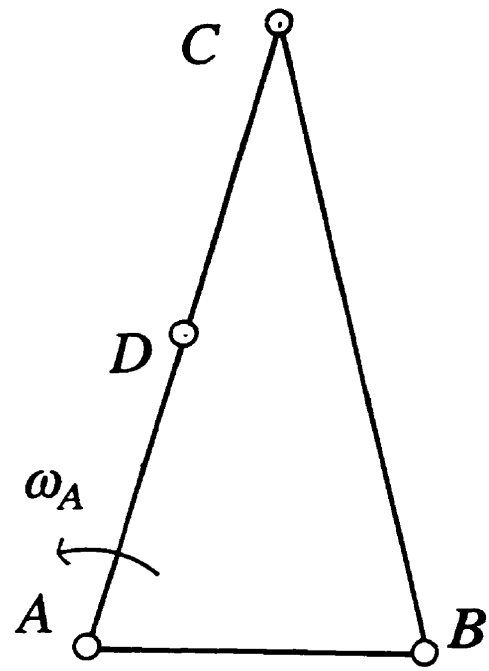

Figure 5.1a. An RI- and IO-type singularity.

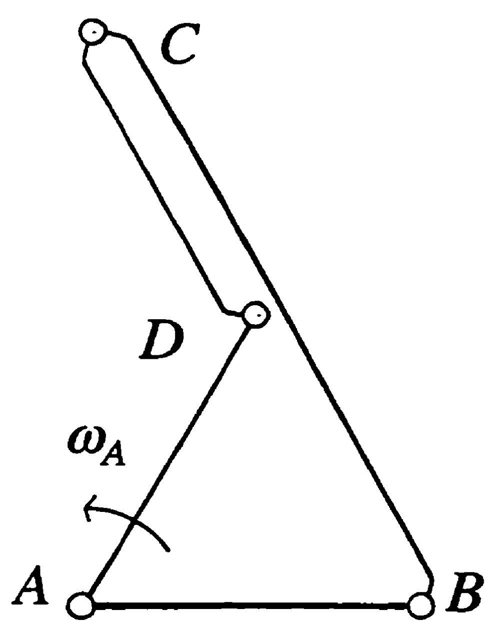

Figure 5.1b. An RO- and II-type singularity.

(1) The loop equations are:

$$\begin{aligned} x_D^2 + y_D^2 &= 1 \\ (x_C - x_D)^2 + (y_C - y_D)^2 &= 1 \\ (x_C - 1)^2 + y_C^2 &= 1 \end{aligned} \tag{5.5}$$

(2) The velocity equation, generic for any four-bar linkage (cf Example 3.2), is of the form:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & \mathbf{S}_A & \mathbf{S}_D & \mathbf{S}_C & \mathbf{S}_B \end{bmatrix} \begin{bmatrix} \tau \\ \omega_A \\ \omega_D \\ \omega_C \\ \omega_B \end{bmatrix} = 0 \tag{5.6}$$

where $\mathbf{S}_P$, $P = A, B, C, D$, are 3-dimensional planar screws, i.e., vectors of the type $\mathbf{S}_P = (1, y_P, -x_P)^T$, and ω_P are the joint velocities. The first equation in (5.6) is the output equation, and the remaining three form a screw equation, which states that the sum of the joint twists in the only loop is zero.

(3) The expression for det $L_I(q)$ leads to the following expression:

$$-y_D x_C + y_C x_D = 0$$

This equation is solved together with the system of Equations (5.5). The solution is $x_D = 1/4$, $y_D = \pm\sqrt{15}/4$, $x_C = 1/2$, $y_C = \pm\sqrt{15}/2$, (Figure 5.1a).

(4) The expression for det $L_O(q)$ leads to the following equation:

$$-y_C(1 - x_D) + y_D(1 - x_C) = 0$$

This equation is solved together with the system of Equations (5.5). The solution is $x_C = 0$, $y_C = \pm\sqrt{3}$, $x_D = 1/2$, $y_D = \pm\sqrt{3}/2$, (Figure 5.1b).

(5) Thus, 4 distinct singular configurations are obtained. They are symmetrical with respect to the line AB.

5.5. Example. Consider the slider shown in Figure 5.2a, with $AC = BC = 1$.

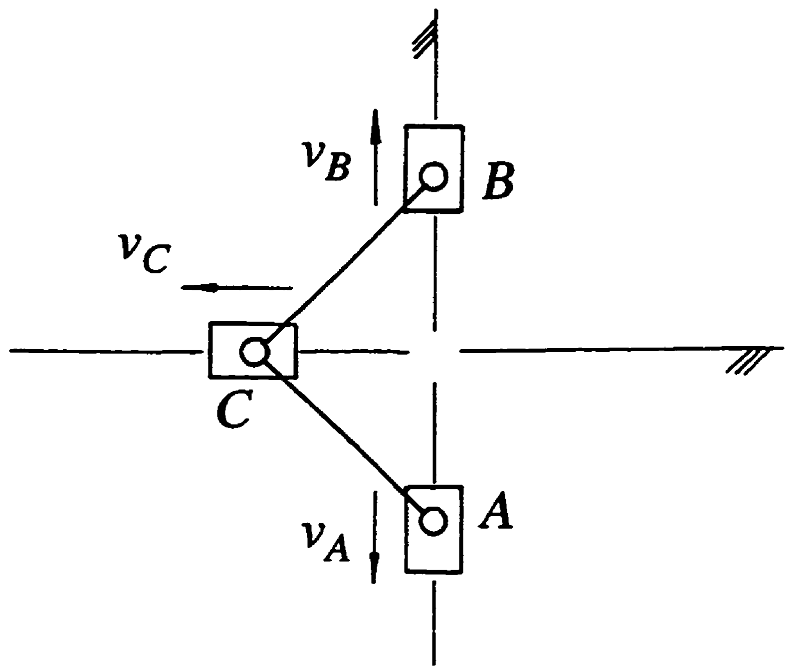

Figure 5.2a. A 1-dof slider.

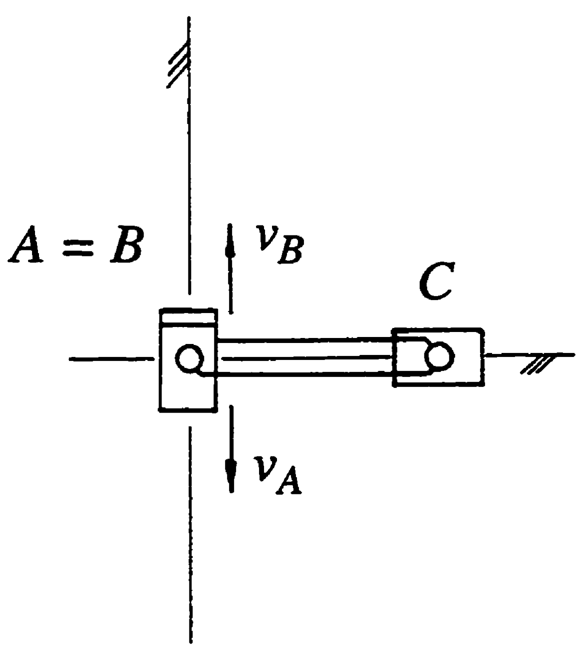

Figure 5.2b.

An (RO, RI, IIM)-class singularity.

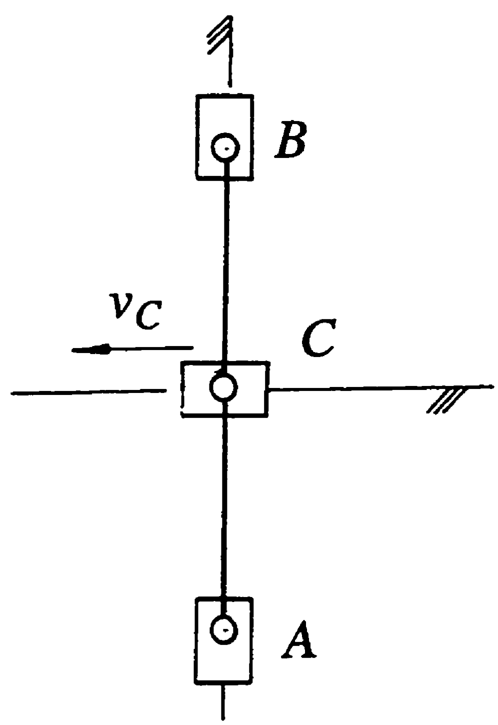

Figure 5.2c.

An (RPM, II, IO)-class singularity

The prismatic joints at A and B are on a line which is perpendicular to the axis of the prismatic joint C. The input is the velocity of point A, v_A. The output, $v = v_B$, is the motion of the point B. The coordinates of points A, B and C are used as position parameters. The base reference frame is chosen with its axes along the lines of the prismatic joints.

(1) The loop equations are the expressions for the constant lengths of AC and BC, as well as for the constant orientation of the prismatic joint axes:

$$\begin{gathered} x_A = x_B = y_C = 0, \\ y_A^2 + x_C^2 = 1, \\ y_B^2 + x_C^2 = 1. \end{gathered}$$

(2) The velocity equation is obtained with a 7×8 matrix L:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & \mathbf{P}_A & \mathbf{S}_A & \mathbf{S}_C & \mathbf{P}_C & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{P}_C & \mathbf{P}_B & \mathbf{S}_B & \mathbf{S}_C \end{bmatrix} \begin{bmatrix} v \\ v_A \\ \omega_A \\ \omega_C^A \\ v_C \\ v_B \\ \omega_B \\ \omega_C^A \end{bmatrix} = 0, \qquad (5.7)$$

where $\mathbf{S}_P$, $P = A, B, C$, are 3-dimensional planar revolute-joint screws ($\mathbf{S}_P = (1, y_P, -x_P)^T$) and $\mathbf{P}_P$, $P = A, B, C$, are prismatic-joint screws: $\mathbf{P}_A = (0, 0, 1)^T$, $\mathbf{P}_B = (0, 0, -1)^T$, $\mathbf{P}_C = (0, -1, 0)^T$.

(3) The system

$$\begin{gathered} \det L_I(q) = y_A x_C = 0, \\ y_A^2 + x_C^2 = 1, \\ y_B^2 + x_C^2 = 1. \end{gathered}$$

is solved and six singularities are obtained: ($y_A = y_B = 0$, $x_C = \pm 1$) (Figure 5.2b), and ($y_A = \pm y_B = \pm 1$, $x_C = 0$), (Figure 5.2c).

(4) The system

$$\det L_O(q) = y_B x_C = 0,$$
$$y_A^2 + x_C^2 = 1,$$
$$y_B^2 + x_C^2 = 1.$$

yields the same six configurations obtained in Step (3).

(5) The singularity set has six elements with the following values of (y_A, y_B, x_C): (0, 0, 1); (0, 0, −1); (1, 1, 0); (−1, −1, 0); (1, −1, 0); (−1, 1,0).

It must be noted that, if singularity identification were attempted by means of an input-output equation, the singularities with $x_C = 0$ would not be detected (as this was already pointed out in Section 3.4).

5.4. Determination of the Singularity Types

The algorithm presented in Section 5.3 can identify all the singularities of a mechanism. However, it cannot classify them, namely determine to which types each singularity belongs. Herein, a comprehensive algorithm that can both identify and classify the singularities of a given mechanism is described.

5.4.1. Finite number of singularities

To classify the singularities of a given mechanism, the singularity conditions (i) to (viii) listed in Section 5.2 must be used. For a finite number of singularities, as could have been determined by the algorithm in Section 5.3, the classification can be carried out by checking each condition for each singularity.

5.6. Example. Consider the four-bar mechanism in Figure 5.1. For the singular configurations given by ($x_D = 1/4$, $y_D = \pm\sqrt{15}/4$, $x_C = 1/2$, $y_C = \pm\sqrt{15}/2$), it is established that: rank $\boldsymbol{L} = 4$, rank $\boldsymbol{L}_p = 3$, rank $\boldsymbol{L}_I = 3$, and rank $\boldsymbol{L}_O = 4$. Using conditions (iii), (vi), (vii) and (viii), it is determined that the singularities obtained in Step 3 belong to both types RI and IO and to no other type, i.e., they are of the (RI, IO) class.

For the singularities given by ($x_C = 0$, $y_C = \pm\sqrt{3}$, $x_D = 1/2$, $y_D = \pm\sqrt{3}/2$), it is established that: rank $\boldsymbol{L} = 4$, rank $\boldsymbol{L}_p = 3$, rank $\boldsymbol{L}_I = 4$, and rank $\boldsymbol{L}_O = 3$. Using conditions (iii), (vi), (vii) and (viii), it is determined that the singularities obtained in Step 4 belong to types RO and II and to no other type, i.e., cell (RO, II) in Table 3.1.

5.7. Example. For the slider in Figure 5.2, it can be found that when $x_C = 0$ (and $y_A = y_B \neq 0$) the ranks are: rank $\boldsymbol{L} = 7$, rank $\boldsymbol{L}_p = 5$, and rank $\boldsymbol{L}_I =$ rank $\boldsymbol{L}_O = 6$. Therefore, from conditions (i), (ii), (iii), (vi), (vii) and (viii), it follows that these singularities belong to the (RPM, IO, II) class (Figure 5.2c).

When $y_A = y_B = 0$ ($x_C \neq 0$), it can be found that: rank $\boldsymbol{L} =$ rank $\boldsymbol{L}_p =$ rank $\boldsymbol{L}_I =$ rank $\boldsymbol{L}_O = 6$. Conditions (iii) to (viii) then imply that the singularity belongs to the (IIM, RI, RO) class (Figure 5.2b).

If a mechanism has infinitely many singularities, the class of each separate singularity can be obtained by calculating the ranks of the four matrices, $\boldsymbol{L}$, $\boldsymbol{L}_p$, $\boldsymbol{L}_I$ and $\boldsymbol{L}_O$. However, in order to find all singularities that belong to each class, the conditions (i) to (viii) must be *solved* for an unknown $\boldsymbol{q}$, to obtain the *sets* of singularities belonging to the corresponding types.

5.4.2. Classification via $\boldsymbol{L}_I$ and $\boldsymbol{L}_O$

Though the solution of Equations (5.3) and (5.4) identifies all the singularities of a mechanism, it does not classify them. In general, by using only matrices $\boldsymbol{L}_I$ and $\boldsymbol{L}_O$, it is not possible to classify all the singularities of a mechanism. However, classification can be accomplished for some mechanisms, and for some of the singularities of other mechanisms. Conditions (vii) and (viii) imply that, if for a given configuration L_I is singular but L_O is nonsingular, the configuration is a singularity of class (RI, IO). Conversely, when a configuration satisfies condition (viii) but not (vii), it must be of the (RO, II) class. It is only when both L_I and L_O are singular that conditions other than (vii) and (viii) need to be considered. Singularities that satisfy both (vii) and (viii) may have substantially different kinematic features, e.g., they may lead to either a loss or a gain in output/input dof. In fact, a configuration where both L_I and L_O are singular may or may not belong to any of the six singularity types.

5.8. ***Example.*** For the four-bar linkage analyzed above, the singularity subsets obtained in Steps (3) and (4) of the identification algorithm in Section 5.3 do not intersect. Therefore, the singularities obtained in Step (3), with a singular L_I, form the (RI, IO) singularity class, while those obtained in Step (4), with a singular matrix L_O, form the (RO, II) class.

5.9. ***Example.*** In the case of the slider, however, all the singularities satisfy both conditions (vii) and (viii) and they cannot be classified without using additional singularity conditions. As it was shown in Sub-section 5.1, the singularities are either of the (RPM, II, IO) class or of the (IIM, RO, RI) class, and therefore for this mechanism conditions (vii) and (viii) cannot resolve whether the singularity belongs to any particular singularity type.

From Example 5.9, it is evident that singularities that satisfy both (vii) and (viii) may have substantially different kinematic features, e.g., they may lead to either a loss or a gain in output/input dof. Therefore, a more refined classification is needed.

5.4.3. Classification algorithm

On the basis of the discussion in Section 5.4.2, if it were known that there are no singularities of the IIM or RPM types, the identification and classification process could be completed by examining only conditions (vii) and (viii). The main strategy of the method described below is, thus, to first identify and classify the IIM and RPM singularities, and then analyze the remaining configurations using the determinants of $\boldsymbol{L}_I$ and $\boldsymbol{L}_O$.

As in Sub-Section 5.3, it is understood that the singularity equations are solved subject to the joint constraints and the loop equations. To simplify the presentation, these operations are not explicitly included in the description of the algorithm. Below, $\{k\}$ stands for "all configurations obtained in Step k of the algorithm."

(1) Find all feasible $\boldsymbol{q}$ satisfying condition (vi).

(2) Find all feasible $\boldsymbol{q}$ satisfying condition (iii).

(3) Classify $\{1\} \cup \{2\}$:

- (3.1) For $\{1\}$, check (iv) and (v). Obtain 4 sets:
  IIM; IIM & II; IIM & IO; IIM & II & IO.
- (3.2) For $\{2\}$, check (i) and (ii). Obtain 4 sets:
  RPM; RPM & RI; IIM & RO; RPM & RI & RO.
- (3.3) Find all the intersections of each set in $\{3.1\}$ and each set in $\{3.2\}$.
  Obtain **10 classes**. (These are the 10 classes that belong to the IIM *and* RPM types, see Table 3.1)
- (3.4) Subtract $\{2\}$ from each set in $\{3.1\}$. Obtain **4 classes.**
  (The 4 classes of IIM, but *not* RPM singularities, see Table 3.1).
- (3.5) Subtract $\{1\}$ from each set in $\{3.2\}$. Obtain **4 classes.**
  (The 4 classes of RPM, but *not* IIM singularities, see Table 3.1).

(4) Find all $\boldsymbol{q}$ satisfying condition (vii). From these subtract $\{1\} \cup \{2\}$.

(5) Find all $\boldsymbol{q}$ satisfying condition (viii). From these subtract $\{1\} \cup \{2\}$.

(6) Intersect {4} and {5}. Obtain **3 classes.**

(Singularities that are neither IIM nor RPM).

Thus, the singularities that belong to each of the 21 classes in Table 3.1 are identified.

5.10. Remark. The operations in Steps (1) and (2) require the identification of the points x for which some rectangular matrix $\boldsymbol{M}(\boldsymbol{x})$ is singular. This can be done by finding all x for which all sub-matrices of maximum dimension have zero determinants, i.e., by solving a system of nonlinear equations. In Steps (3.1) and (3.2) it is required to find sets of the type $\mathcal{R} = \{\boldsymbol{x} \mid \text{rank}\, \boldsymbol{A}(\boldsymbol{x}) < \text{rank}\, \boldsymbol{B}(\boldsymbol{x})\}$. This can be done by presenting $\mathcal{R}$ as the union of the sets $\mathcal{R}_i = \{\boldsymbol{x} \mid \text{rank}\, \boldsymbol{A}(\boldsymbol{x}) < i \leq \text{rank}\, \boldsymbol{B}(\boldsymbol{x})\}$. The sets $\mathcal{R}_i$ can be obtained by solving systems of equations.

5.11. Remark. It can be noted that, since the condition for RPM (or IIM) singularity requires the rank-deficiency of a rectangular matrix, a larger number of equations must be satisfied and the dimension of the solution set will be typically lower than the dimension of the singularity set as a whole. In practice, IIM singularities occur only for mechanisms with specially proportioned link parameters. RPM singularities, when they exist, form sets of low dimensions. The algorithm is organized in such a way that the conditions for RI, RO, II and IO, which may involve the examination of multiple cases, are solved only together with the conditions for IIM (RPM), i.e., for a comparatively small subset of singularities.

5.12. ***Example.*** The algorithm is applied to the slider in Figure 5.2.

(1) Two configurations are obtained: $y_A = y_B = 0$, $x_C = \pm 1$.

(2) Four configurations are obtained: $y_A = \pm y_B = \pm 1$, $x_C = 0$.

(3) (3.1) Both elements of {1} belong to neither {II} nor {IO} (neither (iv) nor (v) are satisfied).

(3.2) Both elements of {2} belong to neither {RI} nor {RO} (neither (i) nor (ii) are satisfied).

(3.3) The intersection of {1} and {2} is empty.

All ten classes of IIM-type and RPM-type singularities are empty.

(3.4) The two elements of {1} form the (IIM, RI, RO) singularity class.

The other three classes of IIM, but *not* RPM singularities are empty.

(3.5) The two elements of {2} form the (RPM, II, IO) singularity class.

The other three classes of RPM, but *not* IIM singularities are empty.

(4 to 6) {4} and {5} are empty. The remaining three classes are empty.

5.13. Example. Consider the mechanism shown in Figure 5.3 ($N = 8$, $n = 2$).

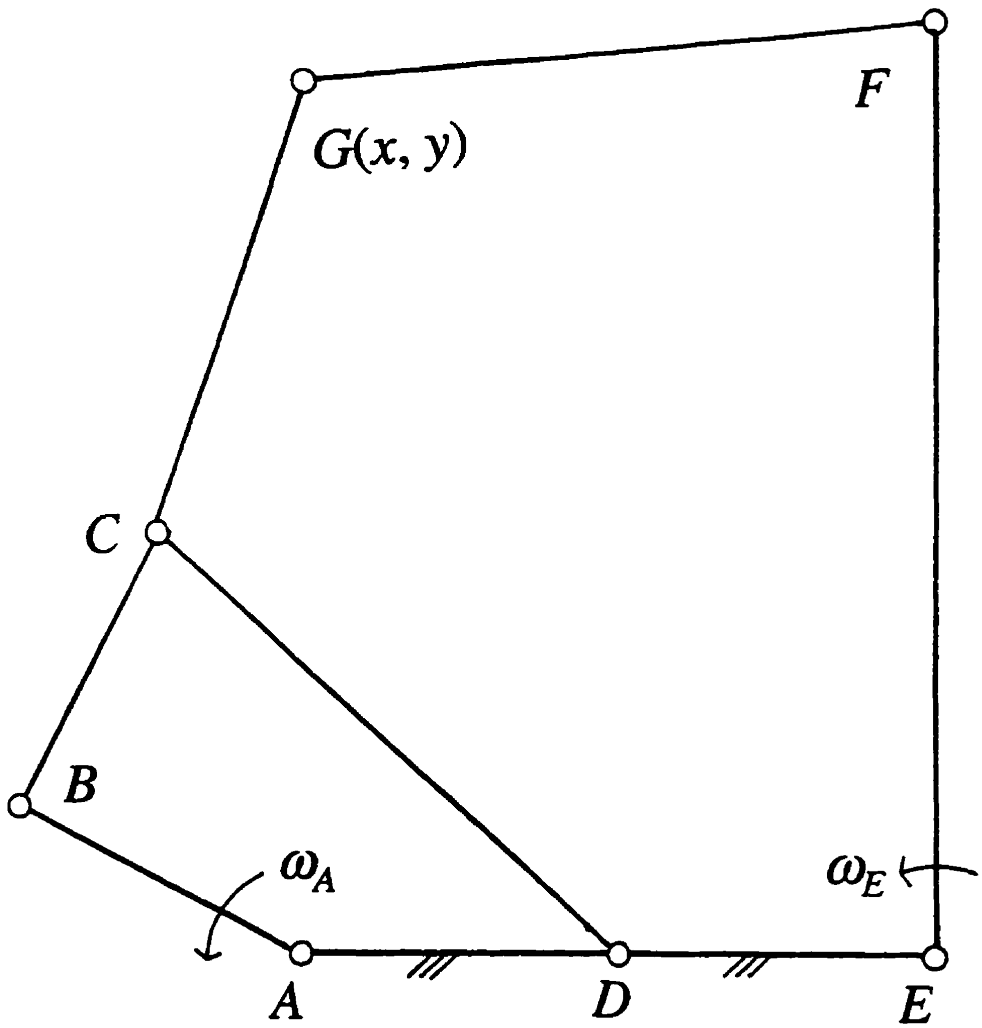

Figure 5.3. A 2-dof planar mechanism.

The inputs are the joint velocities at A and E, the output is the motion of point G. The link dimensions are $AB = AD = BC = DE = 1$, $CD = FG = 2$, $CG = 1.5$, $EF = 3$. The $\boldsymbol{L}$ matrix with dimensions 8×10 is:

$$L = \begin{bmatrix} I_2 & \mathbf{0} & \mathbf{m}_{EG} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{m}_{FG} \\ O & \mathbf{S}_A & \mathbf{0} & \mathbf{S}_B & \mathbf{S}_C & \mathbf{S}_D & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ O & \mathbf{0} & \mathbf{S}_E & \mathbf{0} & \mathbf{0} & \mathbf{S}_D & \mathbf{S}_C & \mathbf{S}_G & \mathbf{S}_F \end{bmatrix} \tag{5.8}$$

where $\mathbf{S}_P$, $P = A, B, \ldots, G$, are 3-dimensional planar screws, $\mathbf{S}_P = (1, y_P, -x_P)^T$, $\mathbf{m}_{PG} = (y_P - y_G, x_G - x_P)^T$, and I_2 is the 2×2 unit matrix. To find all the singularities and establish their types, the procedure described in Remark 5.10 is followed:

(1) Check for IIM singularities. For the given mechanism, it is established that condition (vi) has no solution compatible with the given link lengths.

(2) Check for RPM singularities. The condition (iii) is satisfied only when the determinants of both $[\mathbf{S}_B\mathbf{S}_C\mathbf{S}_D]$ and $[\mathbf{S}_C\mathbf{S}_G\mathbf{S}_D]$ vanish. This yields 8 distinct singular configurations (one of them is shown in Figure 5.4).

(3) (3.2) For each of 8 the configurations in {2}, Conditions (i) and (ii) are checked and it is found that neither is satisfied.

(3.5) The (RPM, IO, II) class consists of the 8 elements of {2}.

(4) Condition (viii) is applied. (viii) is equivalent to the singularity of at least one of the matrices $[\mathbf{S}_B\mathbf{S}_C\mathbf{S}_D]$ or $[\mathbf{S}_C\mathbf{S}_G\mathbf{S}_F]$. The solution of each of these equations (combined with the loop equations) is a 1-dimensional submanifold of the 2-dimensional configuration space. The first manifold has 4 connected components, and the second one has 3 components. All elements of the union of these manifolds, *except* the 8 elements of {2} found in Step 2, are of the types RO and II. One such singularity is shown in Figure 5.5. The corresponding connected component is obtained by moving the linkage, while keeping the joint angle at G constant.

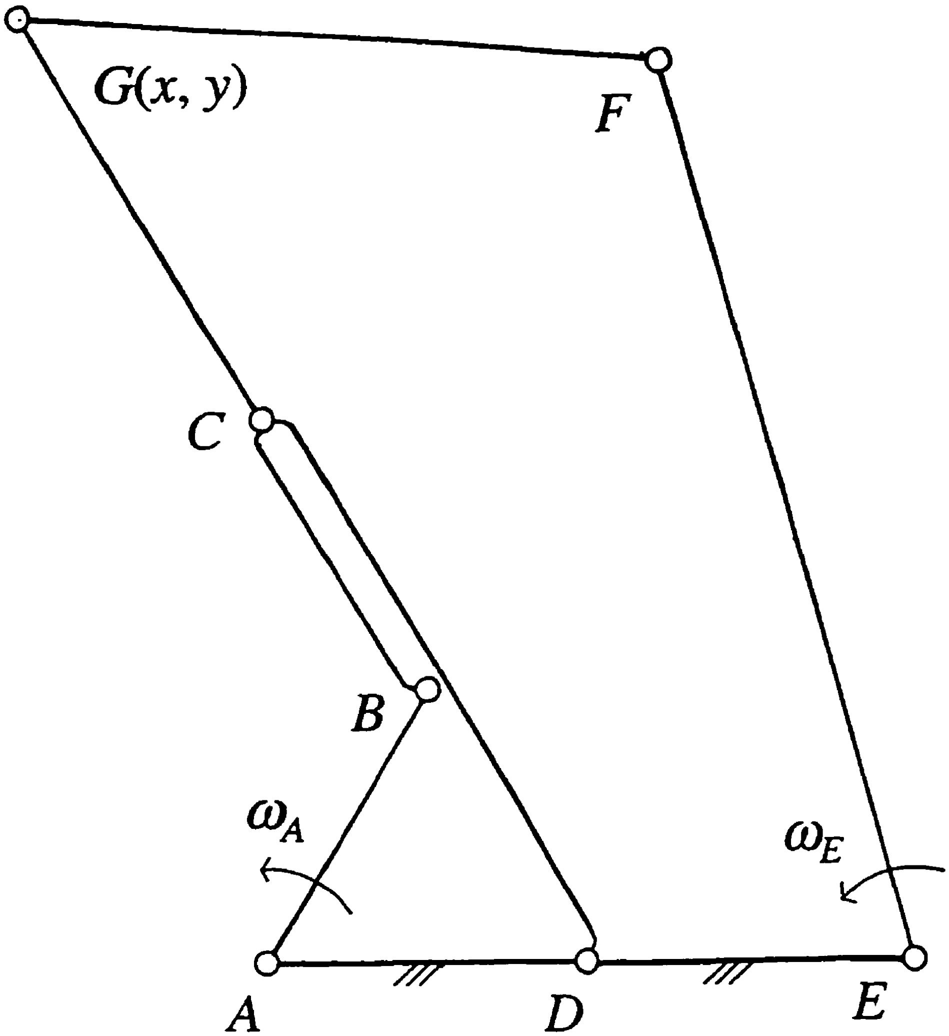

Figure 5.4. An RPM-, IO-, and II-type singularity.

(5) The condition (vii) is applied. (vii) is equivalent to the singularity of at least one of the matrices $[\mathbf{S}_A\mathbf{S}_B\mathbf{S}_C]$, $[\mathbf{S}_G\mathbf{S}_C\mathbf{S}_D]$ or $[\mathbf{S}_E\mathbf{S}_G\mathbf{S}_F]$. The solution for each of these equations (combined with the loop equations) is a 1-dimensional submanifold of the 2-dimensional configuration space. The first and third manifolds have each 2 connected components, while the second one has 4. All elements of the union of these manifolds, *except* the 8 elements of {2}, belong to the types RI, IO. Figure 5.6 provides an example. The connected component corresponding to the shown

configuration is obtained by moving the linkage while keeping the points *B* and *C* fixed.

(6) The intersection of the sets obtained in Steps 4 and 5 consists of 16 configurations Apart from the 8 configurations classified in Step 3.5 as (RPM & II *and* IO)-class singularities, the others are (RI, RO, IO, II)-class singularities. The remaining configurations obtained in Step 4 (or 5) belong to the class (RO, II) (or (RI, IO)).

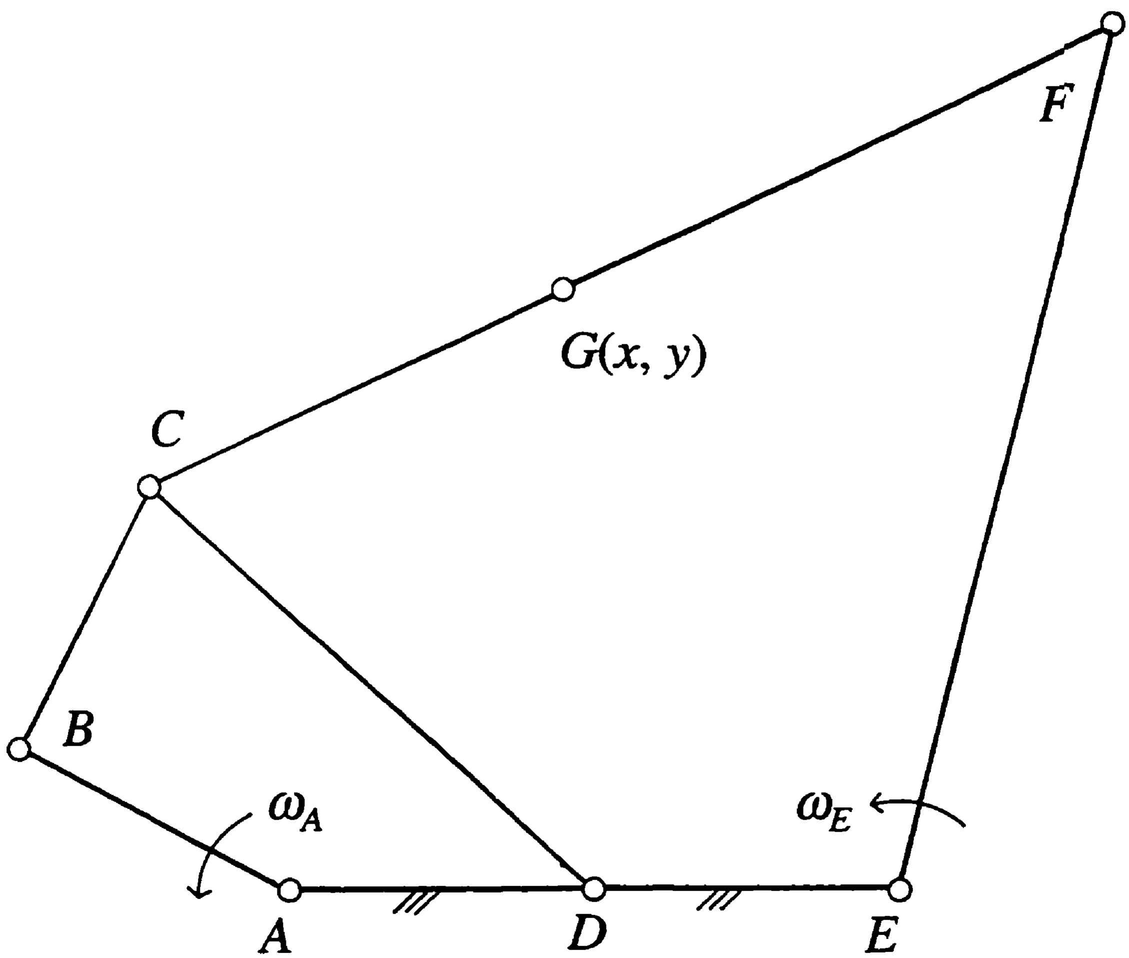

Figure 5.5. An RO- and II-type singularity.

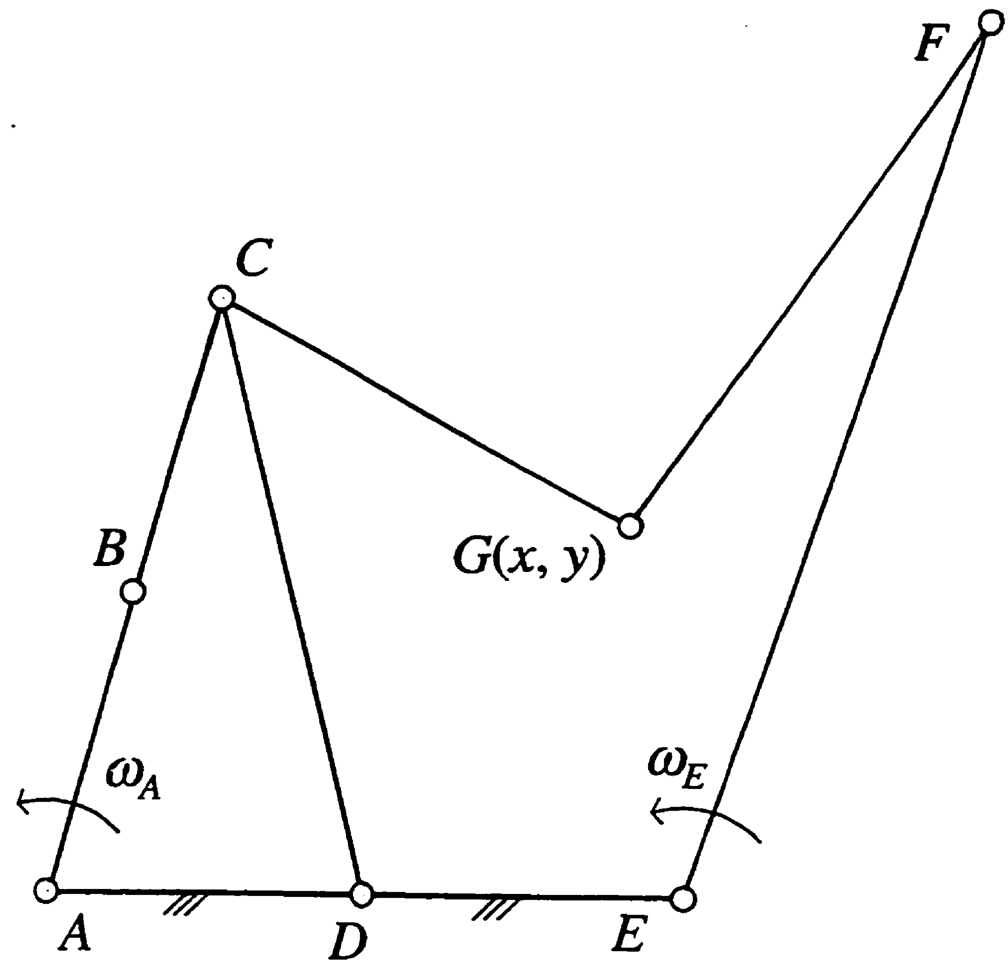

Figure 5.6. An RI- and IO-type singularity.

Thus, four different classes of singularities are obtained for the given mechanism: 8 (RPM, II, IO) singularities, Step (3.5); 8 (RI, RO, IO, II) singularities, Step (6); ∞^1 (RO, II) configurations, Steps (4) and (6); and, ∞^1 (RI, IO) configurations, Steps (5) and (6).

5.5. Mechanisms with High-Dimensional Singularity Sets

Once the loop equations and the velocity equation of a mechanism are derived, the methods described in Section 5.4 reduce the problem of singularity identification and classification to the solutions of systems of algebraic equations. However, since these are systems of nonlinear equations with multiple unknowns, their symbolic solution, if it exists, may be non-trivial. Numerical methods, on the other hand, may be computationally expensive, especially when the mobility of the mechanism is higher than 1, in which case the solution sets are manifolds rather than finite sets. These manifolds can be quite complex as can be seen in (Sefrioui and Gosselin 1994, 1995), (Mayer St-Onge and Gosselin 1995) or Collins and McCarthy 1996) where examples of singularity sets are provided for RO-type singularities of parallel manipulators.

This section addresses the application of the proposed method to complex mechanisms with high-dimensional singularity sets. Two methods for the simplification of this process are discussed and, as an illustration, the comprehensive singularity identification and classification of a 6-dof multi-loop mechanism is performed.

5.5.1. Geometrical solution of the singularity conditions

Geometrical considerations can be used to simplify the solution of the singularity conditions. Since the velocity equation is composed of screw equalities, Conditions (i)–(viii), which require the rank-deficiency of different submatrices of the velocity equation, are equivalent to conditions of linear dependence of certain joint screws. Instead of attempting to analytically solve the nonlinear equations, obtained from the vanishing of different determinants, one can find geometrical conditions for the linear dependence of the columns of the corresponding matrices. For instance, in Step (2) of Example 5.13 the singularity of the matrix $\boldsymbol{L_p}$ is equivalent to the linear dependence of the last six columns of the matrix given in Equation (5.8). However, it can be seen that, if a non-trivial linear combination of these 8-dimensional column vectors equals zero, then both sets of screws $\{\mathbf{S}_B, \mathbf{S}_C, \mathbf{S}_D\}$ and $\{\mathbf{S}_C, \mathbf{S}_D, \mathbf{S}_G\}$ must be linearly dependent. (Indeed, since $\mathbf{m}_{FG}$ is never zero, the coefficient of the last column must be zero. Moreover, since $\mathbf{S}_B$ is always different from $\mathbf{S}_C$, the coefficients of three columns preceding the last one (columns 7, 8, 9 of $\boldsymbol{L}$ in (5.8)) are not all zero. This implies that $\{\mathbf{S}_C, \mathbf{S}_D, \mathbf{S}_G\}$ are linearly dependent. From the properties of planar screws, it then follows that an RPM-type singularity of the mechanism shown in Figure 5.3 occurs when both sets of points $\{B, C, D\}$ and $\{C, D, G\}$ are collinear (Figure 5.4).

Thus, using screw theory, the singularity conditions can be interpreted as geometric criteria, as illustrated by the example analyzed later in Section 5.6. Such a screw-theory based approach provides a better geometrical insight into the problem of singularity identification, and it is not dependent on the specific values of the link parameters. This allows the study of singularities that occur for a given kinematic chain regardless of the values of the link parameters. This geometric approach is similar to the one used by Merlet (1989) to analyze RO-type singularities of parallel manipulators with prismatic actuators.

5.5.2. Simplification of the velocity equation

For complex mechanisms with many loops, the dimension of the velocity equation can be quite large. Sometimes, the velocity equation can be simplified by eliminating some of the passive velocities. It is important, however, to ensure that the resulting equation is a necessary and sufficient condition for the feasibility of the remaining velocities.

Let $\overline{\Omega}^p$ be a vector with components $(N - n - k)$ of the passive-joint velocities of the mechanism and $\overline{\boldsymbol{M}} = \left[\boldsymbol{T}^{\mathrm{T}}, \Omega^{a\mathrm{T}}, \overline{\Omega}^{p\mathrm{T}}\right]$. Also, let the $(N - k) \times (N - k + n)$ matrix $\overline{\boldsymbol{L}}(\boldsymbol{q})$ be a continuous function of $\boldsymbol{q}$. Let $\overline{\boldsymbol{L}}_I$, $\overline{\boldsymbol{L}}_O$ and $\overline{\boldsymbol{L}}_p$ be submatrices of $\overline{\boldsymbol{L}}$, defined in the same way as $\boldsymbol{L}_I$, $\boldsymbol{L}_O$ and $\boldsymbol{L}_p$ were defined as submatrices of $\boldsymbol{L}$ in Section 5.2. For brevity we introduce the notation $\overline{N} = N - k$. Also, we denote by $\hat{\Omega}^p$ the column matrix composed of the k remaining passive velocities. The following proposition can be then proven:

5.14. Proposition. *Suppose that, for every* $\boldsymbol{q}$, *there is a matrix* $\boldsymbol{P}(\boldsymbol{q})$ *such that the velocity equation can be written in the form:*

$$\overline{L}(q)\overline{M} = O, \tag{5.9}$$
$$\hat{\Omega}^{p} = P(q)\overline{M}. \tag{5.10}$$

Then, all singularity conditions derived in Theorems 5.1 and Proposition 5.2 remain true when the matrix $\overline{L}(q)$ *is used instead of* $L(q)$, *i.e.,*

(1) *A configuration,* q, *is nonsingular, if and only if both the matrices* $\overline{L}_I$ *and* $\overline{L}_O$ *are nonsingular at* q.

(2)

(i) $q \in \{\mathrm{RI}\} \Leftrightarrow \operatorname{rank} \overline{L}_O < \operatorname{rank} \overline{L}_p + n$,

(ii) $q \in \{\mathrm{RO}\} \Leftrightarrow \operatorname{rank} \overline{L}_{II} < \operatorname{rank} \overline{L}_p + n$,

(iii) $q \in \{\mathrm{RPM}\} \Leftrightarrow \operatorname{rank} \overline{L}_p < \overline{N} - n$,

(iv) $q \in \{\mathrm{II}\} \Leftrightarrow \operatorname{rank} \overline{L}_I < \operatorname{rank} \overline{L}$,

(v) $q \in \{\mathrm{IO}\} \Leftrightarrow \operatorname{rank} \overline{L}_O < \operatorname{rank} \overline{L}$,

(vi) $q \in \{\mathrm{IIM}\} \Leftrightarrow \operatorname{rank} \overline{L} < \overline{N}$,

(vii) $q \in \{\mathrm{RO}\}$ or $q \in \{\mathrm{RPM}\} \Leftrightarrow q \in \{\mathrm{II}\}$ or $q \in \{\mathrm{IIM}\} \Leftrightarrow \overline{L}_O$ *is singular,*

(viii) $q \in \{\mathrm{RI}\}$ or $q \in \{\mathrm{RPM}\} \Leftrightarrow q \in \{\mathrm{IO}\}$ or $q \in \{\mathrm{IIM}\} \Leftrightarrow \overline{L}_I$ *is singular.*

Proof.

(1) It needs to be proven that q is singular, if and only if at least one of the matrices $\overline{L}_I$ and $\overline{L}_O$ is singular.

As in the proof of Theorem 5.1, we note that Equation 5.9 can be written in any of the following two forms:

$$\overline{L}_I \begin{bmatrix} T \\ \overline{\Omega}^{p} \end{bmatrix} = -\overline{L}_a \Omega^{a}, \tag{5.11}$$

and

$$\overline{L}_O \begin{bmatrix} \Omega^{a} \\ \overline{\Omega}^{p} \end{bmatrix} = -\overline{L}_T T . \tag{5.12}$$

From Equations (5.11) and (5.12), it is evident that all, but the eliminated velocities, can be expressed in terms of the output (input) velocities, if and only if $\overline{L}_I$ (respectively $\overline{L}_O$) is invertible. The eliminated passive velocities, $\hat{\Omega}^p$, are given as a function of the remaining velocities by Equation (5.10) Therefore, according to the definition of singularity in Section 3.3 (Definition 3.5), Theorem 5.1 remains true when stated for the matrices $\overline{L}_I$ and $\overline{L}_O$.

(2)

(i) A configuration, $\boldsymbol{q}$, is of the RI-type only when the velocity equation is satisfied for $\Omega^a \neq 0$ and $\boldsymbol{T} = 0$. Considering the form of the velocity equation given by Equations (5.12) and (5.10), the condition for RI becomes:

$$\overline{L}_O \begin{bmatrix} \Omega^a \\ \overline{\Omega}^p \end{bmatrix} = \boldsymbol{O} \ , \ \Omega^a \neq 0. \tag{5.13}$$

Let d, $d \geq 0$, be defined by $\text{rank}\,\overline{L}_p = \overline{N} - n - d$. Then, $\dim(\text{Ker}\,\overline{L}_O)$ is exactly d, if and only if the left-hand side of (5.2) can be zero only for a zero Ω^a. Therefore, an RI-type singularity is present only when $\dim(\text{Ker}\,\overline{L}_O) > d$. This proves condition (i), since $\text{rank}\,\overline{L}_O = \overline{N} - \dim(\text{Ker}\,\overline{L}_O)$ as well as $\overline{N} - d = \text{rank}\,\overline{L}_p + n$.

(ii) Analogous to (i).

(iii) From Equations (5.9) and (5.10) and the definition of RPM-type singularity, Definition 3.21, it is clear that an RPM-type singularity can occur only when the equation $\overline{L}_p \overline{\Omega}^p = 0$ can be satisfied for a nonzero $\overline{\Omega}^p$. This is so only when $\overline{L}_p$ is singular, i.e., $\text{rank}\,\overline{L}_p < \overline{N} - n$

(iv) Equation (5.11) implies that $\boldsymbol{q} \in \{\text{II}\}$ is equivalent to the existence of a vector $\boldsymbol{v}$, which is in $\text{Im}\,\overline{L}_a$ but not in $\text{Im}\,\overline{L}_I$, i.e., to $\text{Im}\,\overline{L}_a - \text{Im}\,\overline{L}_I = \varnothing$. Since

$\mathrm{Im}\,\overline{L} = \mathrm{Im}\,\overline{L}_a + \mathrm{Im}\,\overline{L}_l$, this in turn is equivalent to $\mathrm{Im}\,\overline{L} - \mathrm{Im}\,\overline{L}_l = \varnothing$, i.e., $\mathrm{rank}\,\overline{L}_l < \mathrm{rank}\,\overline{L}$.

(v) Analogous to (iv).

(vi) According to Definition 3.18, the IIM-type singularity requires that the scalar equations of the velocity equation are linearly dependent. Since the k scalar equations in (5.10) are clearly linearly independent, $\boldsymbol{q}$ can be of the IIM singularity type, if and only if Equation (5.9) is singular, i.e., $\mathrm{rank}\,\overline{L} < \overline{N}$.

(vii) Follows from (ii), (iii), (iv) and (vi).

(viii) Follows from (i), (iii), (v) and (vi). □

5.15. Remark. The above Proposition 5.14 allows us to decrease the dimension of the singularity-identification problem by the elimination of some passive velocities. It must be noted that passive velocities can be eliminated only when the resulting reduced velocity equation (Equation (5.9)) is still a necessary and sufficient condition for the feasibility of the remaining velocities.

Note that the matrix function $\boldsymbol{P}(\boldsymbol{q})$ need not be known explicitly in order to apply the singularity criteria, since they are based solely on the submatrices of $\overline{L}(\boldsymbol{q})$. It is sufficient to make sure that for every configuration the eliminated velocities, $\hat{\Omega}^p$, are determined in a unique way by the remaining joint velocities.

Later in this chapter, in Sub-Section 5.6.2 the process of partial elimination of the passive screws and the derivation of simplified singularity conditions will be illustrated by an example.

The elimination of passive velocities may be executed by algebraic manipulations of the velocity equation, or geometrically by using reciprocal screws. Reciprocal screws have been used by different authors to obtain input-output velocity equations of parallel and hybrid-chain manipulators (Kumar 1990, Angeles 1994, Etamadi-Zanganeh and Angeles 1994, Chapter 4 of this thesis). A similar approach can be used for general closed-loop mechanisms: by multiplying the twist equation for each loop by one or more reciprocal screws part of the passive velocities are eliminated. However, as it was shown in Chapter 4, if the reciprocal screws are not chosen in a correct way, the resulting equation may no longer be a necessary and sufficient condition and would not be suitable for singularity analysis.

5.6. Singularity Analysis of an Exemplary Spatial Mechanism

5.6.1. The mechanism

To illustrate the above techniques, herein, the singularities of the mechanism shown in Figure 5.7 are identified. This is a 6-dof platform manipulator with an asymmetric distribution of the actuated joints (first described in Zlatanov et al., 1992). The output link (the end-effector) is the moving platform *ABC*, the six input joints are: the first three joints of sub-chain *A*, the second and third joint in subchain *B* and the third joint in subchain *C*. The base $A_0B_0C_0$ and the moving platform are equilateral triangles with sides $AB = A_0B_0 = a$. The two nonzero links in each serial subchain have the same length, l. It is assumed that $2l/\sqrt{3} \leq a \leq 2l$.

The velocity equation, obtained using the method outlined in Chapter 3, is:

$$\begin{bmatrix} I_6 & -J_a^A & O & O & -J_p^A & O & O \\ O & J_a^A & -J_a^B & O & J_p^A & -J_p^B & O \\ O & O & J_a^B & -J_a^C & O & J_p^B & -J_p^C \end{bmatrix} \begin{bmatrix} T \\ \Omega^a \\ \Omega^p \end{bmatrix} = O \qquad (5.14)$$

where, for each P, ($P = A, B, C$), J_a^P is a matrix which has as its columns the active joint screws in the serial sub-chain, while J_p^P is composed of the passive screws in the sub-chain. The output is the twist of the moving platform, $\boldsymbol{T} = \mathbf{T}$, the input, $\Omega^a = [\omega_1^A, \omega_2^A, \omega_3^A, \omega_2^B, \omega_3^B, \omega_3^C]^T$, is composed of the six active-joint velocities, and Ω^p

is the vector of the passive velocities. (The spherical joints are modelled by three linearly-independent rotations through their centers). The first six scalar equations in (5.14) are the output equation, while the remaining 12 equations are given by two loop-closure twist equations. The only restrictions imposed on the joint parameters is the condition of non-interference of the different links. In particular, those configurations for which a leg is folded (i.e., where $P = P_0$) will be considered as impossible to achieve.

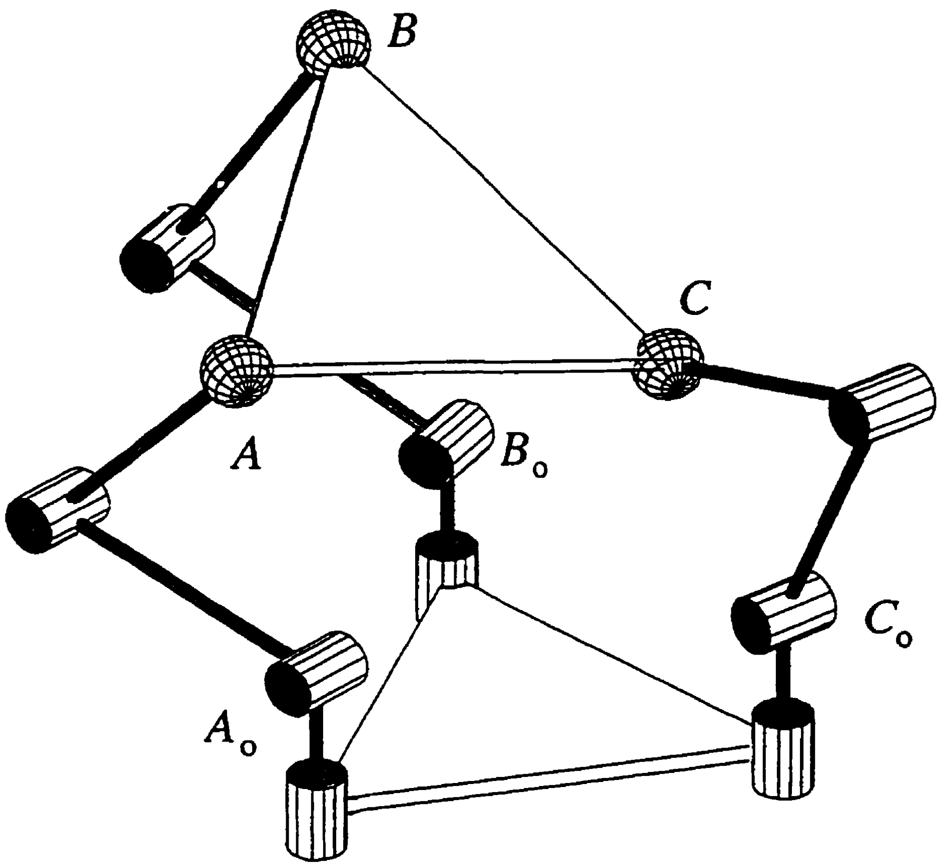

Figure 5.7. A 6-dof hybrid-chain manipulator.

5.6.2. Simplification of the singularity conditions

Following the guidelines from Sub-Section 5.5.2, Equation (5.14) can be simplified by eliminating some of the passive-joint velocities. First, we observe that (5.14) is equivalent to the system of equations:

$$\mathbf{T} = \sum_{i=1}^{6} \mathbf{S}_i^P \omega_i^P \qquad P = A, B, C. \tag{5.15}$$

(Systems of this type are commonly used in the literature to describe the velocity kinematics of parallel manipulators).

Each of the three twist equations in (5.15) can be multiplied (via the so-called reciprocal scalar product) by a screw, **R**, to obtain a scalar equation. If **R** is chosen to be always orthogonal (i.e., reciprocal) to one or more joint screws, then the corresponding joint variables will be eliminated from the resulting equations. If a sufficient number of such reciprocal screws can be found, a new system with a smaller number of variables will be obtained. As it was pointed out in Section 5.5.2, to be suitable for singularity identification, the new system must be equivalent to the old one and the values of the eliminated variables must be uniquely determined for each set of values of the remaining variables. In the case of System (5.15), this can be ensured only if the joint screws of the eliminated velocities in each one sub-chain are linearly independent. Therefore, *all* passive-joint velocities could be eliminated, only if the matrices J_p^P were of maximum rank for all P and for all $\boldsymbol{q}$. This, however, is not true, since for some configurations the passive-joint screws in subchains B and C can become linearly dependent. For example, whenever point B lies on the screw axis $\mathbf{S}_1^B$, the rank of J_p^B is 3 rather than 4. Therefore, it is impossible to properly eliminate all four passive velocities in this subchain. Indeed, if we assume that all other velocities are known, ω_1^B could still have any value, and thus the values of the passive-joint velocities in sub-chain B could not be determined in a unique way.

On the other hand, since the three joint screws corresponding to each spherical joint are always linearly independent, the corresponding nine passive velocities can be safely eliminated. This is done by multiplying each of the three screw equations in (5.9) by the screw "annihilator" of the spherical joint (Angles 1994). In other words, we take the reciprocal product of equation P with three linearly independent screws, all reciprocal to joint-screws $\mathbf{S}_4^P$, $\mathbf{S}_5^P$ and $\mathbf{S}_6^P$. Therefore, these three screws must be linearly independent rotations with axes through P. It is convenient to choose these axes parallel to the axes of the reference frame. Then, in a coordinate system with origin at A and axes parallel to those of the base frame the following system, $\overline{\boldsymbol{LM}} = \boldsymbol{O}$, is obtained:

$$\begin{bmatrix} \boldsymbol{O} & I_3 & -\boldsymbol{M}_{123}^A & \boldsymbol{O} & \mathbf{0} & \mathbf{0} & \boldsymbol{O} \\ -\tilde{\mathbf{b}} & I_3 & \boldsymbol{O} & -\boldsymbol{M}_{23}^B & \mathbf{0} & -\mathbf{m}_1^B & \boldsymbol{O} \\ -\tilde{\mathbf{c}} & I_3 & \boldsymbol{O} & \boldsymbol{O} & -\mathbf{m}_3^C & \mathbf{0} & -\boldsymbol{M}_{12}^C \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \Omega^a \\ \overline{\Omega}^p \end{bmatrix} = \boldsymbol{O}\,. \quad (5.16)$$

In Equation (5.16), $\mathbf{m}_i^P$ is the moment of the screw $\mathbf{S}_i^P$ *with respect to point P*, while $\boldsymbol{M}_{ij...}^P$ is the matrix $[\mathbf{m}_i^P, \mathbf{m}_j^P, \ldots]$. For a 3-dimensional vector $\mathbf{v}$, $\tilde{\mathbf{v}}$ denotes the skew-symmetric matrix with the property: $\mathbf{v} \times \mathbf{w} = \tilde{\mathbf{v}}\mathbf{w}$, for any vector $\mathbf{w}$. The vector $\mathbf{b}$ is parallel to AB and $\mathbf{c}$ is parallel to AC. Only three of the passive joint velocities remain in (5.16), $\overline{\Omega}^p = [\omega_1^B, \omega_1^C, \omega_2^C]^T$.

According to Proposition 5.14 in Sub-Section 5.5.2, Equation (5.16) can be used in the same way as (5.14) or (5.15) for singularity identification. Note that, using the above technique one can easily obtain equations analogous to (5.16) for any hybrid-chain manipulator with passive spherical joints at the moving platform.

Several simplified matrices can be introduced and used for the calculation of the ranks of $\overline{L}_I$, $\overline{L}_O$, $\overline{L}_p$ and $\overline{L}$ (or L_I, L_O, L_p and L).

From Equation (5.14) it can be deduced that rank L = rank $\hat{L} + 6$ (and, therefore, rank $\overline{L}$ = rank $\hat{L} - 3$), where $\hat{L}$ is the the 12×18 matrix,

$$\hat{\boldsymbol{L}} = \begin{bmatrix} J^A & -J^B & O \\ O & J^B & -J^C \end{bmatrix}, \quad (5.17)$$

Above, J^P are the 6×6 sub-chain Jacobians.

From Equation (5.16), it follows that rank $\overline{L_p}$ = rank $\hat{L}_p$, where $\hat{L}_p$ is the 6×6 matrix,

$$\hat{L}_p = \begin{bmatrix} \mathbf{m}_1^B & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{m}_1^C & \mathbf{m}_2^C \end{bmatrix}. \tag{5.18}$$

Also from (5.16), the rank of $\overline{L_I}$ can be expressed by the matrix $\hat{L}_I$,

$$\hat{L}_I = \begin{bmatrix} \tilde{\mathbf{b}} & \mathbf{m}_1^B & \mathbf{0} & \mathbf{0} \\ \tilde{\mathbf{c}} & \mathbf{0} & \mathbf{m}_1^C & \mathbf{m}_2^C \end{bmatrix}, \tag{5.19}$$

for which: rank $\overline{L_I}$ = rank $\hat{L}_I$ + 3

Finally, rank $\overline{L_O}$ = rank $\hat{L}_O$, where $\hat{L}_O$ is obtained by rearranging the columns of $\overline{L_O}$,

$$\hat{L}_O = \mathrm{diag}(M_{123}^A, M_{123}^B, M_{123}^C). \tag{5.20}$$

Thus, for the mechanism in Figure 5.7, the conditions from Section 5.2 can be expressed in terms of the matrices from Equations (5.17) to (5.20) as follows:

(i) $q \in \{RI\} \Leftrightarrow$ rank $\hat{L}_O <$ rank $\hat{L}_p + 6$,

(ii) $q \in \{RO\} \Leftrightarrow$ rank $\hat{L}_I <$ rank $\hat{L}_p + 3$,

(iii) $q \in \{RPM\} \Leftrightarrow$ rank $\hat{L}_p < 3$,

(iv) $q \in \{II\} \Leftrightarrow$ rank $\hat{L}_I <$ rank $\hat{L} - 6$,

(v) $q \in \{IO\} \Leftrightarrow$ rank $\hat{L}_O <$ rank $\hat{L} - 3$,

(vi) $q \in \{IIM\} \Leftrightarrow$ rank $\hat{L} < 12$,

(vii) $q \in \{RO\}$ or $q \in \{RPM\} \Leftrightarrow q \in \{II\}$ or $q \in \{IIM\} \Leftrightarrow \hat{L}_O$ *is singular,*

(viii) $q \in \{RI\}$ or $q \in \{RPM\} \Leftrightarrow q \in \{IO\}$ or $q \in \{IIM\} \Leftrightarrow \hat{L}_I$ *is singular.*

5.6.3. Identification and classification of the mechanism's singularities

5.6.3.1. Summary. We apply the identification algorithm from Section 5.2 to the mechanism in Figure 5.7 using conditions (i)–(viii) listed above, in Sub-Section 5.6.2. At each step, the conditions are resolved through geometric analysis of the screws composing the corresponding matrices. The singularities obtained are summarized in Table 5.1.

	IO	II	IO *and* II	IIM	IO *and* IIM	II *and* IIM	IO *and* II *and* IIM
RI	**YES** Step 6						
RO		**YES** Step 6 5.13-14					
RI *and* RO			**YES** Step 6	**NO** Step 3.1	**YES** Step 3.4 5.10	**NO** Step 3.1	**NO** Step 3.4
RPM			**YES** Step 3.5 5.9†	**NO** Step 3.1			**YES** Step 3.3 5.9
RI *and* RPM			**YES** Step 3.5 5.8†		**YES** Step 3.3 5.9*		**YES** Step 3.3 5.8
RO *and* RPM			**YES** Step 3.5 5.12†			**NO** Step 3.1	**YES** Step 3.3 5.12
RI *and* RO *and* RPM			**YES** Step 3.5 5.11†	**NO** Step 3.1	**NO** Step 3.3	**NO** Step 3.1	**YES** Step 3.3 5.11

† A representative of this class is obtained by a small variation of the configuration in the corresponding figure.

* A representative of this class is obtained by a re-labelling of the corresponding figure.

Table 5.1. Possible singularity classes for the 6-dof mechanism shown in Figure 5.7.

In the table, for each singularity class, the following are denoted: whether the class is non-empty (YES) or empty (NO); the steps in which the singularities of the class are obtained (or it is proven that the class is empty); and, the number of the figure that shows a representative configuration of the class.

It is determined that, the mechanism has singularities belonging to 13 different singularity classes. Seven figures (Figures 5.8–5.14) illustrate the singularities of the mechanism. Except for Figures 5.13 and 5.14, which represent the same (RO, II) singularity class, the figures depict configurations belonging to different singularity classes. One figure (Figure 5.9) is used to illustrate two singularity classes after a re-labelling of the sub-chains (Step 3.3). Four figures (Figures 5.8, 5.9, 5.11 and 5.12) can illustrate four additional classes, if a small perturbation in the depicted configuration is performed (Step 3.5). The remaining two classes, which are not directly illustrated by figures, consist of singularities that are comparatively easy to describe and envision (Steps 5 and 6).

5.6.3.2. The identification procedure. Below, the steps of the identification procedure are detailed.

(1) *IIM-type singularities*

For an IIM-type singularity, the matrix $\hat{L}$ must be rank-deficient. A necessary and sufficient condition for this is the existence of a row vector $[(\Pi\mathbf{A})^{\mathrm{T}}, (\Pi\mathbf{C})^{\mathrm{T}}]$, which is in the kernel of $\hat{L}^{\mathrm{T}}$. ($\Pi\mathbf{A}$ is the screw **A** with its rotational and translational parts interchanged). Equivalently, there must exist screws **A** and **C** reciprocal to all the columns of, respectively, J^A and J^C, while $\mathbf{A} - \mathbf{C}$ is reciprocal to the columns of J^B. This condition is quite restrictive and for generic values of the link parameters no IIM-type configurations exist. In the present example, however, the special choice of congruent triangles for the base and moving platforms assures the existence of such singularities.

Careful geometrical analysis reveals that the set of IIM-type configurations consists of two non-intersecting components.

The first component has ∞^3 configurations, and one of them is shown in Figure 5.8.

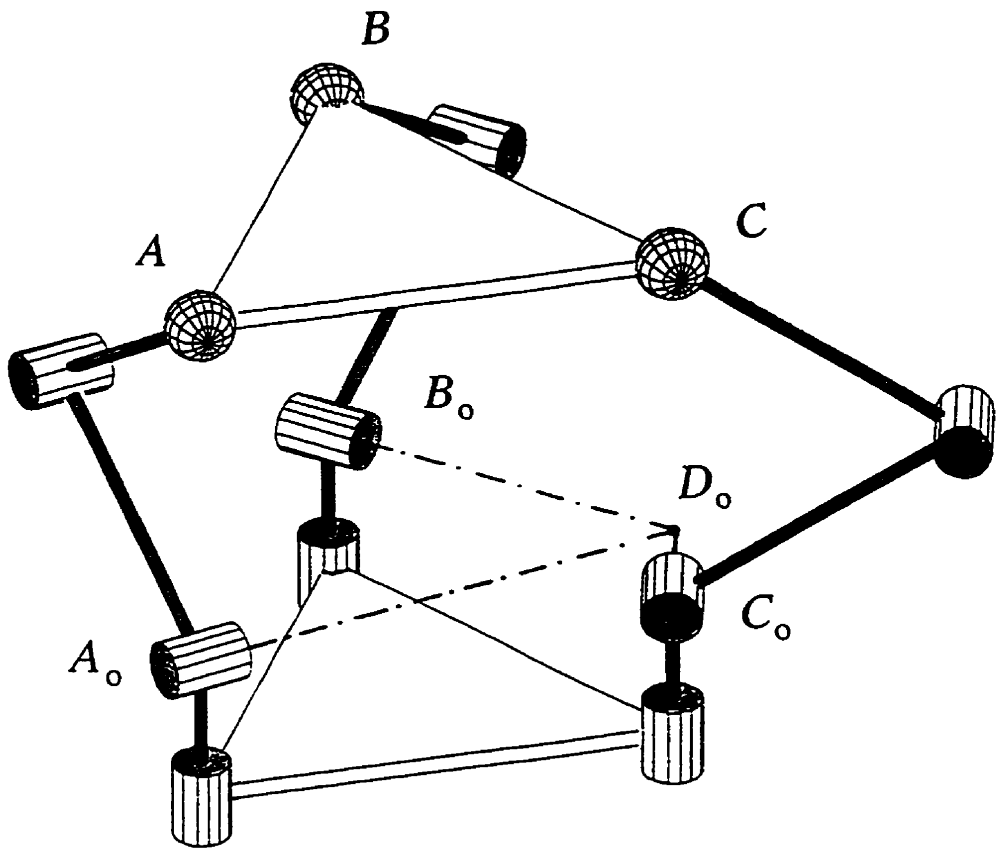

Figure 5.8. A singular configuration of class (RPM, RI, IIM, II, IO).

In this configuration, the points P are on the $\mathbf{S}_1^P$ axes and the three axes $\mathbf{S}_2^P$ intersect in one point, D. The ∞^3 configurations can be obtained by varying the elevation of the moving platform and moving the intersection point, D, in the base plane (D can also be at infinity). Several 2-dimensional manifolds of IIM-type singularities are attached to the 3-dimensional set. One of these can be obtained from the configuration shown in Figure 5.9 by rotating the moving platform about the line *BC* (and varying the elevation of points *B* and *C*).

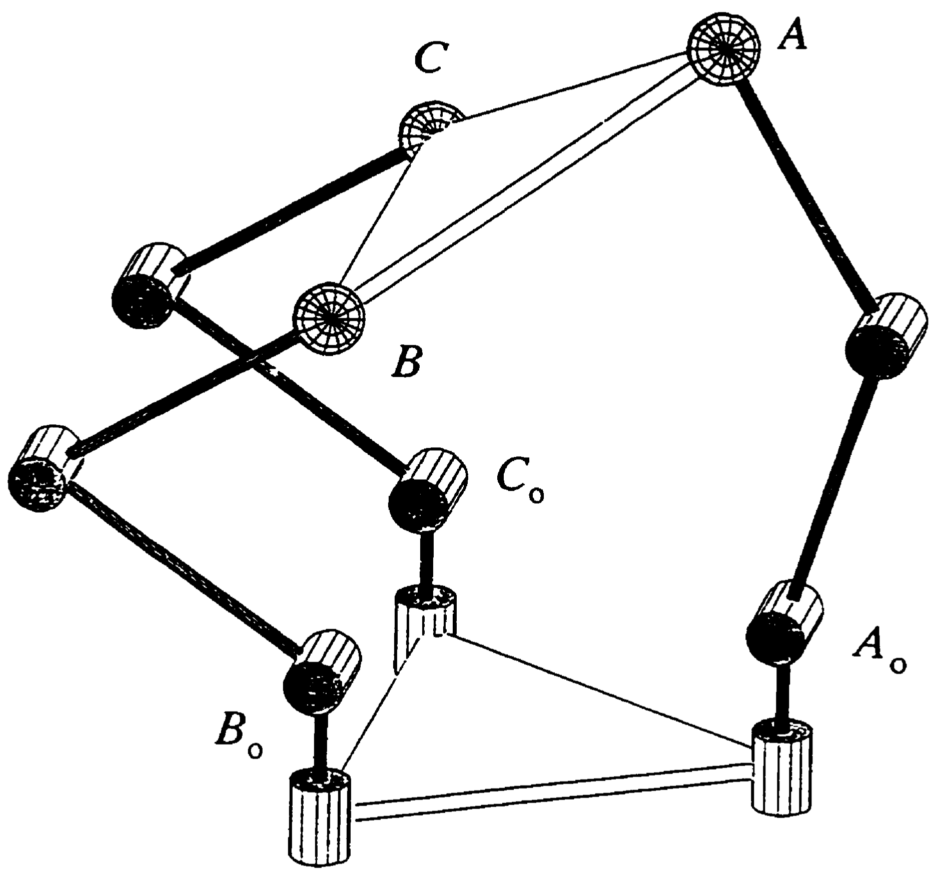

Figure 5.9. A singular configuration of class (RPM, IIM, II, IO).

The second component is 1-dimensional and consists of configurations like the one in Figure 5.10, where the three supporting legs are fully extended and the two platforms are in the same plane.

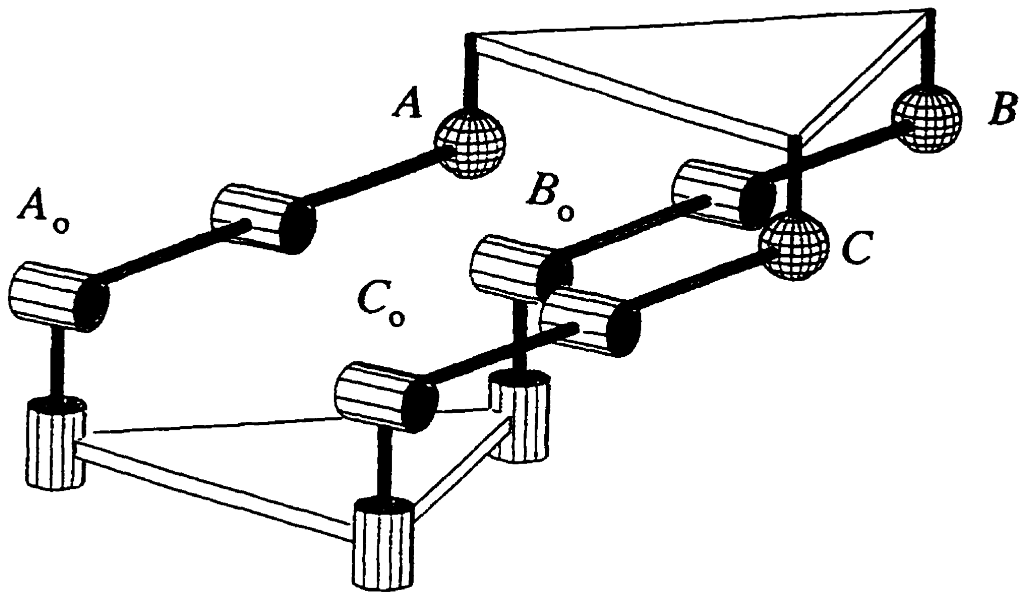

Figure 5.10. A singular configuration of class (RI, RO, IO, IIM).

(2) *RPM-type singularities*

From Equation (5.18) it is evident that $\hat{L}_P$ is singular only when either $\mathbf{m}_1^B$ or $\mathbf{m}_1^C$ are zero, i.e., when either B or C are on the axis of the first-joint screw of the corresponding subchain. Each of these two conditions corresponds to a set of ∞^5 configurations. They intersect in a 4-dimensional set.

(3) *Classification of* $\{1\} \cup \{2\}$

(3.1) It can be observed that in all existing singularities of $\{1\}$ the rank of L decreases by only one, while the rank of $\hat{L}_O$ decreases by at least two. Therefore, $\{1\}$ is a subset of the IO type. Therefore, the six classes belonging to the IIM-type but not the IO-type are empty.

From Condition (iv), it follows that an element of $\{1\}$ is an II-type singularity, if and only if the 6×6 matrix $\hat{L}_I$ has a null-space dimension of at least two. Next, we check whether this condition is satisfied for the different IIM-type singularities as determined in Step (1).

For all the ∞^3 configurations of the type shown in Figure 5.8, where for all three serial subchains point P is on the $\mathbf{S}_1^P$ axis, the condition is satisfied since $\mathbf{m}_1^B$ and $\mathbf{m}_1^C$ are zero vectors.

For the IIM-type configurations with three extended legs (as in Figure 5.10) the condition is not satisfied.

If only two subchains are singular (similarly to Figure 5.9), the condition is always satisfied, when the singular subchains are B and C (as in the figure). When, however, one of the singular subchains is A, then, generally, the matrix A is of rank 5. There are two exceptions. The first is represented in Figure 5.11.

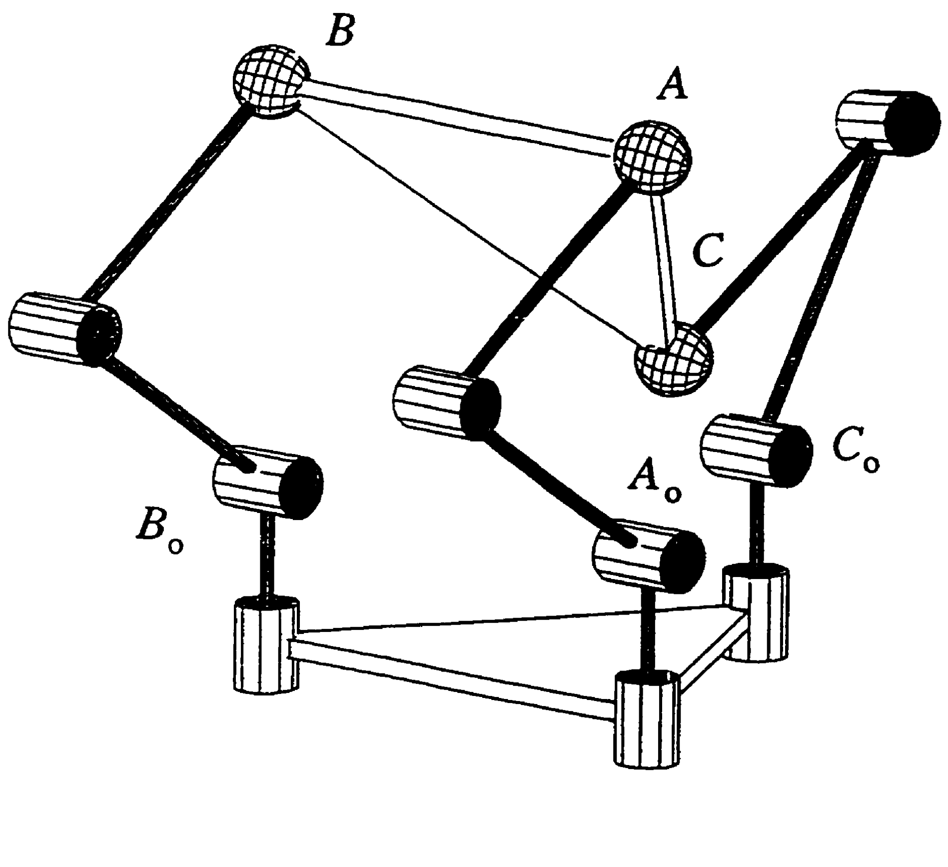

Figure 5.11. A singular configuration of class (RI, RO, RPM, IO, II, IIM).

In Figure 5.11, the singular subchains are A and B and, additionally, the point C_0 lies in the plane ABC. The second exception is shown in Figure 5.12, where not only points B and C are located on screws $\mathbf{S}_1^B$ and $\mathbf{S}_1^C$, but also point A lies in the (vertical) plane defined by the two screws. Each of Figures 5.11 and 5.12 represents, in fact, ∞^1 configurations, since the elevation of point A can vary.

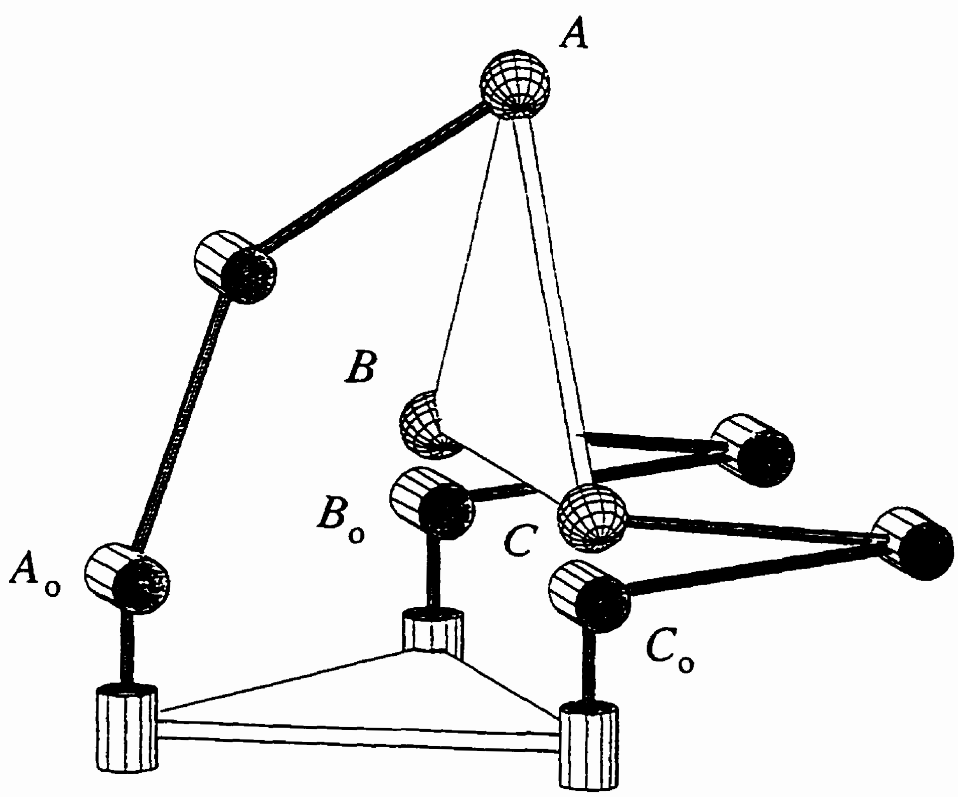

Figure 5.12. A singular configuration of class (RO, RPM, IO, II, IIM).

Thus, the set of singularities belonging to the IIM, IO and II types consists of a main 3-dimensional set (Figure 5.8), a 2-dimensional set (Figure 5.9) and two 1-dimensional sets (Figures 5.11 and 5.12). The set of singularities in the IIM and IO types has two 2-dimensional components (similar to Figure 5.9, with subchain A as one of the singular ones) and one 1-dimensional component (Figure 5.10).

(3.2) According to condition (i) and Equation (5.14), a configuration is an RI-type singularity, if and only if at least one of the following conditions is satisfied: either the subchain A is singular (in any way); or subchain B is fully extended; or subchain C is fully extended.

Condition (ii) and Equation (5.13) imply that an RPM-singularity is also of the RO-type in the following three cases:

(a) When C is on the $\mathbf{S}_1^C$ axis and the plane ABC is perpendicular to $\mathbf{m}_2^C$ (Figure 5.12 is an example, though subchain B need not be singular).

(b) When C is on the $\mathbf{S}_1^C$ axis but point B is not on the $\mathbf{S}_1^B$ axis, and $\mathbf{b} \perp \mathbf{m}_1^B$.

(c) When B is on the $\mathbf{S}_1^B$ axis, while point C is not on the $\mathbf{S}_1^C$ axis, and the point C_0 lies in the plane ABC. (Figure 5.11, though subchain A need not be singular).

Thus, four sets are obtained: ∞^5 RPM-type singularities, ∞^4 RPM and RI-type singularities, ∞^4 RPM and RO-type singularities and RPM, RI and RO-type singularities.

(3.3) The intersections of the subsets of {3.1} and {3.2} give the ten singularity classes (Table 5.1) of configurations that are both IIM and RPM. Of these, only five classes are non-empty for the mechanism under consideration:

(a) (IIM, IO, RPM, RI) has ∞^2 configurations with two singular subchains similarly to Figure 5.9, but sub-chain A must be one of the singular sub-chains.

When the two singular subchains are A and B, point C_0 should not lie in the plane ABC (i.e, unlike Figure 5.12). Alternatively, if the singular subchains are A and C, then the plane ABC should not contain C_0 and A_0.

(b) (IIM, IO, II, RPM) has ∞^2 configurations as in Figure 5.9. The singular subchains must be B and C. The plane ABC must not contain C_0 and B_0 (unlike Figure 5.12).

(c) (IIM, IO, II, RPM, RI) has ∞^3 configurations with three singular subchains as in Figure 5.8.

(d) (IIM ,IO, II, RPM, RO) has ∞^1 configurations like the one depicted in Figure 5.12. The moving plane ABC contains the points C_0 and B_0 and the subchains B and C are singular in the same way as in Figure 5.9.

(e) (IIM, IO, II, RPM, RI, RO) has ∞^1 configurations in two 1-dimensional sets. The first is represented by the configuration in Figure 5.11. It is similar to Figure 5.9 with singular sub-chains A and B, but point C_0 is in the plane ABC, allowing for a RO-type singularity. The second set is similar to the configuration in Figure 5.11, however, the nonsingular subchain must be B rather than A.

(3.4) Only one of the four classes of IIM but not RPM singularities is non-empty: (IIM, IO, RI, RO) consists of ∞^1 configurations as in Figure 5.10.

(3.5) All of the four RPM but not IIM classes are non-empty.

(RPM, II, IO) has ∞^5 configurations. An example for this class can be obtained from the configuration in Figure 5.9 by an arbitrarily small perturbation of the subchain C while subchains A and B remain fixed.

(RPM,RI, II, IO) has ∞^4 configurations and can be illustrated by a variation of Figure 5.8 obtained by maintaining the depicted position of the subchains A and B and slightly perturbing subchain C.

(RPM, RO, II, IO) has ∞^4 configurations. An example is obtained from the configuration in Figure 5.12 by a small rotation of subchain C about $\mathbf{S}_1^C$.

(RPM, RI, RO, II, IO) has ∞^2 configurations and a representative can be obtained from Figure 5.11 by a small rotation of subchain C about $\mathbf{S}_1^C$.

(4) *RO- and II-type singularities*

There are ∞^5 configurations that are of the RO and II types but are not IIM nor RPM-singularities. From Equation (5.19), the conditions for RO-type singularity are:

(a) Either C_0 must be in the plane *ABC* (Figure 5.13), or

(b) The point *A* must be in the plane of subchain *B* (Figure 5.14), i.e., $\mathbf{b} \perp \mathbf{m}_1^B$.

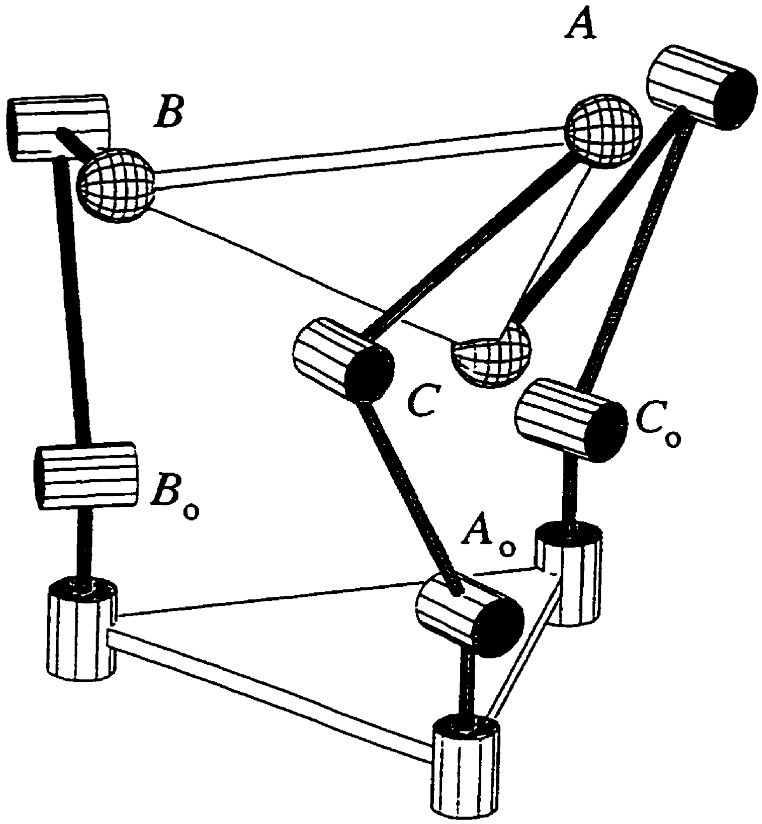

Figure 5.13. A singular configuration of class (RO, II).

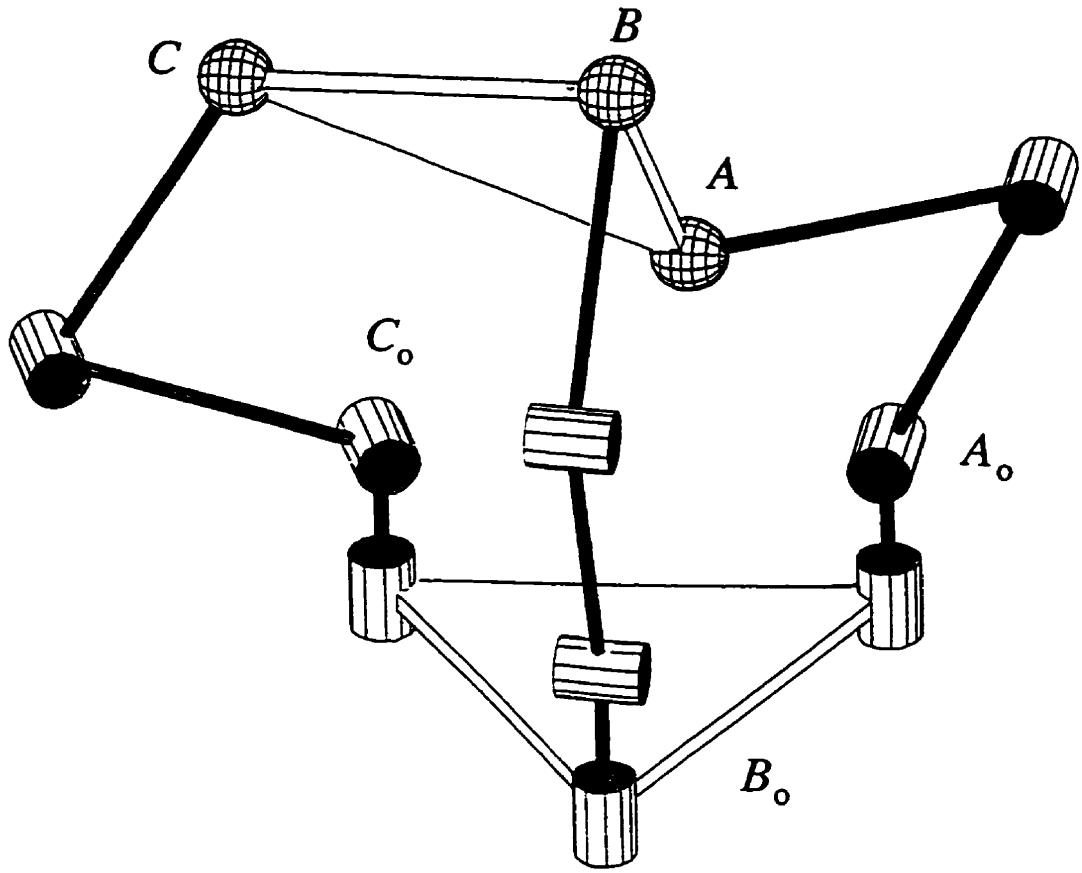

Figure 5.14. A singular configuration of class (RO, II).

(5) *RI- and IO-type singularities*

There are ∞^5 configurations which satisfy (viii) without being RPM or IIM-type. In these configurations, the subchain *A* is singular or one of the other two serial chains is fully extended.

(6) *Classification of* $\{4\} \cup \{5\}$

The last three singularity classes are obtained as the intersection and differences of $\{4\}$ and $\{5\}$. (RI, IO) and (RO, II) have ∞^5 configurations, while (RI, RO, IO, II) is of dimension 4.

Thus, for the the mechanism considered in this example there are 13 different classes of singularities. The remaining 8 classes are empty.

5.7. Summary

It has been shown that the singularities of a general non-redundant mechanism form a set which is divided into 21 singularity classes. Singularities from the same class belong to exactly the same combination of the six fundamental singularity types. On the basis of the velocity-equation formulation of mechanism singularity, this chapter establishes the necessary and sufficient conditions for the occurrence of singularities from each of the six singularity types. By employing the proposed singularity criteria, all the singular configurations of an arbitrary non-redundant mechanism can be identified and classified. This can be achieved via a procedure, described in the Section 5.4, which reveals step-by-step the structure of the singularity set of the mechanism. The configurations belonging to each of the singularity classes are obtained as solution sets of nonlinear algebraic equations. Algebraic and geometric techniques for finding these solution sets are proposed. As a comprehensive example, a 6-dof hybrid-chain manipulator, with asymmetrical distribution of the input joints and a complex singularity set, is studied. Through careful geometric and algebraic analysis, the structure of the singularity set is revealed and configurations from all singularity classes are described and illustrated.

CHAPTER 6

REDUNDANT MECHANISMS

6.1. Introduction

In this chapter, the techniques proposed in Chapters 3 and 5 are further generalized to include mechanisms with redundancy. Mechanism redundancy was defined in Chapter 2. It is present when the dimensions of the input and output space are not equal and therefore either $n > n_O$ or $n_I > n$. The first inequality defines kinematic redundancy. If $n_I > n$ we will say that a dynamic redundancy is present. This phenomenon is often referred to as actuation redundancy in the literature.

In this chapter, it will be assumed that $n_I \geq n \geq n_O$. Thus, non-redundant, dynamically-redundant and kinematically-redundant mechanisms will be treated as special cases of the general assumption. All definitions and propositions in the present chapter will be valid for redundant and non-redundant mechanisms alike. Moreover, the classification framework and the identification methods, proposed herein, when applied to non-redundant mechanisms, must be equivalent to the respective results obtained in Chapters 3 and 5.

The main task of this chapter is to define the singularity types for a redundant mechanism and study their inter-dependence. The difficulty arises from the fact that the definition of the singularity types in Section 3.4 was based on the specific definition of singularity for non-redundant mechanisms, (Section 3.3). This approach allowed for simple definitions that clarified the kinematic implications of each singularity type and emphasized the symmetry of input and output and of I-type and R-type singularities. This symmetry was further exploited in Section 3.6 and made explicit in the symmetric statements of Proposition 3.29 and the diagonal symmetry of the classification table, Table 3.1. As we shall see in the following sections, these symmetries are not entirely preserved in the general case (when redundancy is possible). It will be shown that there are six additional singularity classes (combinations of singularity types), which never occur in non-redundant mechanisms but are possible for some redundant mechanisms. This means that the identification methods, derived for non-redundant mechanisms in Chapter 5, must be modified to be applicable to redundant mechanisms.

In Section 6.2, we briefly discuss the applicability of the velocity-equation and motion-space models of instantaneous kinematics for the case of redundant mechanisms. In Section 6.3, the six singularity types are re-defined in a way relevant to redundant mechanisms. In Section 6.4, we study the interdependence of the singularity types and prove classification theorems for redundant mechanisms. Finally, in Section 6.5, we examine the identification algorithms of Chapter 5 and make the necessary modifications to make these procedures applicable to redundant mechanisms as well.

6.2. Infinitesimal Model of Mechanism Kinematics

The goal of this Section is to generalize Theorem 3.1 and obtain a statement valid for redundant mechanisms as well.

For redundant mechanisms, the notations $\boldsymbol{T}$, $\boldsymbol{\Omega}^a$, $\boldsymbol{\Omega}^p$ are defined, as this was done in Section 3.2 for non-redundant mechanisms. $\boldsymbol{T}$ denotes an output vector (an element of T_qO), $\boldsymbol{\Omega}^a$ is an input vector (an element of T_qI) and $\boldsymbol{\Omega}^p$ is the vector of passive-joint velocities. Also, the tangent spaces T_qO and T_qI, are denoted by $\mathcal{O}$ and $\mathcal{I}$, while $\mathcal{P}$ is the space of all the vectors Ω^p. The dimensions of the vector spaces I, $\mathcal{P}$ and O (and of the vectors Ω^a, Ω^p and T) are n_I, $N - n_I$, and n_O, respectively.

The spaces O, I and $\mathcal{P}$, defined in Section 3.2, can be viewed as spanning an $(N+n_O)$-dimensional space $\mathcal{V} = O \oplus I \oplus \mathcal{P}$. $\mathcal{V}$ is, in fact, the tangent space of the manifold $Q \times O$ at the point $(q, f_O(q))$. The elements of $\mathcal{V}$ are velocity vectors of the form $m = (T, \Omega) = (T, \Omega^a, \Omega^p)^T$. Those velocity vectors, that represent feasible motions, form a subspace of $\mathcal{V}$, the *motion space* at q, denoted by $\mathcal{M}_q$. Its dimension is equal to the instantaneous mobility n_q. All properties of the instantaneous kinematics of the mechanism are determined by the orientation of the subspace $\mathcal{M}_q$ in $\mathcal{V}$.

The maps $p_I : \mathcal{M}_q \to I$, and p_O: $\mathcal{M}_q \to O$, are defined as the restrictions on $\mathcal{M}_q$ of the projections which map $\mathcal{V}$ onto I and O. They map any motion vector into the vector of its input or output, respectively. The ranks of p_I and p_O, i.e., the dimensions of their image spaces, are r_I and r_O, respectively. The dimensions of their null-spaces will be denoted by d_I and d_O, respectively. Additionally, we introduce the notation d_{IO}, defined as:

$$d_{IO} = \dim(\mathrm{Ker}\, p_I \cap \mathrm{Ker}\, p_O).$$

Note that, the maps p_I and p_O (and their ranks) are dependent on the configuration q. For simplicity, this dependency will not be denoted explicitly.

As in Chapter 3, we make the natural assumption that the differential output in any configuration is an explicit linear function of the joint velocities:

$$T = A(q)\Omega. \tag{6.1}$$

The difference with Equation (3.1) is that $A(q)$ is of dimension $n_O \times N$. Equation (6.1) is the output equation of the mechanism.

The derivation of Equations (3.2) and (3.3) is unaffected by redundancy, since this derivation does not depend on the choice of inputs or outputs. Therefore, it remains true that for every configuration, q, there is an $N \times (N - n)$ matrix, $D(q)$, such that the feasible joint velocities, Ω, are given by the solution of the equation:

$$D(q)\Omega = 0. \tag{6.2}$$

Combining the $N - n$ equations of (6.2) with the n_O equations of (6.1) we obtain a system of $N - n + n_O$ linear equations which fully determines the instantaneous kinematics of the mechanism. The definition of the matrix $L(q)$ as:

$$L(q) = \begin{bmatrix} I_{n_O \times n_O} & A_q \\ O_{(N-n) \times n_O} & D_q \end{bmatrix}, \tag{6.3}$$

completes the proof of the following theorem:

6.1. Theorem. *For any given configuration, q, an $(N - n + n_O) \times (N + n_O)$ matrix, $L(q)$, can be found, such that a velocity vector, m, is a feasible motion vector of the mechanism, if and only if*

$$L(q)m = 0. \tag{6.4}$$

Equation (6.4) will be referred to herein as the **velocity equation** *for q.*

6.2. Example. Let us obtain the velocity equation of the simple mechanism shown in Figure 6.1. This is a five-bar linkage with three inputs and a single output. The input velocities are the joint velocities at points A, B and C. The output velocity is the angular velocity of the link ED. The general mobility of the mechanism is two.

There is only one loop and $c = 1$. The loop equation is:

$$\omega_A \mathbf{S}_A + \omega_B \mathbf{S}_B + \omega_C \mathbf{S}_C + \omega_D \mathbf{S}_D + \omega_E \mathbf{S}_E = \mathbf{0}, \tag{6.4}$$

where ω_P, $\mathbf{S}_P$ $(P = A, B, C, D)$ are the joint velocities and the joint screws, respectively. Since only the planar components of the joint screws are nonzero, they are 3-dimensional vectors). The output equation is:

$$\tau = -\omega_E. \tag{6.5}$$

Therefore, the velocity equation is:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & \mathbf{S}_A & \mathbf{S}_B & \mathbf{S}_C & \mathbf{S}_D & \mathbf{S}_E \end{bmatrix} \begin{bmatrix} \tau \\ \omega_A \\ \omega_B \\ \omega_C \\ \omega_D \\ \omega_E \end{bmatrix} = 0. \tag{6.6}$$

If point A is the origin and the x-axis is along AE, Equation (6.6) can be written as:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & x_B & x_C & x_D & x_E \\ 0 & 0 & y_B & y_C & y_D & 0 \end{bmatrix} \begin{bmatrix} \tau \\ \omega_A \\ \omega_B \\ \omega_C \\ \omega_D \\ \omega_E \end{bmatrix} = 0. \tag{6.7}$$

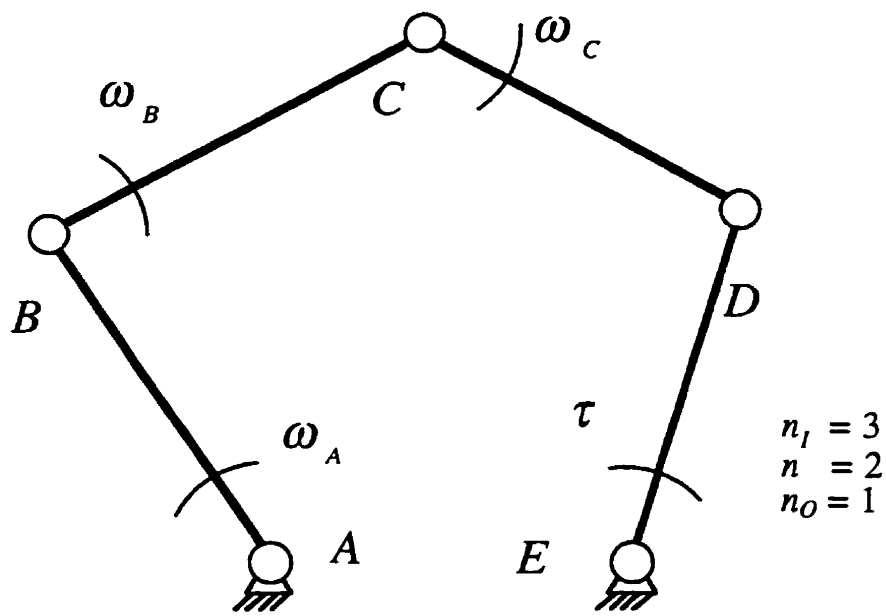

Figure 6.1. A 2-dof redundant mechanism.

6.3. ***Example.*** Another mechanism, this time with two independent loops is shown in Figure 6.2. In this case, we have: $n_I = n = 2$, $n_O = 1$. The input velocities are the joint velocities at joints A and E. The output is the motion of the slider G.

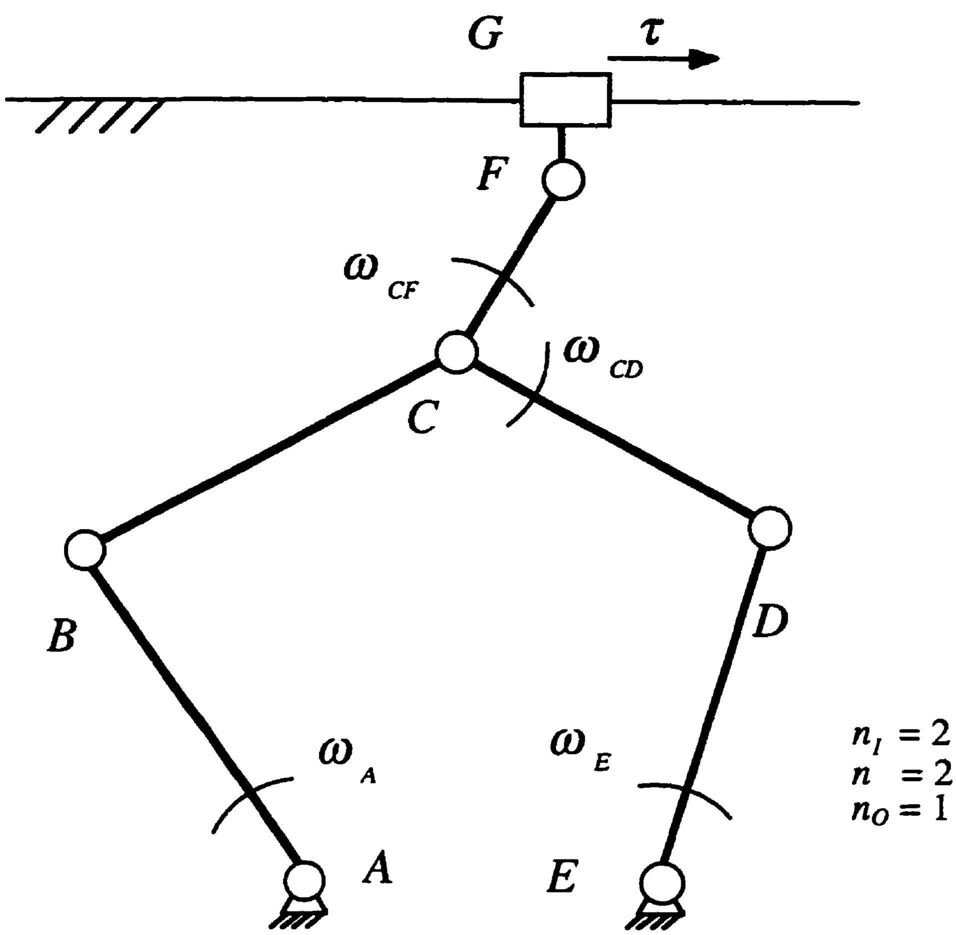

Figure 6.2. A 2-dof redundant mechanism.

The output equation is simple:

$$\tau = -v_G \,, \tag{6.8}$$

where v_G is the joint velocity of the base link with respect to the link *FG*. We use the loops *ABCDE* and *ABCFG*. The two loop twist equations, together with Equation (6.8), lead to the following velocity equation:

$$
\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & \mathbf{S}_A & \mathbf{S}_E & \mathbf{S}_B & \mathbf{S}_D & \mathbf{S}_C & 0 & 0 & 0 \\ 0 & \mathbf{S}_A & 0 & 0 & 0 & 0 & \mathbf{S}_C & \mathbf{S}_F & \mathbf{S}_G \end{bmatrix} \begin{bmatrix} \tau \\ \omega_A \\ \omega_E \\ \omega_B \\ \omega_D \\ \omega_{CD} \\ \omega_{CF} \\ \omega_F \\ v_G \end{bmatrix} = 0. \qquad (6.9)
$$

6.3. Singularity and Singularity Types for Redundant Mechanisms

In this Section we give new, more general definitions for the six singularity types, first defined in Chapter 3. Before that, we must examine how the instantaneous formulation of the definition of singularity, as given in Sections 3.3 for non-redundant mechanisms, changes when redundancy is possible.

6.3.1. Singularity

6.4. ***Proposition.*** *A configuration, $\boldsymbol{q}$, is nonsingular, if and only if*

$$r_I = n = n_q \text{ and } r_O = n_O .$$

Proof. The proposition follows directly from the general definition of singularity, Definition 2.31.

Indeed, $\boldsymbol{q}$ is a regular point of the configuration space manifold, D, if and only if $n = n_q$. On the other hand, a configuration is nonsingular, only if f_I and f_O are nonsingular at $\boldsymbol{q}$. Since r_I and r_O are, in fact, rank $D_q f_I$ and rank $D_q f_O$, a configuration is nonsingular, only if $r_I = \min(n_I, n_q)$ and $r_O = n_O$. □

6.5. Remark. It should be noted that the definition of singularity by means of the forward and inverse kinematics is not applicable in the case of redundant mechanisms. Indeed, if Definition 3.5 is applied for a mechanism with $n_I > n_O$, every nonsingular configuration (according to the general definition) will qualify as singular, since the inverse kinematics is not solvable in the sense of Definition 3.5(2). For a truly redundant mechanism, all configurations with solvable forward and inverse kinematics (i.e., configurations that satisfy the nonsingularity conditions of Definitions 3.5(1) and 3.5(2)) must be singular. Indeed, a solvable IIKP would imply $r_I < n_O < \min(n_I, n_q)$, and therefore a singular configuration.

The observations in Remark 6.5 show that the singularity of redundant mechanisms cannot be understood in terms of forward and inverse kinematics. However, it can be modelled by means of the motion-space formulation, as given in Section 3.6. Indeed, Proposition 6.4 is a generalization of Proposition 3.25 for non-redundant mechanisms since the two propositions are identical when $n_I = n_O$.

The motion-space formulation is used in the presentation of the six singularity types, which is given in the following sub-sections. The definitions are generalizations of the definitions in Section 3.4. For each of the definitions a generalization of one of the statements in Proposition 3.26 is provided.

6.3.2. Redundant Input

6.6. Definition. *A configuration is a singularity of* **redundant input** *(RI) type, if there exist at least* $n - n_O + 1$ *linearly independent input vectors,* $\Omega_1^a, \ldots, \Omega_{n-n_O+1}^a$, *such that each of them satisfies the velocity equation for a zero-output,* $\boldsymbol{T} = 0$ *(and some vector of passive-joint velocities,* Ω^p*), i.e., for every* i*, the following equation is satisfied for some* Ω^p*:*

$$L\begin{bmatrix}0\\ \Omega_i^a\\ \Omega^p\end{bmatrix}=0. \tag{6.10}$$

6.7. Proposition. *A necessary and sufficient condition for an RI-type singularity is the inequality:*

$$\dim(\mathrm{Ker}\,\boldsymbol{p}_O) - \dim(\mathrm{Ker}\,\boldsymbol{p}_I \cap \mathrm{Ker}\,\boldsymbol{p}_O) > n - n_O,$$

or equivalently,

$$d_O - d_{IO} > n - n_O. \tag{6.11}$$

Proof. The condition (6.11) is satisfied, if and only if there is a subspace of $\mathrm{Ker}\,\boldsymbol{p}_O$, $\mathcal{S}$, of dimension greater than $n - n_O$, which is complementary to $\mathrm{Ker}\,\boldsymbol{p}_I \cap \mathrm{Ker}\,\boldsymbol{p}_O$. In other words, $\mathcal{S}$ is such that $\mathcal{S} + (\mathrm{Ker}\,\boldsymbol{p}_I \cap \mathrm{Ker}\,\boldsymbol{p}_O) = \mathrm{Ker}\,\boldsymbol{p}_O$ and $\mathcal{S} \cap (\mathrm{Ker}\,\boldsymbol{p}_I \cap \mathrm{Ker}\,\boldsymbol{p}_O) = 0$, i.e., $\mathcal{S} \oplus (\mathrm{Ker}\,\boldsymbol{p}_I \cap \mathrm{Ker}\,\boldsymbol{p}_O) = \mathrm{Ker}\,\boldsymbol{p}_O$. This observation proves the Proposition, since any basis of $\mathcal{S}$ provides the "redundant input" vectors required by Definition 6.6, while when the existence of such vectors is given, their linear envelope provides the subspace $\mathcal{S}$ needed to establish Equation (6.11). □

6.8. Remark. A comparison of the above Definition 6.6 (and Proposition 6.7) with Definition 3.6 (and Proposition 3.27, (i)) shows that the definition of RI-type singularity has been modified in order to include mechanisms with redundancy. While in the non-redundant case an RI-type singularity is associated with the existence of at least one motion with zero output and nonzero input, in the redundant case, a whole space of such motions, with dimension larger than the degree of kinematic redundancy, $n - n_O$, is required.

The reason for this difference is that when $n > n_O$, the existence of only one motion with zero output is no longer a sufficient condition for the occurrence of singularity. Indeed, fixing the output to zero removes only n_O freedoms, which is not sufficient to immobilize a mechanism with mobility higher than n_O. Thus, even in a nonsingular configuration $n - n_O$ "redundant-input" motions are expected to exist. This can be illustrated with the configuration shown in Figure 6.1. A nonsingular configuration is shown, which satisfies the "non-redundant" Definition 3.6. When the output link is fixed, the remaining links form a four-bar linkage with mobility of one. It can be said that according to Definition 3.6, every configuration of a kinematically-redundant mechanism would be singular and belong to the RI type.

On the other hand, Definition 6.6 is correct in the sense that it describes a singular configuration, as this is shown in the next proposition.

6.9. Proposition. *(Correctness of Definition 6.7)*

All configurations belonging to the RI-type are singular.

Proof. Equation (6.11) implies

$$d_O > d_O - d_{IO} > n - n_O.$$

Since $d_O = \dim(\mathrm{Ker}\, \boldsymbol{p}_o) = \dim \mathcal{M}_q - \dim(\mathrm{Im}\, \boldsymbol{p}_o) = n_q - r_O$, we have:

$$n_q - r_O > n - n_O,$$

and therefore, either $n_q > n$ or $r_O < n_O$. According to Proposition 6.4 this implies that the configuration is singular. □

6.10 Example. As a simple example of an RI-type singularity for a redundant mechanism, let us consider the configuration in Figure 6.3. The mechanism in the figure has the same kinematic chain and choice of inputs and outputs as the one considered in Example 6.2 and shown in Figure 6.1.

In the configuration shown in Figure 6.3, the output velocity is zero and the remainder of the linkage forms a flattened four-bar linkage with mobility two. The maximum number of linearly-independent inputs is two, which is greater than $n - n_O$. Indeed, the two "redundant input" vectors can be chosen by fixing, respectively, the joint velocity at A and at B to be zero. Checking Equation (6.11), the mobility with fixed input *and* output, d_{IO}, is zero, while the mobility with fixed output is $d_O = 2$. The difference, $d_O - d_{IO} = 2$, is greater than the degree of kinematic redundancy, $n - n_O = 1$.

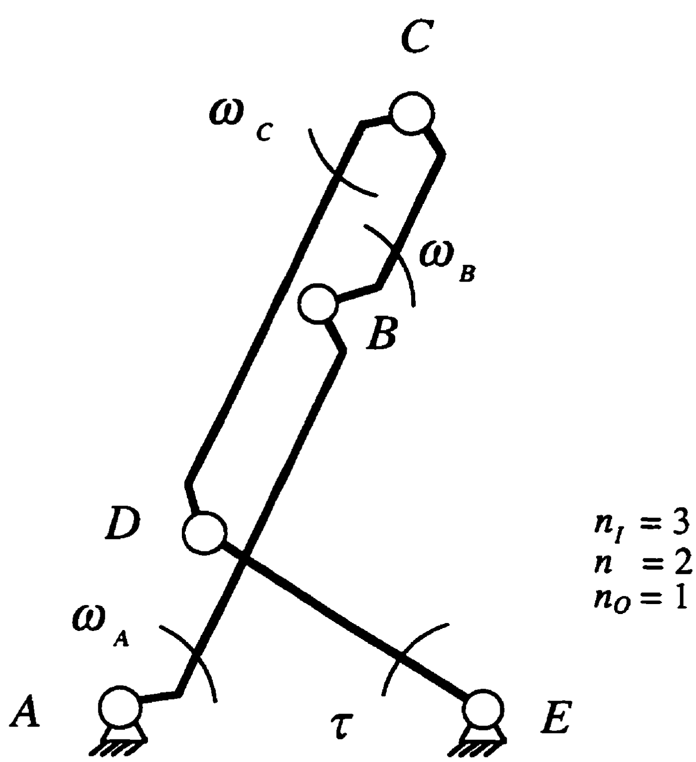

Figure 6.3. A singular configuration of class (RI, IO).

6.3.3. Redundant Output

6.11. Definition. *A configuration is a singularity of* **redundant output** *(RO) type, if there exist a nonzero output, $\boldsymbol{T} \neq 0$, and a vector of passive-joint velocities, Ω^p, which satisfy the velocity equation for a zero-input, $\Omega^a = 0$:*

$$\boldsymbol{L}\begin{bmatrix} \boldsymbol{T} \\ 0 \\ \Omega^p \end{bmatrix} = 0. \tag{6.12}$$

6.12. Proposition. *A necessary and sufficient condition for an RO-type singularity is the inequality:*

$$\dim(\text{Ker}\, \boldsymbol{p}_I) - \dim(\text{Ker}\, \boldsymbol{p}_I \cap \text{Ker}\, \boldsymbol{p}_O) > 0,$$

or equivalently,

$$d_I - d_{IO} > 0. \tag{6.13}$$

Proof. Equation (6.13) is equivalent to $\text{Ker}\, \boldsymbol{p}_I - \text{Ker}\, \boldsymbol{p}_O \neq \varnothing$. This condition, stating that there are motion vectors with zero input and nonzero output, is equivalent to the requirement of Definition 6.10. □

6.12. Remark. Comparing the above formulation of the RO-type with Definition 3.9 and Proposition 3.26, we note that for this singularity type the definition does not change when redundancy is introduced. As a result, for redundant mechanisms the RO-type definition does not mirror the RI-type definition as closely as in the non-redundant case. As we shall see later, this leads to a loss of the input-output symmetry in the redundant-mechanism singularity classification.

The reason for keeping Definition 3.9 is that, unlike Definition 3.6, it ensures that the configuration is singular even when the mechanism is redundant. (See Proposition 6.14 below). In fact, for redundant mechanisms the requirement for an RO-type configuration is even harder to satisfy. Indeed, for dynamically-redundant mechanisms ($n_I > n$), d_I will be smaller, since when the inputs are fixed to be zero, a greater number of dof may be lost. If the mechanism has nonzero mobility when the inputs are zero (i.e., $d_I > 0$), then a kinematically-redundant mechanism will be more likely to have a higher d_{IO}, since it has fewer outputs. For example, one can note that a five-bar mechanism with three inputs (Figures 6.1 and 6.3, Examples 6.2 and 6.10) can have no RI-type singularities. In any configuration, if the first three joints are locked no link can move.

6.14. Proposition. *(Correctness of Definition 6.11)*

All RO-type configurations are singular.

Proof. Since $d_{IO} \geq 0$, the inequality (6.13) implies that $d_I > 0$. From $d_I = n_q - r_I$ it follows:

$$n_q - r_I = (n_q - n) + (n - r_I) > 0.$$

This is possible only if $n_q > n$ or $\min(n_q, n_I) \geq n > r_I$. According to Proposition 6.4, this implies that the configuration is singular. □

6.15. Example. Figure 6.4 depicts an RO-type singular configuration. The shown kinematic chain was introduced in Example 6.3. It can be seen that even when the joint velocities at points A and E are zero, the output slider can still move. It is easy to see that, in this configuration, $d_I = 1$ and $d_{IO} = 0$. Therefore Equation (6.12) is satisfied.

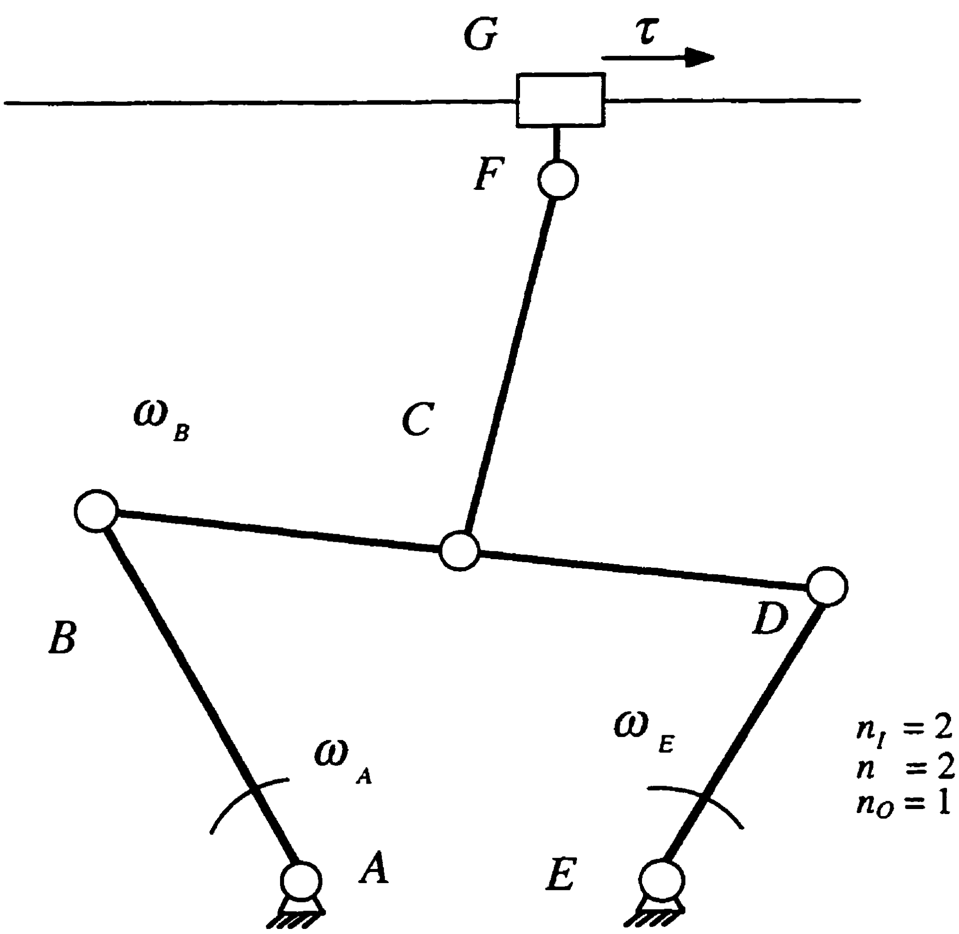

Figure 6.4. A singularity of class (RO, II).

6.3.4. Impossible Input

6.16. Definition. *A configuration is a singularity of* **impossible input** *(II) type, if there exists a subspace of I, S, of dimension higher than the degree of dynamic redundancy, $n_I - n$, such that for every vector $\Omega^a \neq 0$ in S the velocity equation cannot be satisfied for any choice of $\mathbf{T}$ and Ω^p.*

6.17. Proposition. *A necessary and sufficient condition for an II-type singularity is the inequality:*

$$r_I < n,$$

or equivalently,

$$n_q - n < d_I. \tag{6.14}$$

Proof. First, we note that $r_I < n$ is equivalent to (6.14) because of $r_I = n_q - d_I$.

The inequality $r_I < n$ holds if and only if there is a subspace *of I, S,* with dimension $n_I - n$ or more, such that $S \oplus \text{Im } p_I = I$. Since none of the nonzero elements of S is in $\text{Im } p_I$, S satisfies the requirements Definition 6.16. □

6.18. Remark. Similarly to the Definition of the RI type, the definition of II-type singularities is different in the redundant and non-redundant cases. A comparison of Definition 6.16 and Definition 3.12 or Proposition 3.26(iii) shows that in the redundant case the definition is more restrictive. It is no longer sufficient to establish the existence of a single "impossible input" vector, rather an "impossible input" subspace must be present. This means that II-type configurations are more "rare" for dynamically-redundant mechanisms. For example, the 5-bar mechanism with three inputs shown in Figures 6.1 and 6.3 cannot have an II-type singularity since d_I is obviously zero (and therefore, by Equation (6.14), an II-type singularity is not present).

As in the case of the RI-type definition (see Remark 6.8), the requirement of the II-type non-redundant-case Definition 3.12 does not guarantee that the configuration is singular, moreover, for dynamically-redundant mechanisms this condition is satisfied in all configurations. Indeed, there can be no more than n linearly independent feasible inputs, and when $n_I > n$, there must exist input vectors that are not feasible for the mechanism.

Definition 6.16 ensures that the configuration is singular by requiring that II-type configurations have "more" impossible inputs than a nonsingular configuration. When $n_I = n$, this is the same as Definition 3.12 and both definitions are equivalent to the inequality $r_I < n$ (see Propositions 3.27(i) and 6.17).

6.19. Proposition. *(Correctness of Definition 6.16)*

All II-type configurations are singular.

Proof. A configuration is an II-type singularity, if and only if $r_I < n$. Since we have $n \leq \min(n_q, n_I)$, it follows that f_I is not of maximum rank and is therefore singular at this configuration. □

6.20. Example. Consider again the configuration shown in Figure 6.4, which was used as an example for an RO-type singularity (Example 6.15). We will show that this is an II-type singularity as well. As was mentioned in Example 6.15, it is easy to see that $d_I = 1$. It can also be shown that $n_q = 2$. (Indeed, if $\omega_A = \omega_B = 0$, then point C has no velocity, and therefore all joint velocities must be zero. Therefore, $2 = n \leq n_q \leq 2$). Then, $n_q - n = 0$ and $n_q - n < d_I$ and according to Proposition 6.17 this implies an II-type singularity. This can be established directly by noting that when one of the input velocities is zero, the other one must be zero as well and therefore there is a one-dimensional impossible input space. (When $\omega_A = 0$, point B is fixed and then in the four-bar linkage $BCDE$ the joint at E is locked.)

This is an example without dynamic redundancy ($n_I = n = 2$). If we increase n_I to 3 by assuming that the joint at point E is active, then in the same configuration (Figure 6.4),

d_I decreases to zero (i.e., when the three inputs are fixed to zero, all joints are locked) and there can be no II-type singularity.

6.21. Example. Let us consider an example of a mechanism with dynamic redundancy in an II-type singularity. Figure 6.5 shows a mechanism with the same kinematic chain as the mechanisms in Figures 6.2 and 6.4.

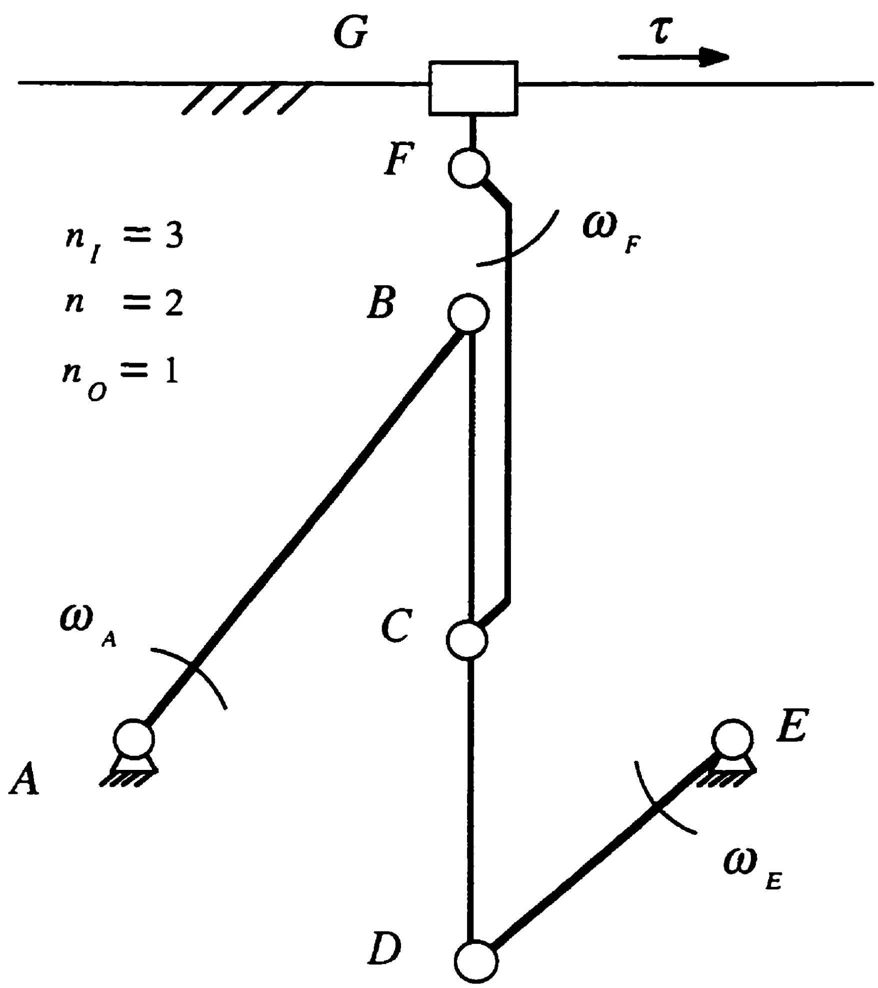

Figure 6.5. A singular configuration of class (RO, II).

However, one additional joint is active, namely the joint at point E. The configuration is such that the points B, C, D, F, G are aligned (and this line is perpendicular to the prismatic-joint axis). In this configuration, $d_I = 1$ and $n = n_q = 2$. According to Equation (6.14), the configuration is an II-singularity. It can also be shown that $d_O = 1$, and according to Proposition 6.12 an RO-type singularity is also present.

6.3.5. Impossible Output

6.22. Definition. *A configuration is a singularity of* **impossible output** *(IO) type, if there exists a vector* ***T*** *for which the velocity equation cannot be satisfied for any combination of* Ω^a *and* Ω^p.

6.23. Proposition. *A necessary and sufficient condition for an IO-type singularity is the inequality:*

$$r_O < n_O,$$

or equivalently,

$$n_q - n_O < d_O. \tag{6.15}$$

Proof. First, $r_O < n_O$ is equivalent to (6.15) because of $r_O = n_q - d_O$.

The inequality $r_O < n_O$ holds, if and only if there is at least one output vector, which corresponds to no feasible instantaneous motion, i.e., it is equivalent to Definition 6.22. □

6.24. Remark. Comparing the above formulation of the IO-type with Definition 3.15 and Proposition 3.26, we note that for this singularity type the definition does not change when redundancy is introduced. As a result, for redundant mechanisms the IO-type definition does not mirror the II-type definition as closely as in the non-redundant case. This leads to a loss of the input-output symmetry in the redundant-mechanism singularity classification. The reason for keeping Definition 3.15 is that, unlike Definition 3.12, it ensures that the configuration is singular even when the mechanism is redundant.

6.25. Proposition. *(Correctness of Definition 6.23)*

All IO-type configurations are singular.

Proof. The inequality $r_O < n_O$ is a necessary and sufficient condition for the singularity of the output map f_O and therefore implies that the mechanism is in a singular configuration. □

6.26. Example. The configuration shown in Figure 6.3, and discussed in Example 6.10, is an IO-type singularity. It can be seen that the joint at point E is locked. (Indeed point D cannot have a velocity component parallel to the line along A, B and C.) Also, since $d_O = 2$, $n_q = 2$ and $n_O = 1$ (see Example 6.10), Equation 6.23 is satisfied.

6.3.6. Increased Instantaneous Mobility

6.27. Definition. *A configuration is a singularity of* **increased instantaneous mobility** *(IIM) type, if* rank $L < N - n + n_O$.

6.28. Proposition. *An IIM-type is present, if and only if $n < n_q$.*

Proof. Since the output equations are linearly independent, the sum rank $L + n_O$ does not depend on the choice of input parameters. Therefore, just like in the non-redundant case, L is singular if and only if $n < n_q$. □

6.29. Remark. Clearly, an IIM-type singularity occurs if and only if the configuration is a singular point of the configuration space of the mechanism, D. Therefore, it does not depend on the choice of the active joint or the output link. IIM is a property of the kinematic chain and is therefore not influenced by redundancy. Thus, the configurations of non-redundant mechanisms, that have been shown to be IIM-type singularities in previous chapters, can be used as examples for the redundant case. It suffices to assume that some of the passive joints are active or redefine the output. This cannot be done for the other singularity types, since they are affected by the way the input and output are chosen.

6.30. Example. Consider, for example, the configuration shown in Figure 6.6.

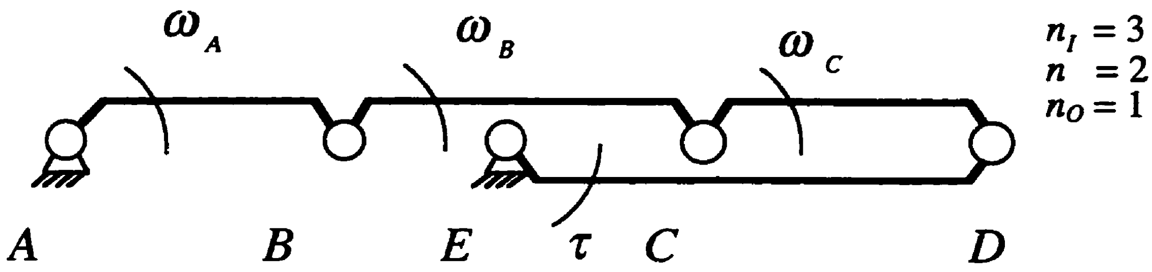

Figure 6.6. An (RI, IIM)-class singular configuration.

The mechanism is similar to the one shown in Figures 6.1 and 6.3, i.e., a five-bar linkage with three input joints and a single output. The flattened configuration is essentially the same as the one used in Example 4.25 to illustrate IIM-type singularity for (non-redundant) HCMs, although in that case there were two inputs and a two-dimensional output. Since in both cases we have $2 = n < n_q = 3$, an IIM-type singularity is present for the redundant mechanism as well. For the mechanism in Figure 6.5, we also have: $n_I = 3$, $n = 2$, $n_O = 1$ and $d_{IO} = 0$, $d_I = 0$, $d_O = 2$. Applying the singularity-type definitions in this section (and Propositions 6.7, 6.12, 6.17 and 6.23) we conclude that the configuration shown also belongs to the RI type but does not belong to types RO, IO, or II (it does not belong to the RPM-type either, as can be seen by applying Definition 6.31 below). This indicates that the combinations of singularity types for redundant mechanisms obey rules different from the ones for non-redundant mechanisms, revealed in Section 3.7. For instance the present example proves that Proposition 3.29(ix) does not hold in the redundant case.

6.3.7. Redundant Passive Mobility

6.31. Definition. *A configuration is a singularity of* **redundant passive motion** *(RPM) type, if there exists a nonzero passive-joint-velocity vector,* $\Omega^P \neq 0$, *which satisfies the velocity equation for a zero input and a zero output, i.e.,*

$$L \begin{bmatrix} 0 \\ 0 \\ \Omega^P \end{bmatrix} = 0. \tag{6.16}$$

6.32. Proposition. *An RPM-type singularity is present, if and only if*

$$d_{IO} > 0. \tag{6.17}$$

Proof. The inequality (6.17) holds, if and only if the intersection $\mathrm{Ker}\, \boldsymbol{p}_I \cap \mathrm{Ker}\, \boldsymbol{p}_O$ has a dimension of at least one. Therefore, there is a nonzero motion vector, which is mapped into zero by both $\boldsymbol{p}_I$ and $\boldsymbol{p}_O$, i.e., a nonzero instantaneous motion with zero input and zero output. □

6.33. Remark. The above definition is identical with Definition 3.21. As we will prove in the next Proposition 6.34, the definition requirement ensures that the configuration is singular, so there is no need to modify the definition for the redundant case. In general, the chances for the existence of an RPM-type singularity improve when the combined total of the inputs and outputs is decreased, and vice versa. Therefore, dynamic and kinematic redundancy have a different effect on RPM-type singularities. A smaller number of outputs facilitates the occurrence of RPM-type singularities, while an increase in the number of active joints makes it more difficult for RPM-type configurations to occur.

6.34. Proposition. *(Correctness of Definition 6.31)*

All RPM-type configurations are singular.

Proof. Since $d_I \geq d_{IO}$, the Inequality (6.17) implies $d_I > 0$. Now the proof can proceed as in the proof of Proposition 6.14. From $d_I = n_q - r_I$ it follows:

$$n_q - r_I = (n_q - n) + (n - r_I) > 0.$$

This is possible only if $n_q > n$ or $\min(n_q, n_I) \geq n > r_I$. According to Proposition 6.4, this implies that the configuration is singular. □

6.35. Example. As an illustration of the RPM singularity type, we use another configuration of the kinematic chain described in Examples 6.3, 6.15 and 6.21. The configuration considered here, shown in Figure 6.7, is very similar to the one in Figure 6.5, but this time the point G is not aligned with B, C, D, F. Similarly to Examples 6.3 and 6.15, it is assumed herein that the mechanism has only two active joints, namely A and E.

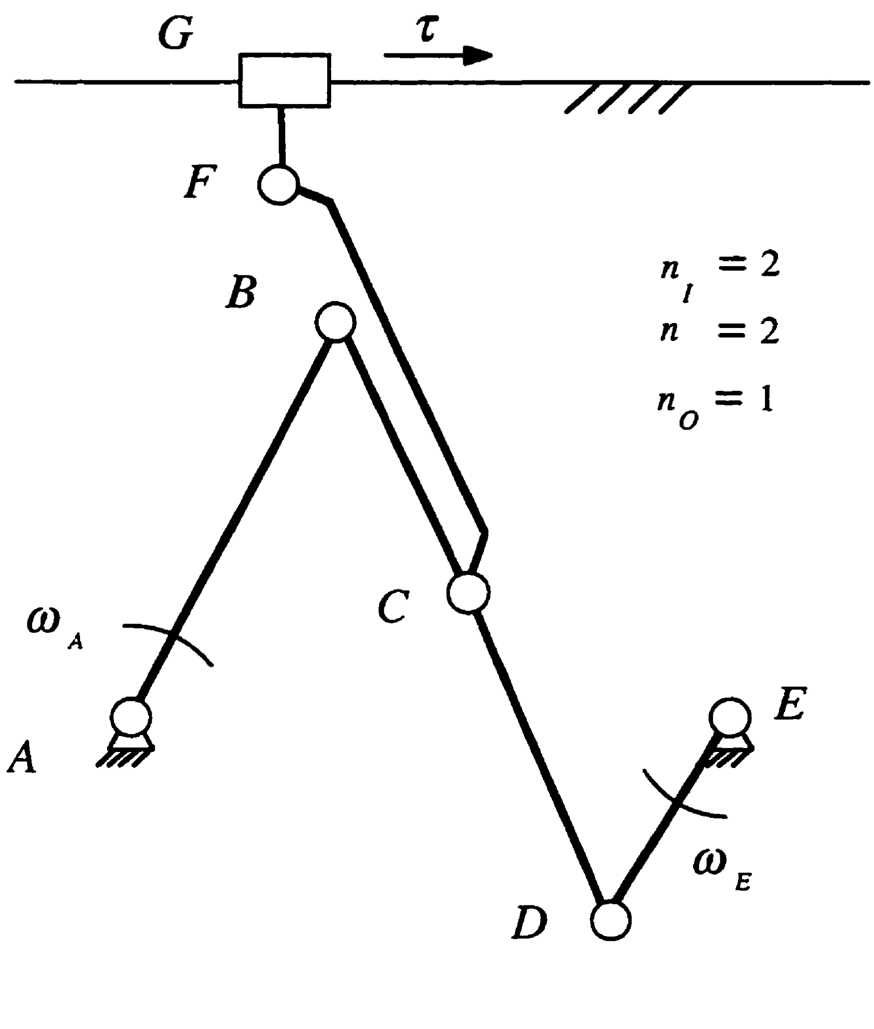

Figure 6.7. A singular configuration of type (RPM, II).

It is verified that $d_{IO} = d_I = d_O = 1$ and $n_q = n = 2$. Then, it is easy to check that the only singularity types that are present are of the RPM and II types. We note that the singularity class (RPM, II) is not among the ones occurring in non-redundant mechanisms (see Table 3.1). The passive motion, which can take place with the input and output equal to zero, occurs with an instantaneous motion of point C along a line normal to the line BD. It can be noted that, if the joint F were active as well, the configuration would no longer be of the RPM type.

Finally, to summarize the present section, we give Table 6.1.

Type	Condition
RI	$d_O - d_{IO} > n - n_O$
RO	$d_I - d_{IO} > 0$
II	$d_I > n_q - n$
IO	$d_O > n_q - n_O$
IIM	$n_q - n > 0$
RPM	$d_{IO} > 0$

Table 6.1. Definitions of the singularity types for mechanisms with redundancy.

6.4. Classification of Singularities

6.4.1. Combinations of singularity types

As a first step to a comprehensive general classification of the singularities of arbitrary mechanisms, we study the interdependence of the six singularity types. A singular configuration never belongs to a single singularity type but rather to a *combination of singularity types*. The following proposition provides the rules which will allow us to distinguish the possible combinations from the impossible ones.

6.36. Proposition

(i) $q \in \{\mathrm{RI}\} \Rightarrow q \in \{\mathrm{IO}\}$ or $q \in \{\mathrm{IIM}\}$,

(ii) $q \in \{\mathrm{RO}\} \Rightarrow q \in \{\mathrm{II}\}$ or $q \in \{\mathrm{IIM}\}$,

(iii) $q \in \{\mathrm{II}\} \Rightarrow q \in \{\mathrm{RO}\}$ *or* $q \in \{\mathrm{RPM}\}$,

(iv) $q \in \{\mathrm{IO}\} \Rightarrow q \in \{\mathrm{RI}\}$ *or* $q \in \{\mathrm{RPM}\}$,

(v) $q \in \{\mathrm{RPM}\} \Rightarrow q \in \{\mathrm{II}\}$ *or* $q \in \{\mathrm{IIM}\}$,

(vi) $q \in \{\mathrm{IIM}\} \Rightarrow q \in \{\mathrm{RI}\}$ *or* $q \in \{\mathrm{RPM}\}$,

(vii) $q \in \{\mathrm{RO}\} \Rightarrow q \in \{\mathrm{II}\}$ *or* $q \in \{\mathrm{RI}\}$,

(viii) $q \in \{\mathrm{IO}\} \Rightarrow q \in \{\mathrm{II}\}$ *or* $q \in \{\mathrm{RI}\}$,

(ix) $q \in \{\mathrm{II}\}$ *and* $q \in \{\mathrm{RI}\} \Rightarrow q \in \{\mathrm{IO}\}$ *or* $q \in \{\mathrm{RO}\}$.

Proof

(i) An RI-type singularity is given. Assume that there is no IIM-type singularity. Thus, we have $d_O - d_{IO} > n - n_O$ and $n = n_q$. Therefore,

$$d_O \geq d_O - d_{IO} > n - n_O = n_q - n_O.$$

Then, the inequality $d_O > n_q - n_O$ implies an IO-type singularity (Proposition 6.23).

(ii) If q is an RO-type singularity, then $d_I - d_{IO} > 0$. We assume that q is not an IIM-type singularity, i.e., $n = n_q$. Therefore,

$$d_I \geq d_I - d_{IO} > 0 = n_q - n.$$

However, $d_I > n_q - n$, implies an II-type singularity.

(iii) $q \in \{\text{II}\}$ implies $d_I > n_q - n$. Assume the configuration is not an RPM-type singularity, or , equivalently, that $d_{IO} = 0$. Therefore , we can write:

$$d_I - d_{IO} = d_I > n_q - n \geq 0,$$

which, according to Proposition 6.12, implies that q is an RO-type singularity.

(iv) The condition for an IO-type singularity is $d_O > n_q - n_O$ (Proposition 6.23). We assume that the singularity is not of the RPM type, i.e., $d_{IO} = 0$. Thus, we have

$$d_O - d_{IO} = d_O > n_q - n_O \geq n - n_O,$$

i.e., $d_O - d_{IO} > n - n_O$, which is the condition defining an RI-type singularity.

(v) An RPM-type singularity is given, thus, $d_{IO} > 0$. Assume q is not an IIM-type singularity, i.e., $n_q = n$. Since d_I is always at least as large as d_{IO}, it follows that:

$$d_I \geq d_{IO} > 0 = n_q - n.$$

This proves that $d_I > n_q - n$, i.e., the configuration is an II-type singularity.

(vi) An IIM-type singularity is equivalent to $n_q > n$. We assume $d_{IO} = 0$, (i.e., that the configuration is not an RPM-type singularity). Then,

$$d_O - d_{IO} = d_O = n_q - r_O \geq n_q - n_O .> n - n_O$$

Above, we have used $n_O \geq r_O$ (the rank of a map cannot exceed the dimension of the target space). Thus, the inequality $d_O - d_{IO} > n - n_O$ is obtained and this ensures the presence of an RI-type singularity at q.

(vii) It is given that $d_I > d_{IO}$ (RO-type singularity). Let us assume that the configuration does not belong to the II type, hence $d_I \leq n_q - n$. Then, we can write:

$$d_O - d_{IO} > d_O - d_I \geq d_O - (n_q - n) = n - (n_q - d_O) = n - r_O \geq n - n_O.$$

Above, the first two inequalities follow from $d_I > d_{IO}$ and $d_I \leq n_q - n.$, respectively. The last inequality uses $r_O \leq n_O$. As a result, it is established that $d_O - d_{IO} > n - n_O$, i.e., the configuration belongs to the RI type.

(viii) q is an IO-type singularity. This requires $d_O > n_q - n_O$. If the configuration is not an II-type singularity as well, then $d_I \leq n_q - n$. We have:

$$d_O - d_{IO} > (n_q - n_O) - d_{IO} \geq (n_q - n_O) - d_I \geq$$
$$(n_q - n_O) - (n_q - n) = n - n_O.$$

The second inequality uses $d_I \geq d_{IO}$. Once again, we obtain $d_O - d_{IO} > n - n_O$, hence the configuration belongs to the RI type.

(ix) It is given that q belongs to both the II and RI singularity types. This implies two inequalities, $d_I > n_q - n$ and $d_O - d_{IO} > n - n_O$, respectively. If we assume that q is not of the IO type as well, we must also have $d_O = n_q - n_O$. Using these conditions the following sequence of inequalities can be can be written:

$$d_I - d_{IO} > (n_q - n) - d_{IO} = (n_q - n) + n_O - n_O - d_{IO} =$$
$$(n_q - n_O) - d_{IO} - (n - n_O) = d_O - d_{IO} - (n - n_O) > 0.$$

This proves $d_I > d_{IO}$ and therefore the occurrence of an RO-type singularity. □

6.37. Remark. The above proposition is analogous to Proposition 3.29 for non-redundant mechanisms. Comparing the two, one can note that only part of the statements of Proposition 3.29 could be proven for redundant mechanisms. Six of the ten points in Proposition 3.29 are present in Proposition 6.36 . The remaining four statements, namely (v), (vi), (vii) and (ix), have been weakened and transformed into Statements (v), (vi) and (ix) of Proposition 6.36. In fact, we already presented proof that these four parts of Proposition 3.29 are not correct for redundant mechanisms. In Example 6.30, a configuration which belongs only to the types IIM and RI was shown (Figure 6.6). This example disproves Proposition 3.29 (vi) and (ix). Another configuration, introduced with Example 6.35 (Figure 6.7), belongs to the RPM and II types and to no other type. Therefore, Proposition 3.29 (v) and (x) do not hold for redundant mechanisms either.

6.4.2. General singularity classification

The goal of this section is to classify the set of all singularities of all mechanisms. As in Chapter 3, this set is divided into classes, using as a criterion the combination of singularity types to which a configuration belongs. More precisely, two configurations are considered "equivalent", i.e., they belong to the same singularity class, when they belong to exactly the same singularity types. This is a relation of equivalence which divides the set of all singularities into non-intersecting classes. In this Sub-section we identify the combinations for which there exist configurations and which therefore correspond to a non-empty singularity class. Thus, by listing all non-empty singularity classes, we develop a comprehensive singularity classification .

6.38. Proposition. *Let* q *be a singular configuration. Then,* q *belongs to at least one of the types IO, II, and IIM.*

Proof. According to the definition of mechanism singularity in Chapter 2, a configuration, q, is singular in (at least) one of three cases: q is a singular point of C; q is a singular point of f_O; and, q is a singular point of f_I.

The first case, when the configuration space is singular, is equivalent to the presence of a singularity of the IIM type.

When f_O is singular, we have $r_O = \text{rank} f_O < n_O$, which is equivalent to IO.

Finally, when q is a singular point of f_I, we have $r_I = \text{rank} f_I < \min(n_I, n_q)$. If we assume that the configuration is not an IIM-type singularity, i.e, $n = n_q$, then, it follows that $r_I < \min(n_I, n_q) \le n_q = n$. However, $r_I < n$ implies an II-type singularity (Proposition 6.17). ☐

6.39. Proposition. *Let* q *be a singular configuration. Then,* q *belongs to at least one of the types RO, RI, and RPM.*

Proof. From Proposition 6.38, it follows that each singularity belongs to at least one of the I-types. From Proposition 6.36 (iii), (iv) and (vi), it is evident that a singularity of any

I-type (i.e., II, IO or IIM) must belong to at least one of the R-types (i.e., RI, RO or RPM) as well. □

6.40. Theorem. *Let* S *be an arbitrary combination of some of the six singularity types. There exists a mechanism with a configuration, q, which belongs to all types* in S *and to no other types, if and only if* S *is marked with "Y" in Table 6.2.*

	IO	II	IO *and* II	IIM	IO *and* IIM	II *and* IIM	IO *and* II *and* IIM
RI	Y			Y	Y		
RO		Y					
RI *and* RO			Y	Y	Y	Y	Y
RPM		Y	Y	Y		Y	Y
RI *and* RPM			Y	Y	Y		Y
RO *and* RPM		Y	Y			Y	Y
RI *and* RO *and* RPM			Y	Y	Y	Y	Y

Table 6.2. Possible combinations of singularity types for redundant mechanisms.

Proof. To prove the theorem, we need to establish that (i) all combinations not marked with "Y" in the table can never occur and (ii) there exist mechanisms and configurations with the marked singularity-class combinations.

(i) There are six singularity types and therefore there are $2^6 = 64$ combinations (one of them is the nonsingular combination). From Propositions 6.38 and 6.39 we conclude that it is sufficient to consider the ones that include at least one I-type and one R-type. These combinations are represented by the 49 cells of Table 6.2. The cell in the *i*-th row and *j*-th column of the table corresponds to a combination of all singularity types listed to the left of the *i*-th row and on the top of the *j*-th column.

We must show that the combinations corresponding to blank cells of the table are impossible. This is proven with the help of Proposition 6.36. Each of the 22 empty cells represents a combination of singularity types which, if it occurred in some configuration, would violate (at least) one of the statements of Proposition 6.36. Table 6.3 illustrates which statement each blank cell violates.

	IO	II	IO *and* II	IIM	IO *and* IIM	II *and* IIM	IO *and* II *and* IIM
RI	**Y**	*(iii)*	*(iii)*	**Y**	**Y**	*(iii)*	*(iii)*
RO	*(ii)*	**Y**	*(iv)*	*(vi)*	*(iv)*	*(vi)*	*(iv)*
RI *and* RO	*(ii)*	*(i)*	**Y**	**Y**	**Y**	**Y**	**Y**
RPM	*(v)*	**Y**	**Y**	**Y**	*(viii)*	**Y**	**Y**
RI *and* RPM	*(v)*	*(i)*	**Y**	**Y**	**Y**	*(ix)*	**Y**
RO *and* RPM	*(ii)*	**Y**	**Y**	*(vii)*	*(viii)*	**Y**	**Y**
RI *and* RO *and* RPM	*(ii)*	*(i)*	**Y**	**Y**	**Y**	**Y**	**Y**

Table 6.3. Impossible combinations of singularity types for redundant mechanisms.

(ii) We need to give an example for each of the 27 remaining combinations. However, we already know that 21 of them are possible since it was proven that they occur for non-redundant mechanisms. Two additional combinations were established in Examples 6.30 (RI, IIM) and 6.35 (II, RPM). The remaining four combinations, namely (RI, IO, IIM), (RO, RPM, II), (RPM, II, IIM) and (RI, RPM, IIM) are illustrated by the four examples which follow.

6.41. Example. The configuration shown in Figure 6.8 is a singularity belonging to the types IO, IIM and RI but to no other type. The mechanism is similar to the one used in Examples 6.3, 6.15, 6.21, and 6.35, however in the present case the joint F is prismatic and assumed to be active, i.e., an input joint. It can be established by inspection that $d_{IO} = 0$, $d_I = 0$, $d_O = 3$ and $n_q = 3$. Using the inequalities in Table 6.1, it is easy to establish that the combination of singularity types for this configuration is (RI, IO, IIM).

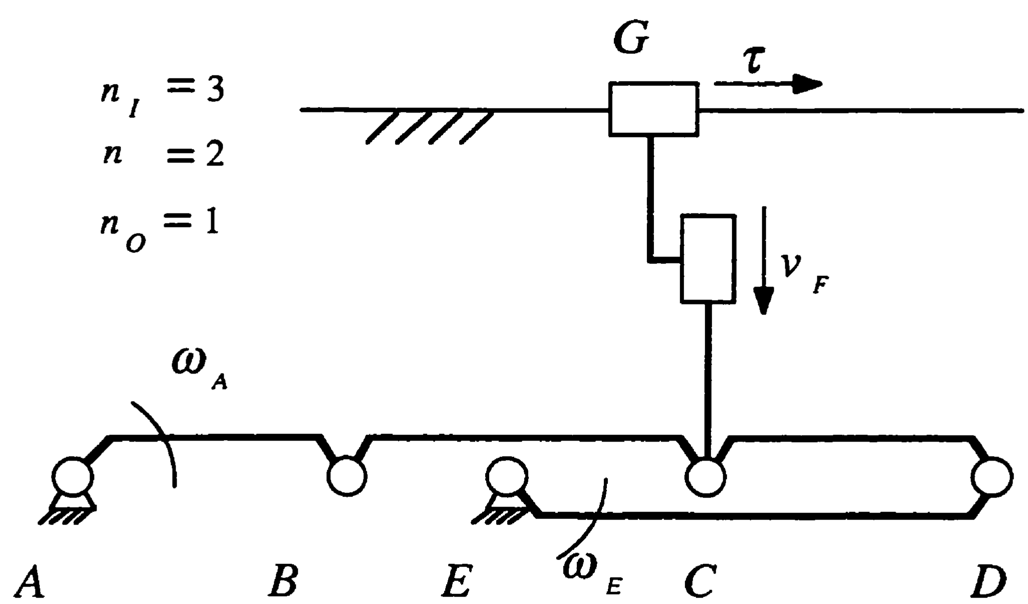

Figure 6.8. An (RI, IO, IIM)-class configuration

6.42. Example. The configuration shown in Figure 6.9 is the same as in Figure 6.5 (discussed in Example 6.21). However, in the present example, the joint F is passive, thus $n_I = n = 2$. It is found that $d_{IO} = 1$, $d_I = 2$, $d_O = 1$ and $n_q = 2$. According to the defining conditions given in Table 6.1, the configuration is of the singularity-type combination (RPM, RO, II).

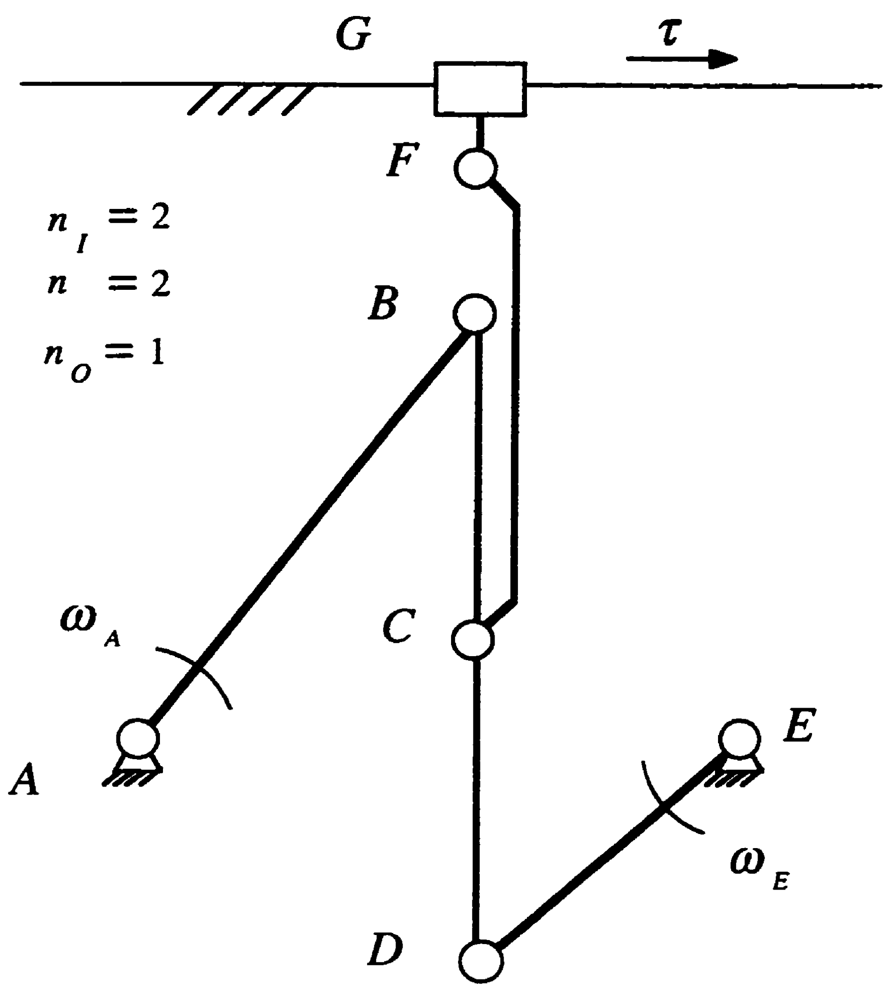

Figure 6.9. An (RO, RPM, II)-class singular configuration.

6.43. Example. In Figure 6.10 we present yet another variation of the five-bar-and-slider mechanism. Here, all rotary joints are aligned and two of them, *A* and *C*, coincide. The joint *F* is active (as well as joints *B* and *D*), therefore $n_I = 3$. By inspection, it is established that $d_{IO} = 1$, $d_I = 1$, $d_O = 3$ and $n_q = 3$. This implies that the configuration belongs only to types RI, RPM and IIM.

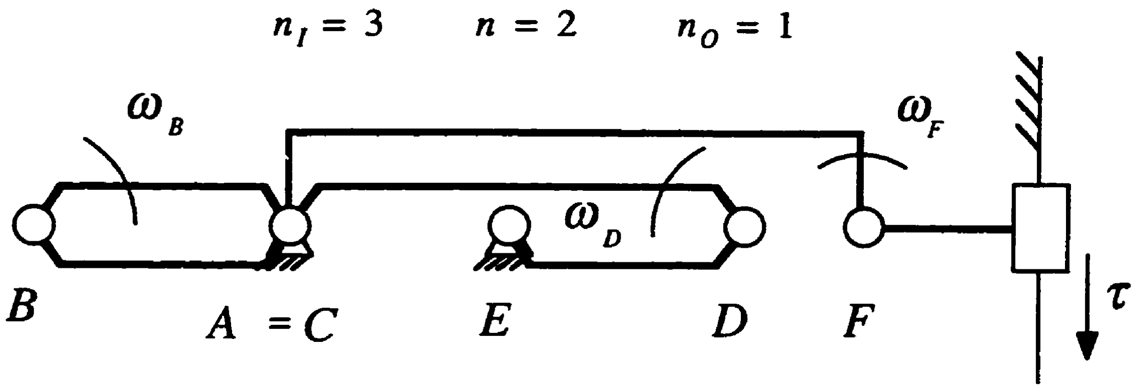

Figure 6.10. An (RI, RPM, IIM)-class singular configuration.

6.44. Example. Finally, let us consider the mechanism shown in Figure 6.11.

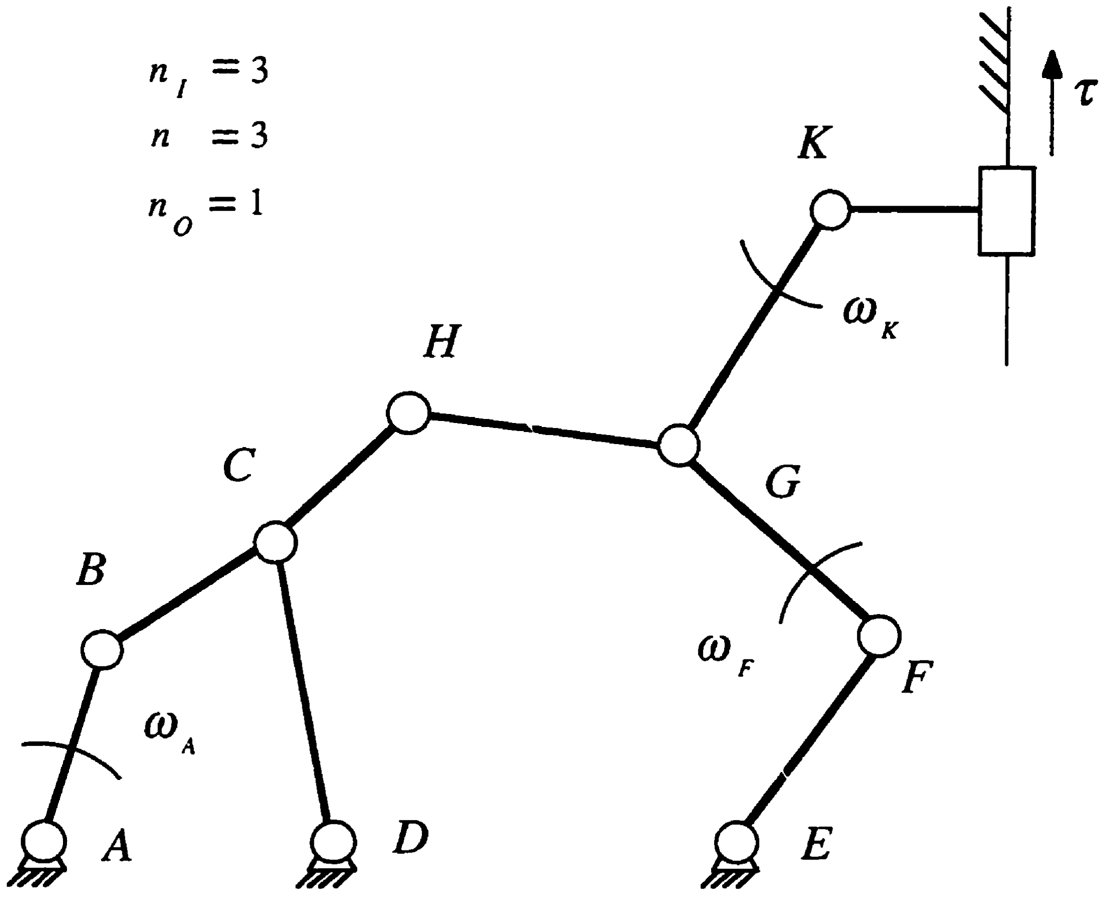

Figure 6.11. A 3-dof redundant planar mechanism.

The output is the velocity of the slider, i.e., $n_O = 1$. The joints at points *A*, *F* and *K* are active and thus $n_I = n = 3$. In the configuration shown in Figure 6.12, it is not difficult to see that $d_{IO} = 3$, $d_I = 3$, $d_O = 4$ and $n_q = 5$. This translates into a singularity of the (RPM, II, IIM)-class.

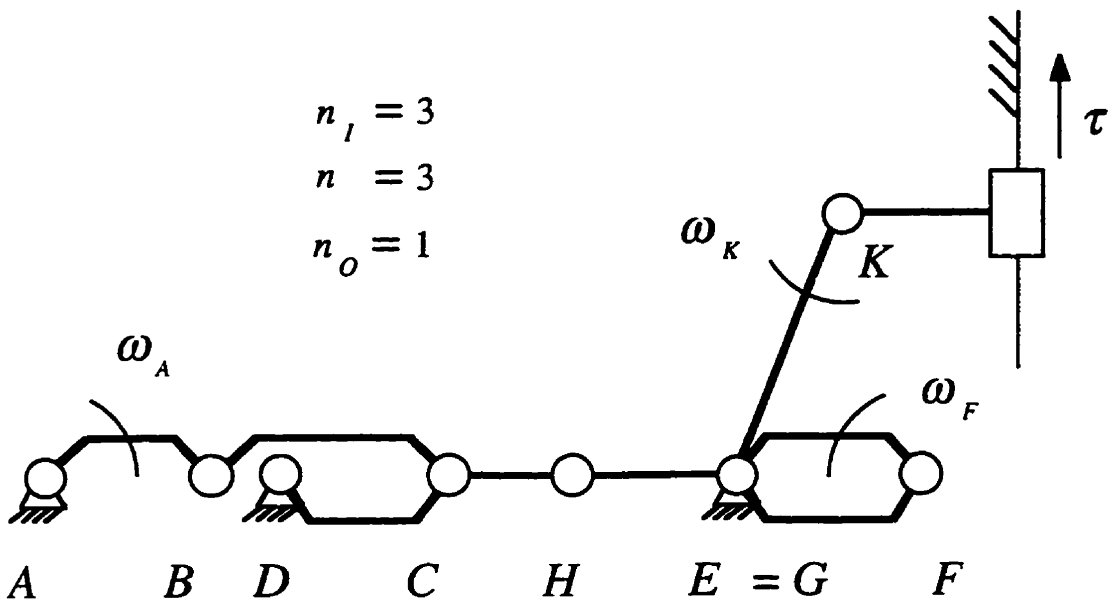

Figure 6.12. An (RPM, II, IIM)-class singular configuration.

With this the proof of Theorem 6.40. is completed. □

6.4.3. Singularity classifications for partially non-redundant mechanisms

In this Sub-section, we modify the general classification established with Theorem 6.40 and obtain classifications for two important special cases of redundant mechanisms. Herein, we are concerned with partially non-redundant mechanisms, i.e., mechanisms for which at least one of the two non-redundancy equalities, $n = n_O$ and $n = n_I$, holds. A mechanism is referred to as *kinematically non-redundant* when $n = n_O$ (and $n \leq n_I$). Alternatively, a mechanism is *dynamically non-redundant* when $n = n_I$ (and $n \geq n_O$).

Most redundant mechanisms appearing either in applications or in the literature belong to one of these groups, therefore the two classifications presented below are of interest.

When one of the non-redundancy conditions holds, some of the statements of Proposition 3.29 which are generally not true for redundant mechanisms can be proven. This rules out some singularity combinations and as a result two classifications with 24 non-empty classes each are obtained.

6.45. Proposition. *Let the mechanism be kinematically non-redundant, i.e.,* $n = n_O$.

(i) $q \in \{\text{RPM}\} \Rightarrow (q \in \{\text{II}\} \text{ and } q \in \{\text{IO}\}) \text{ or } q \in \{\text{IIM}\}$,

(ii) $q \in \{\text{II}\} \Rightarrow q \in \{\text{IO}\} \text{ or } q \in \{\text{RO}\}$.

Proof

(i) Let q be an RPM-type singularity but not an IIM-type singularity. Proposition 6.36 (v) implies that the configuration belongs to the II type. It remains to establish that q is an IO-type singularity as well. $q \in \{\text{RPM}\}$ implies $d_{IO} > 0$. Since there is no IIM-type singularity, we must have$n = n_q$. Then, we can write:

$$d_O \geq d_{IO} > 0 = n_q - n = n_q - n_O,$$

where the last equality uses the kinematic non-redundancy. Thus, it is established that $d_O > n_q - n_O$, which is equivalent to the presence of an IO-type singularity.

(ii) $q \in \{\text{II}\}$ implies $d_I > n_q - n$. We assume that there is no IO-type singularity, therefore, $d_O = n_q - n_O$. We need to show that $d_I - d_{IO} > 0$.

To establish this we will use the inequality sequence in the proof of Proposition 6.36 (ix). We notice that this sequence can be used to prove $d_I - d_{IO} > 0$, even when in the last inequality of the sequence the sign ">" is replaced with "≥", i.e., when the sequence is modified as follows:

$$d_I - d_{IO} > (n_q - n) - d_{IO} = (n_q - n) + n_O - n_O - d_{IO} =$$
$$(n_q - n_O) - d_{IO} - (n - n_O) = d_O - d_{IO} - (n - n_O) \geq 0.$$

Therefore, to be able to use the above sequence we need only $d_O - d_{IO} \geq n - n_O$ (since the other equalities and inequalities in the sequence were already established in the proof of Proposition 6.36 (v)). However, this last inequality is implied by the kinematic non-redundancy, since

$$d_O - d_{IO} \geq 0 = n - n_O. \qquad \square$$

6.46. Proposition. *Let the mechanism be dynamically non-redundant, i.e.,* $n = n_r$.

(i) $q \in \{\mathrm{IIM}\} \Rightarrow (\, q \in \{\mathrm{RI}\}$ and $q \in \{\mathrm{RO}\}\,)$ or $q \in \{\mathrm{RPM}\}$,

(ii) $q \in \{\mathrm{II}\} \Rightarrow q \in \{\mathrm{IO}\}$ or $q \in \{\mathrm{RO}\}$.

Proof

(i) Let q be an IIM-type but not an RPM-type singularity. From Proposition 6.36 (vi) it follows that the configuration belongs to the RI-type. It remains to establish that q is an RO-type singularity as well. The IIM-type singularity is characterized by $n < n_q$. When there is no RPM-type singularity, we have $d_{IO} = 0$. Then, it follows that:

$$d_I - d_{IO} = d_I = n_q - r_I > n - r_I = n_I - r_I \geq 0.$$

This yields $d_I > d_{IO}$, which is equivalent to the presence of an RO-type singularity.

(ii) Assuming that there is no IO-type singularity, we have $d_O = n_q - n_O$. and (from the given RI-type singularity) $d_O - d_{IO} > n - n_O$.

As in Proposition 6.45 (ii), to prove the statement we will use a variation of the inequality sequence in the proof of Proposition 6.36 (ix). We notice that this sequence can be used to prove $d_I - d_{IO} > 0$ even if in the first inequality of the sequence the sign ">" is replaced with "≥", namely:

$$d_I - d_{IO} \geq (n_q - n) - d_{IO} = (n_q - n) + n_O - n_O - d_{IO} =$$
$$(n_q - n_O) - d_{IO} - (n - n_O) = d_O - d_{IO} - (n - n_O) > 0.$$

Therefore, to be able to use the above sequence we need only to establish the additional (first) inequality, $d_I \geq n_q - n$. This, however, is implied by the dynamic non-redundancy, since

$$d_I = n_q - r_I \geq n_q - n_I = n_q - n. \qquad \square$$

6.47. Theorem. *Let* S *be an arbitrary combination of some of the six singularity types. There exists a kinematically non-redundant mechanism with a configuration,* q*, which belongs to all types in* S *and to no other types, if and only if* S *is marked with "Y" in Table 6.4.*

	IO	II	IO *and* II	IIM	IO *and* IIM	II *and* IIM	IO *and* II *and* IIM
RI	Y			Y	Y		
RO		Y					
RI *and* RO			Y	Y	Y	Y	Y
RPM			Y	Y			Y
RI *and* RPM			Y	Y	Y		Y
RO *and* RPM			Y			Y	Y
RI *and* RO *and* RPM			Y	Y	Y	Y	Y

Table 6.4. Possible combinations of singularity types for kinematically non-redundant mechanisms.

Proof. Similarly to the proof of Theorem 6.40, the present proof has of two parts.

(i) To prove that the blank cells correspond to impossible configurations, we can use Proposition 6.40. The 22 configurations marked with blank cells in Table 6.2 are impossible for any mechanisms, including kinematically non-redundant mechanisms. There are three additional blank cells in Table 6.4, namely (RPM, II), (RPM, II, IIM) and (RO, RPM, II). These singularity-type configurations are disproved by Proposition 6.45.

(ii) We can use Theorem 3.30, to establish that 21 of the combinations are possible. (Since non-redundant mechanisms are a special case of kinematically non-redundant mechanisms). The remaining three are (RI, IIM), (RI, IO, IIM) and (RI, RPM, IIM). In the proof of Theorem 6.40, the singularity classes were illustrated with kinematically redundant mechanisms. Below we present examples of mechanisms with $n = n_O$.

6.48. Example. To prove the existence of (RI, IIM) singularities we consider a four-bar linkage, Figure 6.13. The output is defined as usual, while dynamic redundancy is introduced by assuming that the joint at point B is active (in addition to joint A). Thus, we have $n_I = 2$, $n = n_O = 1$. In the flattened configuration shown in the figure, the parameters determining the singularity types are $d_{IO} = 0$, $d_I = 0$, $d_O = 1$, $n_q = 2$. according to Table 6.1, the configuration belongs only to the types RI and IIM.

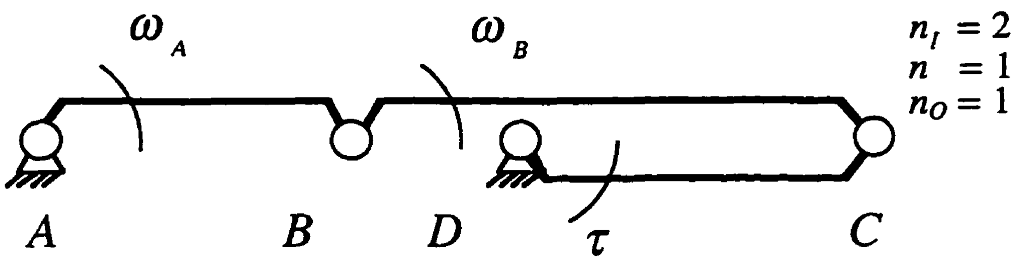

Figure 6.13. An (RI, IIM)-class singular configuration.

6.49. Example. Let us consider the five-bar linkage in its flattened configuration shown in Figure 6.14. The output is the position of point C, while there are three input joints at A, B and E. Therefore, $n_I = 3$, $n = n_O = 2$. It is checked that $d_{IO} = 0$, $d_I = 0$, $d_O = 2$, $n_q = 3$. This implies a singularity of the (RI, IO, IIM) class.

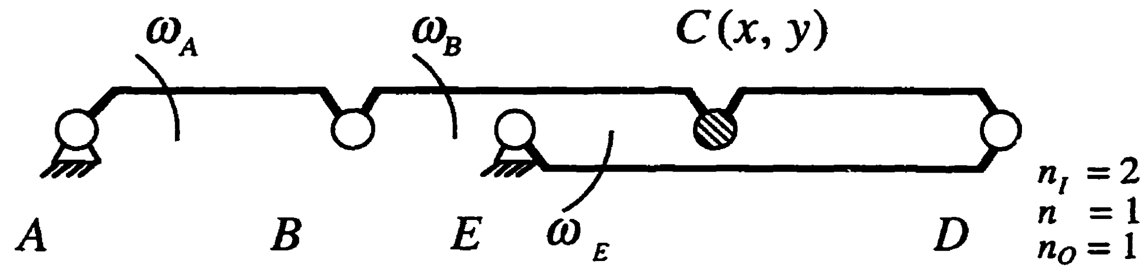

Figure 6.14. An (RI, IO, IIM)-class singularity.

6.50. Example. We consider the configuration shown in Figure 6.15. The figure resembles Figure 6.10, however the present mechanism is composed using a four-bar linkage rather than a five-bar. As a result, the mobility is 1 rather than 2. The output is the motion of the slider, the input joints are B and E. Thus, $n_I = 2$, $n = n_O = 1$. We establish that $d_{IO} = 1$, $d_I = 1$, $d_O = 2$, $n_q = 3$, which implies a singularity-type combination (RI, RPM, IIM).

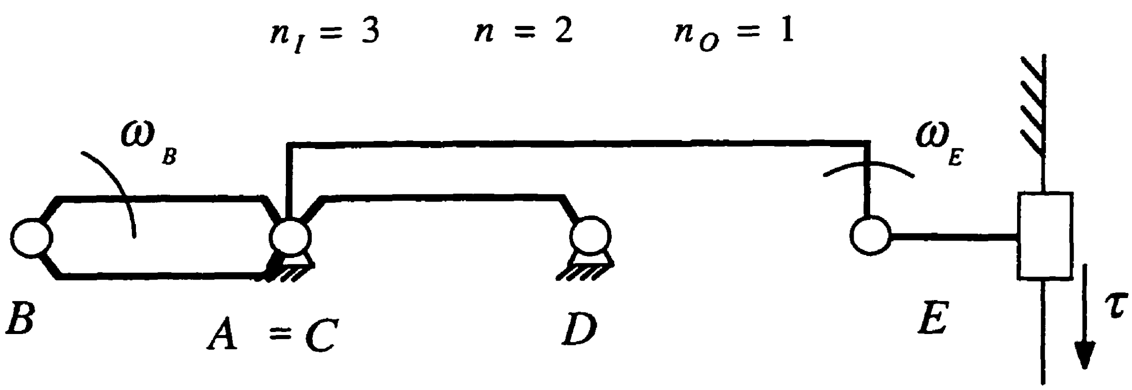

Figure 6.15. A singular configuration of class (RI, RPM, IIM).

This completes the proof of Theorem 6.47. □

6.51. Theorem. *Let* S *be an arbitrary combination of some of the six singularity types. There exists a dynamically non-redundant mechanism with a configuration,* q*, which* *belongs to all types in* S *and to no other types, if and only if* S *is marked with "Y" in Table 6.5.*

	IO	II	IO *and* II	IIM	IO *and* IIM	II *and* IIM	IO *and* II *and* IIM
RI	**Y**						
RO		**Y**					
RI *and* RO			**Y**	**Y**	**Y**	**Y**	**Y**
RPM		**Y**	**Y**	**Y**		**Y**	**Y**
RI *and* RPM			**Y**		**Y**		**Y**
RO *and* RPM		**Y**	**Y**			**Y**	**Y**
RI *and* RO *and* RPM			**Y**	**Y**	**Y**	**Y**	**Y**

Table 6.5. Possible combinations of singularity types for dynamically non-redundant mechanisms.

Proof. Similarly to the previous Theorem 6.47, we need to prove that three combinations are impossible and establish that three other combinations are possible.

(i) The three classes, which are impossible for dynamically non-redundant mechanisms, but are possible for a general (redundant) mechanism are (RI, IIM), (RI, IO, IIM) and (RI, RPM, IIM). Indeed, if such singularities were to exist for some mechanisms this would contradict Proposition 6.46.

(ii) The three classes, which occur for dynamically non-redundant mechanisms, but are impossible for non-redundant mechanisms are (RPM, II), (RO, RPM, II) and (RPM, II, IIM). The existence of singularities from these classes is confirmed by Examples 6.35, 6.42 and 6.44, respectively. □

6.5. Singularity Identification

In this Section, we address the problem of singularity identification, already introduced in Chapter 5. Herein, the techniques developed in Chapter 5 for the singularity identification of non-redundant mechanisms are generalized and made applicable to redundant mechanisms as well.

6.5.1. Conditions for singularity

The velocity equation, $\boldsymbol{Lm} = \boldsymbol{0}$, can be written in the form:

$$\boldsymbol{L}_T\boldsymbol{T} + \boldsymbol{L}_a\boldsymbol{\Omega}^a + \boldsymbol{L}_p\boldsymbol{\Omega}^p = \boldsymbol{0}.$$

As in the non-redundant case, we define two sub-matrices of $\boldsymbol{L}$, namely $\boldsymbol{L}_I = [\boldsymbol{L}_T\boldsymbol{L}_p]$ and $\boldsymbol{L}_O = [\boldsymbol{L}_a\boldsymbol{L}_p]$. When the mechanism is redundant, these matrices are rectangular. The dimensions of $\boldsymbol{L}_I$ and $\boldsymbol{L}_O$ are: $(N - n + n_O) \times (N - n_I + n_O)$ and $(N - n + n_O) \times N$, respectively.

6.52. Lemma.

(i) $n_q = N + n_O - \text{rank}\, \boldsymbol{L}$,

(ii) $d_O = N - \text{rank}\, \boldsymbol{L}_O$,

(iii) $d_I = N - n_I + n_O - \text{rank}\, \boldsymbol{L}_I$,

(iv) $d_{IO} = N - n_I - \text{rank}\, \boldsymbol{L}_p$.

Proof

(i) By definition, n_q is the dimension of the space of feasible instantaneous motions. This space is isomorphic to the the space of solutions of the velocity equation, i.e., the space of all $\boldsymbol{m}$, such that $\boldsymbol{Lm} = \boldsymbol{0}$. Therefore, $n_q = \dim(\mathrm{Ker}\, \boldsymbol{L})$. Since $\boldsymbol{L}$ has $(N + n_O)$ columns, $\dim(\mathrm{Ker}\, \boldsymbol{L}) + \mathrm{rank}\, \boldsymbol{L} = N + n_O$, which proves (i).

(ii) By definition, $d_O = \dim(\mathrm{Ker}\, \boldsymbol{p}_O)$, is the dimension of the space of feasible instantaneous motions with zero output. This space coincides with the space of vectors $\boldsymbol{m} = (\boldsymbol{0}, \boldsymbol{\Omega}^a, \boldsymbol{\Omega}^p)$, which are solutions of the velocity equation. Therefore, d_O is, in fact, the dimension of the space of solutions of the equation $[\boldsymbol{L}_a \boldsymbol{L}_p]\boldsymbol{x} = \boldsymbol{0}$, i.e., $d_O = \dim(\mathrm{Ker}\, \boldsymbol{L}_O)$. Now, (ii) follows from the following equality:

$$\mathrm{rank}\, \boldsymbol{L}_O + \dim(\mathrm{Ker}\, \boldsymbol{L}_O) = N.$$

(iii) We know that $d_I = \dim(\mathrm{Ker}\, \boldsymbol{p}_I)$. $\mathrm{Ker}\, \boldsymbol{p}_I$ is the space of motions with zero input. It is isomorphic to the space vectors $\boldsymbol{m} = (\boldsymbol{T}, \boldsymbol{0}, \boldsymbol{\Omega}^p)$ such that $\boldsymbol{Lm} = \boldsymbol{0}$, or equivalently to the solution space of $\boldsymbol{L}_I \boldsymbol{x} = \boldsymbol{0}$. Therefore, $d_I = \dim(\mathrm{Ker}\, \boldsymbol{L}_I)$ and since $\boldsymbol{L}_I$ has $(N - n_I + n_O)$ columns, we have (iii).

(iv) By definition, $d_{IO} = \dim(\mathrm{Ker}\, \boldsymbol{p}_I \cap \mathrm{Ker}\, \boldsymbol{p}_O)$, i.e., d_{IO} measures the dimension of the space of motions with zero input and zero output. These motions are, in fact, given by the solutions of the equation $\boldsymbol{L}_p \boldsymbol{\Omega}^p = 0$, and therefore

$$d_{IO} = \dim(\mathrm{Ker}\, \boldsymbol{L}_p) = (N - n_I) - \mathrm{rank}\, \boldsymbol{L}_p.$$

□

6.53. Proposition.

(i) $\boldsymbol{q} \in \{\mathrm{RI}\} \Leftrightarrow \mathrm{rank}\, \boldsymbol{L}_O < \mathrm{rank}\, \boldsymbol{L}_p + n_I - (n - n_O)$,

(ii) $\boldsymbol{q} \in \{\mathrm{RO}\} \Leftrightarrow \mathrm{rank}\, \boldsymbol{L}_I < \mathrm{rank}\, \boldsymbol{L}_p + n_O$,

(iii) $\boldsymbol{q} \in \{\mathrm{RPM}\} \Leftrightarrow \mathrm{rank}\, \boldsymbol{L}_p < N - n_I$,

(iv) $\boldsymbol{q} \in \{\mathrm{II}\} \Leftrightarrow \mathrm{rank}\, \boldsymbol{L}_I < \mathrm{rank}\, \boldsymbol{L} - (n_I - n)$,

(v) $\boldsymbol{q} \in \{\mathrm{IO}\} \Leftrightarrow \mathrm{rank}\, \boldsymbol{L}_O < \mathrm{rank}\, \boldsymbol{L}$,

(vi) $\boldsymbol{q} \in \{\mathrm{IIM}\} \Leftrightarrow \mathrm{rank}\, \boldsymbol{L} < N - n + n_O$,

(vii) $\boldsymbol{q} \in \{\mathrm{IO}\}$ or $\boldsymbol{q} \in \{\mathrm{IIM}\} \Leftrightarrow \boldsymbol{L}_O$ *is singular*,

(viii) $\quad q \in \{\mathrm{RO}\}$ or $q \in \{\mathrm{RPM}\} \Leftrightarrow L_I$ *is singular.*

Proof.

(i) An RI-type singularity is present if and only if $d_O - d_{IO} > n - n_O$ (Table 6.1). Applying Lemma 6.52, we have

$$d_O - d_{IO} = (N - \mathrm{rank}\, L_O) - (N - n_I - \mathrm{rank}\, L_p) = \mathrm{rank}\, L_p + n_I - \mathrm{rank}\, L_O.$$

Therefore the defining inequality for the RI type is equivalent to:

$$\mathrm{rank}\, L_p + n_I - \mathrm{rank}\, L_O > n - n_O,$$

which is equivalent to the inequality in (i).

(ii) From the Lemma 6.52, we obtain:

$$d_I - d_{IO} = (N - n_I + n_O - \mathrm{rank}\, L_I) - (N - n_I - \mathrm{rank}\, L_p) = \mathrm{rank}\, L_p - \mathrm{rank}\, L_I + n_O.$$

A necessary and sufficient condition for the RO-type singularity is the inequality: $d_I > d_{IO}$ (Proposition 6.12). From Lemma 6.52, this is equivalent to

$$N - n_I + n_O - \mathrm{rank}\, L_I > N - n_I - \mathrm{rank}\, L_p,$$

i.e.,

$$\mathrm{rank}\, L_p + n_O > \mathrm{rank}\, L_I.$$

(iii) An RPM-type singularity occurs when $d_{IO} > 0$. According to Lemma 6.52 (iv), this is true exactly when

$$N - n_I - \mathrm{rank}\, L_p > 0,$$

or, equivalently, when $\mathrm{rank}\, L_p < N - n_I$.

(iv) The II singularity type is defined with the inequality $d_I > n_q - n$. Lemma 6.52 implies that this inequality is equivalent to:

$$N - n_I + n_O - \mathrm{rank}\, L_I > N + n_O - \mathrm{rank}\, L.$$

The above is obviously equivalent to the condition in (iv).

(v) The necessary and sufficient condition for an IO-type singularity is $d_O > n_q - n_O$. In this, we substitute the expressions for d_O and n_q from Lemma 6.52:

$$N - \mathrm{rank}\, L_O > N + n_O - \mathrm{rank}\, L - n_O,$$

i.e.,

$$\text{rank } L_O < \text{rank } L.$$

(vi) Equivalent to Definition 6.27.

(vii) Follows from (v) and (vi). Indeed it is always true that

$$\text{rank } L_O \leq \text{rank } L \leq N - n + n_O.$$

When the matrix L_O is nonsingular, we have equalities and neither an IO- nor an I-type singularity are possible. When L_O is singular, at least one of the above inequalities must be "<" and therefore either an IO- or IIM-type singularity is present.

(viii) Follows from (iii) and (iv). The maximum rank of L_I is $N - n_I + n_O$. For any configuration, it is true that:

$$\text{rank } L_I \leq n_O + \text{rank } L_P \leq N - n_I + n_O.$$

L_I is singular, if and only if "<" can be replaced in at least one of the above inequalities. From (iii) and (iv) it is evident that L_I is singular, if and only if either an RO- or an RPM-type singularity is present.

□

6.54. Proposition. *For all mechanisms (including redundant ones), a configuration,* $\boldsymbol{q}$*, is nonsingular if and only if both the matrices* L_I *and* L_O *are nonsingular at* $\boldsymbol{q}$*.*

Proof. When either L_I or L_O is singular, it is clear from Proposition 6.53 (vii) and (viii) that the configuration is singular. It remains to show that when the matrices are both nonsingular the configuration must be nonsingular. We assume that $\boldsymbol{q}$ is singular. Then, $\boldsymbol{q}$ belongs to at least one of the types: IO, II, IIM. An IO- or IIM-type singularity implies that L_O is singular (Proposition 6.53 (vii). Therefore, $\boldsymbol{q}$ must be an II-type singularity. Then, by Proposition 6.53 (iv) and since $\text{rank } L \leq N - n + n_O$, we have:

$$\text{rank } L_I < \text{rank } L - (n_I - n) \leq N - n_I + n_O,$$

i.e., L_I has less than maximum rank. □

6.55. Remark. A comparison of Propositions 6.53-4 with Theorem 5.1 and Proposition 5.2 for non-redundant mechanisms shows that the matrices L_I and L_O continue to play a key role in the identification of singularities and their types. Theorem 5.1 remains true in the redundant case as proven by Theorem 6.54. The necessary and sufficient conditions for the different singularity types, as established by Proposition 6.53 (i)-(vi), are modified versions of the conditions (i)-(vi) in Proposition 5.2. As was the case with the definitions in Section 6.3, the singularity conditions for redundant mechanisms lack the input-output symmetry of the corresponding results for non-redundant mechanisms. In addition, we note that statements (vii) and (viii) in Proposition 6.52 are weaker than the corresponding points in Proposition 5.2. As we shall see in the next Sub-section these variations of the singularity conditions require some changes in the identification methods as well.

6.56. Proposition. *Let the mechanism be kinematically non-redundant, i.e., let* $n = n_O$. *Then,*

$$q \in \{\mathrm{RI}\} \text{ or } q \in \{\mathrm{RPM}\} \Leftrightarrow L_O \text{ is singular}.$$

Proof. When $n = n_O$, the matrix L_O is square of dimensions $N \times N$, therefore, it is singular when rank $L_O < N$. For any configuration the following inequalities hold:

$$\operatorname{rank} L_O \le n_I + \operatorname{rank} L_P \le N.$$

Therefore, L_O is singular, if and only if either

$$\operatorname{rank} L_O < n_I + \operatorname{rank} L_P \qquad or \qquad \operatorname{rank} L_P < N - n_I.$$

This, according to Proposition 6.53 (i) and (iii) is equivalent to RI or RPM. □

6.57. Proposition. *Let the mechanism be dynamically non-redundant, i.e., let* $n = n_I$. *Then,*

$$q \in \{\mathrm{II}\} \text{ or } q \in \{\mathrm{IIM}\} \Leftrightarrow L_I \text{ is singular}.$$

Proof. When $n = n_I$ the matrix $\boldsymbol{L}_I$ is square of dimensions $(N - n + n_O)$, therefore it is singular when rank $\boldsymbol{L}_O < N - n + n_O$. For any configuration, the following inequalities hold:

$$\text{rank } \boldsymbol{L}_I \leq \text{rank } \boldsymbol{L} \leq N - n + n_O.$$

Therefore, $\boldsymbol{L}_I$ is singular if and only if either

$$\text{rank } \boldsymbol{L}_I < \text{rank } \boldsymbol{L} \quad \textit{or} \quad \text{rank } \boldsymbol{L} < N - n + n_O.$$

The first of the above inequalities is equivalent to the presence of an II-type singularity according to Proposition 6.53 (iv) (note that $n_I - n = 0$). The second inequality the condition for IIM. □

6.5.2. Identification and classification methods

The singularity conditions derived in Sub-section 6.5.1 can be used to identify and classify all the singularities of a specific mechanism. Herein, we discuss the methods for achieving this goal. The algorithms are similar to the ones proposed in Chapter 5 for the singularity analysis of non-redundant mechanisms, however some modifications are necessary due to the variations in the singularity conditions.

When the goal is only to find the singularities, without necessarily determining to which class each singularity belongs, an algorithm analogous to the one presented in Sub-section 5.3.1 can be used. The only modification will be necessary by the fact that the matrices $\boldsymbol{L}_I$ and $\boldsymbol{L}_O$ are rectangular for redundant mechanisms. Therefore, in Steps 3 and 4 the determinants of $\boldsymbol{L}_I$ and $\boldsymbol{L}_O$ cannot be used. Instead, each of these steps will include the solution of a system of equations, namely that all minors of maximum dimension are equal to zero.

For the determination of the singularity classes of a given redundant mechanism a modified version of the algorithm presented in Sub-section 5.4.3 is used:

(1) Find all feasible $\boldsymbol{q}$ satisfying condition (vi).

(2) Find all feasible q satisfying condition (iii).

(3) Classify {1} ∪ {2}:

(3.1) For {1}, check (iv) and (v). Obtain 4 sets:

IIM; IIM&II; IIM&IO; IIM&II&IO.

(3.2) For {2}, check (i) and (ii). Obtain 4 sets:

RPM; RPM&RI; IIM&RO; RPM&RI&RO.

(3.3) Find all the intersections of each set in {3.1} and each set in {3.2}.

Obtain **12 classes**. (These are the 12 classes that belong to the IIM *and* RPM types, see Table 6.2)

(3.4) Subtract {2} from each set in {3.1}.

Obtain **2 classes**, namely

(RI, RO, II, IIM), (RI, RO, II, IO, IIM),

and 2 sets, namely

IIM and IIM&IO, both with no RPM.

(3.4.1) Check (ii) for the 2 sets.

Obtain **4 classes**, namely

(IIM, RI), (IIM, RI, RO),

(IO, IIM, RI) and (IO, IIM, RI, RO).

(3.5) Subtract {1} from each set in {3.2}.

Obtain **2 classes**, namely

(RI, RPM, II, IO) and (RI, RO, RPM, II, IO),

and 2 sets, namely

RPM and RPM&RO, both with no IIM.

(3.5.1) Check (v) for the 2 sets.

Obtain **4 classes**, namely

(RPM, II), (RPM, II, IO),

(RO, RPM, II) and (RO, RPM, II, IO).

(4) Find all $\boldsymbol{q}$ satisfying condition (vii). From these subtract $\{1\} \cup \{2\}$.

(5) Find all $\boldsymbol{q}$ satisfying condition (viii). From these subtract $\{1\} \cup \{2\}$.

(6) Intersect $\{4\}$ and $\{5\}$. Obtain **3 classes**, namely (RI, IO), (RO, II) and (RI, RO, IO, II).

6.6. Summary

In this chapter, a general framework for the singularity analysis of redundant mechanisms was developed. This was achieved by the generalization of the ideas introduced in Chapters 3 and 5 for non-redundant mechanisms. The six singularity types, were re-introduced with new, generalized definitions which remain relevant even when the mechanism is redundant. Using the motion-space model of instantaneous kinematics, the interdependence of the singularity types was examined. A comprehensive classification of the singular configuration of arbitrary mechanism was obtained. It was shown that, there are 27 different singularity classes, which can occur for various redundant mechanisms.

Furthermore, the problem of singularity identification and classification of specific mechanisms was addressed. New necessary and sufficient conditions for the occurrence of each of the singularity types were derived. The algorithms for the singularity analysis of non-redundant mechanisms, introduced in Chapter 5, were modified in a way that allows their application to redundant mechanisms.

CHAPTER 7

CONCLUSIONS

7.1. Summary and Contributions of the Thesis

This thesis presents a new, general, approach to the study of mechanism singularity. Unlike many previous works, this investigation is not limited to a narrow class of mechanisms. On the contrary, the central objective has been to address the problems of mechanism singularity in a most general setting, namely, to consider arbitrary singular configurations of both non-redundant and redundant mechanisms with arbitrary kinematic chains. Hence, the theoretical results of the thesis provide general insight into the kinematics of mechanical systems, while the proposed methods for singularity analysis and identification are applicable to all mechanisms, including ones with multiple closed loops and a high number of degrees of freedom. In fact, the dissertation places a special emphasis on the study of mechanical devices with complex kinematic chains, thus contributing to those increasingly important areas of robotics research and application (such as platform manipulators, walking machines, grasping), where non-serial, high-dof architectures play a central role.

The main contributions of this work can be briefly summarized as follows. The thesis contains a re-formulation of mechanism kinematics in the geometric and topological language of a novel *mathematical model*. Mechanical singularity has been examined in the terms of this model and thus a general yet rigorous mathematical *definition of singular* *configurations* for arbitrary mechanisms has been proposed. When the mathematical model is applied to the relationship between the joint and output velocities, a new, unifying *framework for the interpretation and classification of mechanism singularities* is obtained. This framework, based on the newly introduced six singularity types, is applicable for arbitrary non-redundant as well as redundant mechanisms. Furthermore, in the terms of this framework, mathematical tools, such as *singularity criteria and identification methods* have been developed for the study of the singularity set of both non-redundant and redundant systems. The analysis and classification of the singularities of hybrid-chain manipulators has been examined in detail, which has resulted in new *mathematical tools for the kinematic analysis of HCMs.*

Our mathematical model of mechanism position kinematics was introduced in Chapter 2. There, general, abstract kinematic systems were defined as families of smooth curves on manifolds. Kinematic chains were introduced as kinematic systems with specific configuration spaces, which can be described in terms of a connectivity graph and a joint-type distribution function, while articulated systems were defined as kinematic chains with a given link-geometry map. This allowed the definition of a mechanism as an articulated system where two subsystems, namely, the input and output systems were identified. The maps between the configuration space of the mechanism and the configuration spaces of these two systems were defined as the input and output maps of the mechanism. The local geometrical properties of the configuration space as well as the input and output maps were then used to define singularity: At a nonsingular configuration, the mechanism configuration space must locally be a smooth manifold, while the two maps must be smooth and regular.

The local nature of singularity was used to re-state the singularity definition from Chapter 2 in terms of the velocity kinematics (first, in Chapter 3, for non-redundant mechanisms and later, in Chapter 6, for arbitrary mechanisms). The examination of the various possibilities for the degeneration of the instantaneous kinematics led to the definition of six different types of singularity, namely, singularities of redundant input (output), impossible input (output), increased instantaneous mobility and redundant passive motion. The interdependence of the six types was studied and a classification theorem was proved establishing that the non-redundant-mechanism singularities can be divided into 21 distinct classes, each class containing only kinematically similar singularities.

In Chapter 4, hybrid-chain manipulators were studied as an example of the potential for application of the general framework, developed in Chapters 2 and 3, to specific mechanisms. (HCMs are a specific, yet quite general, type of mechanisms that includes many complex parallel-like manipulator architectures which find increasingly wide applications.) Efficient criteria for the detection and classification of the singular configurations of HCMs were presented. This was achieved with the help of an improved method for the elimination of passive-joint velocities from the velocity equation. Such innovations in the methods for velocity analysis of parallel manipulators were necessary since the existing techniques were shown to fail at certain singular configurations. A classification theorem for HCM singularity was proved and it was established that the singularities of HCMs can be divided into 15 distinct classes, while 6 other singularity classes, though occurring in general non-redundant mechanisms, are impossible for HCMs.

Methods for the identification of the singular configurations of any non-redundant mechanism and the description of the division of the singularity set of the mechanism into classes were presented in Chapter 5. This identification and classification problem was solved by the methodical application of six criteria for the occurrence of the singularity types, derived in the same chapter. Special attention is given to the application of the proposed methods to the analysis of mechanisms with complex chains. Two techniques for the simplification of the process of identification and singularity-class description for multi-loop high-dof mechanisms were proposed and applied to a 6-dof example mechanism.

The validity of the classification framework of Chapter 3 and the identification methods of Chapter 5 were further generalized in Chapter 6, where it was shown that the propositions and methods obtained in the former chapters can be applied, with some modifications, to mechanisms with redundancy. The effects of dynamic and/or kinematic redundancy on each of the results obtained for non-redundant mechanisms was examined. The classification theorem proved for redundant mechanisms established that the singularities of all mechanisms (redundant or not) can be divided into 27 distinct classes. Six of these occur only for redundant mechanisms, three classes being associated with dynamic redundancy while the other three are caused by kinematic redundancy.

7.2. Possibilities for Future Work

Obviously, there remain many unsolved problems related to the kinematic singularity of mechanisms. Herein, we suggest areas of continued investigation based on the general approach presented in this dissertation.

7.2.1. Generic singularities

As demonstrated in the previous chapters, mechanisms have a large variety of substantially different singularities. The pattern of the locations of the singular configurations can be very complicated even for simpler classes of mechanisms like serial chains. It would be very difficult to characterize the global properties of the singularity set without imposing any restrictions on the geometry of the mechanisms considered. Therefore, it is desirable to establish a comparatively simple description valid for a comparatively large subset of mechanisms. Ideally, one would like to prove that the singularity set of almost every mechanism forms "nice" topological spaces, such as smooth manifolds. The words "almost every mechanism" can be made rigorous by using the notion of generic properties, which is formally defined below.

Suppose a kinematic chain, K, is given (i.e., a graph and a joint-type distribution) and the input joints and the output link are specified. Thus, the spaces Q, I, and O (defined in Chapter 2) are given. The space D and the maps f_I and f_O will then depend on the choice of the link geometry γ. However, the link geometry can be described by the parameters that determine the relative position of the joint axes in each link. These parameters are angles and distances that can be chosen in the spirit of the Denavit-Hartenberg symbolism. Therefore, each mechanism geometry with the given architecture is specified by a unique point, α, in a space, A, of the type $\mathbf{R}^k \times T^m$ (T^m is the m-dimensional torus).

To say that a property, P, is true for almost every mechanism or, equivalently, that P is a generic property, will be understood to mean that, for every architecture, the union of the points α for which P is satisfied is a dense and open subset of A. Thus, if a mechanism satisfies P, the property will be preserved under small perturbation of the link parameters, and if for a mechanism P is not true, this may be corrected by a small change of the mechanism.

It is proposed to find a dense set in A for which the singularity set of the corresponding mechanisms has a comparatively simple structure. Such mechanisms can be called *generic*. The non-generic mechanisms form thin sets (with measure zero) which divide the space A into classes of generic mechanisms.

For serial chains, this problem can be solved by applying to the output map the results on the singularities of the so-called "one-generic" maps (Golubitsky and Guillemin 1973). For arbitrary chains, the problem is more complicated since not only the properties of the output map, but also the properties of the input map and the structure of the configuration space are important. Moreover, the requirement for f_O to be one-generic is not suitable since it can be shown that the mechanisms with such output maps correspond to a non-dense set in A. Furthermore, one cannot expect a dense subset of the mechanisms with a given architecture to have a structure of the singularity set as simple as the one of generic serial chains. A conjecture can be made that the generic singularity set of a mechanism consists of a number of smooth manifolds which intersect transversally.

7.2.2. Automatic singularity analysis

For all but the simplest mechanisms the singularity set contains infinitely many configurations and therefore to locate the singularities implies the task of obtaining a good description of a multi-dimensional subspace of the mechanism's configuration space. This could be done by either obtaining simplified symbolic equations for the singularity set or by providing an algorithm able to trace numerically and represent graphically the projections and cross-sections of this set.

The procedures in Section 5.4 describe an algorithm for the automatic identification of the singularity set, however significant kinematic and computational problems remain to be solved before a "black box" can emerge for singularity analysis. Some of these issues are briefly outlined below.

The first step in an algorithm for singularity analysis must be the automatic generation of the loop equations. It is desirable to make use of symbolic methods designed to take advantage of possible closed form solutions (Kecskeméthy 1993). On the other hand, since an algebraic solution cannot be guaranteed, a representation that is suitable for numerical iterative solution should be preferred. In particular, the position parameters should be chosen in such a way that the resulting equations are polynomial.

The next step is the (automatic) formulation of the singularity conditions. As it was shown in Sections 5.3 and 5.4 these conditions involve the rank-deficiency of some (polynomial) matrix function of $\boldsymbol{q}$. According to Davenport et al. (1993) for such matrices (functions of multiple variables) the Cramer rule is a more efficient way for symbolic computation of the determinants than any process of Gaussian elimination (transforming the matrix into a triangular form). However, if the kinematic nature of $\boldsymbol{L}(\boldsymbol{q})$ is taken into account, the matrix could be simplified and the computation of singularity conditions for the submatrices be made easier. The strategies for passive-joint screw elimination by reciprocal screws developed for hybrid chains in Chapter 4 and discussed in Section 5.5 could be helpful.

Finally, once the systems of algebraic equations have been generated, the goal would be to extract maximum information about their solution sets. These sets (algebraic varieties) are subsets of the singularity set. This investigation may involve symbolic simplification of the equations or their numerical solution. (On the other hand, some interesting properties of the solution set may be deduced without solving the equations by applying algebraic-geometry tools (Merlet, 1993)). Ideally, one would like to obtain a stratification of the singularity set, which would decompose the set into non-intersecting manifolds consisting of singularities of the same class.

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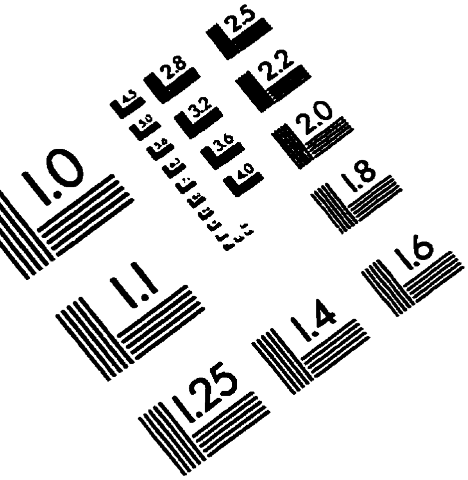
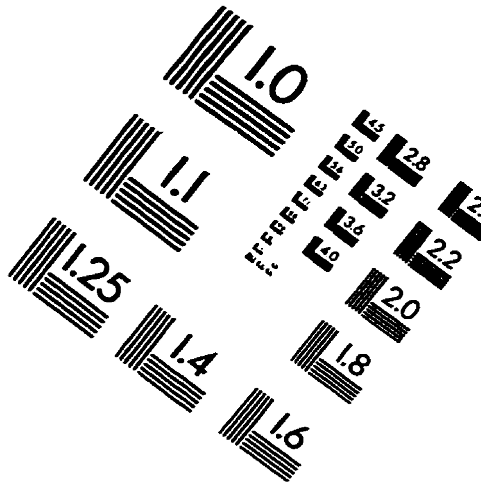
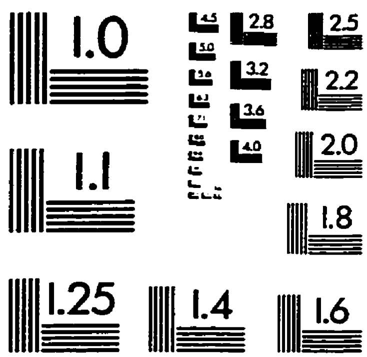
IMAGE EVALUATION TEST TARGET (QA-3)







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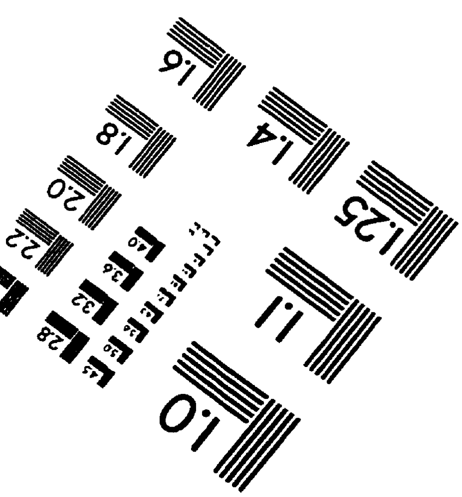

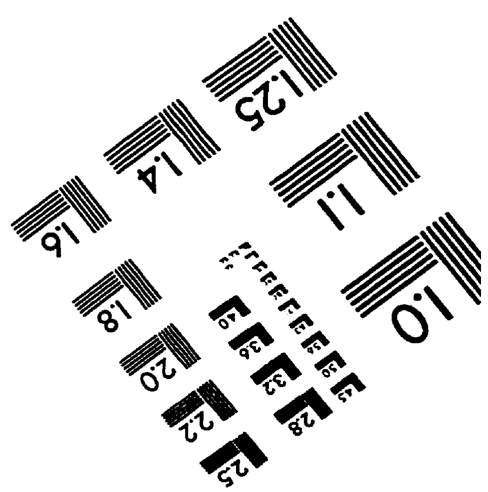

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