CO-irredundant Ramsey Numbers

by

JILL SIMMONS
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Supervisors: Dr. Ernest Cockayne and Dr. Gary MacGillivray
(Department of Mathematics and Statistics)

Abstract

Given a graph $G = (V, E)$, a subset of vertices S is *CO-irredundant* if for any vertex v in S, the closed neighbourhood of v is not contained in the union of the open neighbourhoods of the vertices of $S - \{v\}$. The *CO-irredundant Ramsey number* $t(l, m)$ is the least value of n such that any n-vertex graph G either has a CO-irredundant vertex subset of at least m vertices, or its complement $\overline{G}$ has a CO-irredundant vertex subset of at least l vertices. The existence of these numbers is guaranteed by Ramsey's theorem. We prove that $t(4,5) = 8$, $t(4,6) = 11$, $t(4,7) = 14$, $t(3,m) = m$, and $t(3,3,m) = 2m - 1$ or $2m - 2$ for m odd or even respectively. We also prove that $t(n_1, \ldots, n_k) = R(F_1, \ldots, F_k)$ where $n_i \in \{3, 4\}$ and $F_i = P_3(C_4)$ if $n_i = 3(4)$. Bounds will be given for $t(5,5)$.

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Jill Simmons

Chapter 1

Introduction

In 1930, a paper written by Frank Ramsey introduced a result which would become the foundation of a vast amount of literature on what is referred to as Ramsey type problems. A special case of Ramsey's theorem says: Given two positive integers, l and m, there exists a smallest integer n such that for any graph G on n vertices, either G contains an independent set of m vertices or $\overline{G}$ contains an independent set of l vertices. This number n is denoted by $r(l,m)$ and is called a Ramsey number, or *classical Ramsey number*. The classical Ramsey numbers have proven extremely difficult to evaluate, most of the progress being obtained in the last decade. Slight changes to the definiton by Chvátal and Harary [10] led to generalized Ramsey theory for graphs, which is an area of research of great interest with many published results. The purpose of this thesis is to present a new generalization and to calculate some nontrivial values.

In 1978, Cockayne, Hedetniemi and Miller [13] introduced irredundant vertex sets which include independent sets, and this led to the definition of irredundant Ramsey numbers. CO-irredundance extends the concept of irredundance. A set of vertices S is *CO-irredundant* if for each vertex v in S, the closed neighbourhood of v is not contained in the union of the open neighbourhoods of the vertices in $S - \{v\}$. This permits the following generalization of the Ramsey numbers which is the subject of this work: Given two positive integers, l and m, there exists a smallest integer n such that for any graph G on n vertices, either G contains a CO-irredundant set of m vertices or $\overline{G}$ contains a CO-irredundant set of l vertices. This new number n is called a *CO-irredundant Ramsey number* and is denoted by $t(l, m)$. The existence of these numbers is guaranteed by Ramsey's theorem.

Chapter 2 provides an introduction to all graph theoretic concepts relevant to this thesis, as well as a selection of results on independence, domination, irredundance, CO-irredundance, Ramsey theory, and generalized Ramsey theory.

Chapter 3 is dedicated to the calculation of several CO-irredundant Ramsey numbers. We will see the simple result that $t(3, m) = m$ and it will be shown that several of the CO-irredundant Ramsey numbers may be obtained from the generalized graph Ramsey numbers. We will also prove that $t(4, 5) = 8$, $t(4, 6) = 11$, $t(4, 7) = 14$, and $t(3, 3, m) = 2m - 1$ or $2m - 2$ for m odd or even respectively. Bounds will be given for $t(5, 5)$.

Further CO-irredundant Ramsey numbers are probably within reach, but their evaluation will no doubt be difficult. The use of computer programs may prove useful, but currently no computer is fast enough to evaluate the smallest unknown classical Ramsey number, $r(5,5)$.

Chapter 2

Preliminaries

This chapter summarizes the graph theoretic definitions used in this thesis. It also provides an introduction to irredundance and CO-irredundance. as well as an introduction to Ramsey theory. For further discussion of basic graph theory. the reader is referred to Bondy and Murty [4].

2.1 Graph Theory

A *graph* $G = (V, E)$ consists of a nonempty set V of *vertices* and a set E of unordered pairs of distinct vertices from V, called *edges*. When more than one graph is being discussed, $V(G)$ and $E(G)$ will be used to denote the vertex set and edge set of the graph G. For the remainder of this section let G and H be graphs.

If the pair of vertices (u, v) is an edge in $E(G)$, then we write $uv \in E(G)$. The vertices u and v may be referred to as *ends* of the edge uv, and we say that u and v are *adjacent*. In addition, we say that the edge uv is *incident* to u and to v. Two vertices u and v are called *nonadjacent* if $uv \notin E(G)$. Similarly, two edges are adjacent if they have a vertex in common, and nonadjacent otherwise.

A *subgraph* H of G is a graph whose vertex set is a subset of the vertex set of G and whose edge set is a subset of the edge set of G. In other words, a graph H is a subgraph of G if and only if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. We write $H \subseteq G$ to show that H is a subgraph of G. If $H \subseteq G$ and $V(H) = V(G)$ then H is a *spanning subgraph* of G.

Often we are interested in a specific substructure of a graph. Suppose V' is a nonempty subset of $V(G)$. The subgraph of G which has vertex set V' and edge set consisting of all edges of G with both ends in V' is called the *subgraph of G induced by* V'. This induced subgraph of G is denoted $G[V']$. Similarly, we can define a subgraph of G induced by an edge subset of $E(G)$: If $E' \subseteq E(G)$ then the *spanning subgraph induced by* E', denoted $G[E']$, has vertex set $V(G)$ and edge set E'.

The *union* of G and H, denoted $G \cup H$, is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$.

There are many structures within a graph which are given a special name. Some simple structures of great importance will be defined here. A $v_0 - v_n$ *walk* in G is an alternating sequence of vertices and edges starting with v_0 and ending with v_n: $v_0e_1v_1e_2 \dots e_nv_n$ where $e_i = v_{i-1}v_i$ for $i = 1, 2, \dots, n$. Since all graphs which are considered in this thesis are simple (no multiple edges, no loops, undirected edges), we can simply write a $v_0 - v_n$ walk as a sequence of vertices: $v_0v_1 \dots v_n$. A special kind of walk, a *path*, has all distinct vertices. We say that the path $v_0v_1 \dots v_n$ is a path from v_0 to v_n or that it is a $v_0 - v_n$ path. The graph which is precisely a path on n vertices is called P_n, and we say that a graph G contains a P_n if P_n is a subgraph of G. A *cycle* is a walk in which all vertices are distinct except $v_0 = v_n$. The graph which is a cycle on n vertices is called C_n. If n is odd (even) we say C_n is an *odd cycle (even cycle)*. A graph is called *connected* if there exists a $u - v$ path for any pair of distinct vertices u and v.

A *complete graph* is a graph in which every pair of distinct vertices are adjacent. The complete graph on n vertices is denoted K_n. A *clique* in a graph G is a subgraph of G which is a complete graph. An *independent set* (of vertices) is a set $V' \subseteq V(G)$ such that $G[V']$ contains no edges. A vertex $v \in V' \subseteq V(G)$ is said to be *isolated* in V' if v is not adjacent to any vertex in V'. An *independent set of edges* is a set of edges in which no two edges have a vertex in common, that is, a set of mutually nonadjacent edges.

Two graphs G and H are called *isomorphic* if there exists a function $f : V(G) \to V(H)$ such that f is one-to-one and onto and $uv \in E(G)$ if and only if $f(u)f(v) \in E(H)$, and we write $G \cong H$.

The complement of G, denoted $\overline{G}$, has $V(\overline{G}) = V(G)$ and $E(\overline{G})$ contains precisely the unordered pairs of distinct vertices which are not in $E(G)$, that is $uv \in E(\overline{G})$ if and only if $uv \notin E(G)$. A graph G is *self complementary* if G is isomorphic to $\overline{G}$. An important fact to notice is that a clique in G is an independent set in $\overline{G}$.

The *degree* of a vertex $v \in V(G)$, $deg_G(v)$, is the number of vertices in $V(G)$ which are adjacent to v, or equivalently the number of edges in $E(G)$ incident to v. We will write $deg(v)$ if it is clear from the context which graph is being discussed. The following well-known result will be frequently used

Theorem 2.1.1 *Let G be a graph. Then*

$$\sum_{v \in V(G)} deg(v) = 2|E(G)|.$$

A simple result which follows from Theorem 2.1.1 is that the number of odd degree vertices in a graph must be even.

The *minimum degree* of G, denoted $\delta(G)$, is the minimum value of $deg(v)$ taken over all $v \in V(G)$. The *maximum degree* of G, denoted $\Delta(G)$, is the maximum value of $deg(v)$ taken over all $v \in V(G)$.

2.2 Irredundance and CO-irredundance

Before introducing the definitions of irredundant sets and CO-irredundant sets, it is important to understand how they originated. We require several new definitions.

The *open neighbourhood* of a vertex v in G is the set of all vertices adjacent to v in G. We use $N_G(v)$ to represent the open neighbourhood of v in G. When it is clear from the context which graph is being discussed, the open neighbourhood of v will simply be written $N(v)$. The *closed neighbourhood* of v in G is given by

$N_G[v] = N_G(v) \cup \{v\}$. Again, $N_G[v]$ will be written $N[v]$ when it is clear what graph is being discussed. Open and closed neighbourhoods are also defined for vertex subsets. For $X \subseteq V(G)$, the open and closed neighbourhoods of X are given by

$$N(X) = \bigcup_{x \in X} N(x)$$

and

$$N[X] = \bigcup_{x \in X} N[x].$$

Given $X \subseteq V(G)$ and $x \in X$, the *private neighbourhood of x relative to X* is

$$pn(x, X) = N[x] - N[X - \{x\}].$$

It is appropriate that the elements of $pn(x, X)$ be called private neighbours of x as (informally) all vertices in $pn(x, X)$ are neighbours of x and not neighbours of any other vertex in X.

A set $D \subseteq V(G)$ is a *dominating set* of G (and is said to dominate G) if each vertex in $V - D$ is adjacent to a vertex in D. Further, D is a *minimal dominating set* if no proper subset of D dominates G.

The following proposition shows how dominating sets are related to private neighbourhoods:

Theorem 2.2.1 *[27] A dominating set D is a minimal dominating set if and only if $pn(d, D) \neq \emptyset$ for all $d \in D$.*

When a dominating set D is not minimal, there is some vertex $v \in D$ such that $D - \{v\}$ is still a dominating set, which implies $pn(v, D) = \emptyset$. We can call this vertex

v *redundant* in D as it does not dominate any vertex which is not already dominated by another vertex in D. This leads to the definition of an irredundant set which is (informally) a set containing no redundant vertices.

Formally, a set $X \subseteq V$ for which $pn(x, X) \neq \emptyset$ for all $x \in X$ is called an *irredundant set*. An irredundant set X is *maximal irredundant* if no proper superset of X is irredundant. Note that an irredundant set need not be dominating.

Irredundance was introduced in 1978 by Cockayne, Hedetniemi, and Miller [13]. Since then the subjects of domination, independence and irredundance have been widely studied; the bibliography in [23] contains over a thousand papers on these topics.

The following simple result relates domination and independence:

Theorem 2.2.2 *[2]*

i) S is maximal independent if and only if S is independent and dominating.

ii) If X is maximal independent, then X is minimal dominating.

The next result is a similar theorem relating domination and irredundance. Note that part (i) is immediate from Theorem 2.2.1 and the definition of an irredundant set.

Theorem 2.2.3 *[13]*

i) S is minimal dominating if and only if S is irredundant and dominating.

ii) If X is minimal dominating, then X is maximal irredundant.

The *domination number* and *upper domination number* of a graph G are denoted by $\gamma(G)$ and $\Gamma(G)$ respectively, and are the smallest and largest number of vertices in a minimal dominating set. Similarly, the *independence number* and *upper independence number* (*irredundance number* and *upper irredundance number*) are denoted by $i(G)$ and $\beta(G)$ ($ir(G)$ and $IR(G)$) and are the smallest and largest number of vertices in a maximal independent set (maximal irredundant set). From Theorems 2.2.2 and 2.2.3 it can be seen that

$$ir(G) \leq \gamma(G) \leq i(G) \leq \beta(G) \leq \Gamma(G) \leq IR(G).$$

Farley and Schacham [18] defined another vertex subset property by generalizing the definition of an irredundant set. Recall that a set X is irredundant if and only if

$$N[x] - N[X - x] \neq \emptyset, for \;\; all \;\; x \in X.$$

Farley and Schacham changed the second closed neighbourhood in the definition of an irredundant set to an open neighbourhood, giving: A set X is called *CO-irredundant* if and only if

$$N[x] - N(X - x) \neq \emptyset, for \;\; all \;\; x \in X.$$

The "CO" in the name CO-irredundant represents the fact that the neighbourhoods in the definition are <u>C</u>losed and <u>O</u>pen respectively. CO-irredundance is not yet well-studied, but it is mentioned briefly in [19], [20], and [24].

We denote $N[x] - N(X - x)$ by $PN(x, X)$, and we say $PN(x, X)$ is the *private neighbourhood of x with respect to X*. It may at first seem confusing that both

$PN(x,X)$ and $pn(x,X)$ are called the private neighbourhood of x with respect to X. However, it will always be clear from the context whether we are referring to a private neighbour in the irredundant sense or in the CO-irredundant sense. Furthermore, when more than one graph is being discussed, the notation $pn(x,X,G)$ and $PN(x,X,G)$ will be used to denote the private neighbourhoods of x with respect to X in G.

The difference between an irredundant set and a CO-irredundant set can be clearly seen from the following characterization of $pn(x,X)$ and $PN(x,X)$.

Theorem 2.2.4 *Vertex $u \in pn(x,X)$ if and only if*

(i) $u = x$ and x is isolated in $G[X]$ or

(ii) $u \in V - X$ and $N(u) \cap X = \{x\}$

Moreover, $u \in PN(x,X)$ if and only if (i) or (ii) holds or

(iii) $u \in X$ and $N(u) \cap X = \{x\}$.

The characterization in Theorem 2.2.4 shows that $pn(x,X) \subseteq PN(x,X)$, and since $x \in pn(x,X)$ for any vertex x of an independent set X, we deduce

$$X \ \ independent \Longrightarrow X \ \ irredundant \Longrightarrow X \ \ CO-irredundant$$

Thus if $COIR(G)$ is the largest cardinality of a maximal CO-irredundant set in G, then

$$\beta(G) \leq IR(G) \leq COIR(G).$$

Although irredundance implies CO-irredundance, a maximal irredundant set need not be maximal CO-irredundant. For example, in P_5 with vertex sequence $v_1, v_2, \ldots, v_5$ the set $\{v_2, v_4\}$ is minimal dominating and therefore maximal irredundant by Theorem 2.2.3. However, the set $\{v_1, v_2, v_4\}$ is a CO-irredundant set, and thus $\{v_2, v_4\}$ is not maximal CO-irredundant.

The next few results show that CO-irredundant sets have several properties similar to those of irredundant sets.

Theorem 2.2.5 *CO-irredundance is a hereditary property.*

Proof Let $T \subseteq S \subseteq V$ where S is a CO-irredundant set of G. For $t \in T, \emptyset \neq PN(t, S) \subseteq PN(t, T)$, as $N[t] - N(S - t) \subseteq N[t] - N(T - t)$. Thus $PN(t, T) \neq \emptyset$. ∎

The following theorem is simple but important, as it will be constantly used in Chapter 3.

Theorem 2.2.6 *If $S \subseteq U \subseteq V$ and S is CO-irredundant in $G[U]$, then S is CO-irredundant in G.*

Proof For $s \in S, \emptyset \neq PN(s, S, G[U]) \subseteq PN(s, S, G)$. ∎

A set $S \subseteq V(G)$ is called *total dominating* if and only if every vertex in $V(G)$ is adjacent to a vertex in S. The following theorem relating total domination and CO-irredundance is similar to Theorem 2.2.3 which related domination and irredundance.

Theorem 2.2.7

i) S is minimal total dominating if and only if S is CO-irredundant and total dominating

ii) If S is minimal total dominating, then S is maximal CO-irredundant.

Proof

i) ($\Rightarrow$) Suppose S is minimal total dominating. Then for each $s \in S$, $N(S-\{s\}) \neq V$. Since S is total dominating, $N(S) = V = N[S]$. Thus there exists $u \in N[S] - N(S - \{s\}) = PN(s, S)$ and hence S is CO-irredundant.

($\Leftarrow$) Let S be CO-irredundant and total dominating. For $s \in S$, there exists $u \in N[s] - N(S - \{s\})$. But $u \notin N(S - \{s\})$ so u has no neighbour in $S - \{s\}$. Thus $S - \{s\}$ is not a total dominating set. Therefore S is minimal total dominating.

ii) Let S be minimal total dominating. S is certainly CO-irredundant by i). Suppose there exists y such that $S \cup \{y\}$ is CO-irredundant. Then there exists $v \in PN(y, S \cup \{y\}) = N[y] - N(S)$. Therefore $N(S) \neq V$, a contradiction which shows that S is maximal CO-irredundant. ∎

2.3 Ramsey Theory

Ramsey theory refers to a large body of results in mathematics concerning the idea that when any large enough structure of a certain type is partitioned, some class of the partition contains a substructure of some prescribed type.

The *pigeonhole principle* states that if m objects are partitioned into n classes, then some class contains at least $\lceil \frac{m}{n} \rceil$ objects. This concept is very simple, but a generalization called Ramsey's theorem leads to some very deep results. The *pigeonhole principle* guarantees that when we partition objects into classes we get a class with many objects. Ramsey's famous theorem [29] guarantees a similar result:

Theorem 2.3.1 (Ramsey's Theorem)

Let r, k be positive integers ≥ 2 and $n_1, n_2, \ldots, n_k$ be positive integers $\geq r$. There exists a smallest integer n such that for any ordered partition of the r-subsets of $\{1, 2, \ldots, n\}$ into k classes, there is a subset of size n_i all of whose r-subsets are in the i^{th} class of the partition, for some i. This number n is denoted $R(n_1, n_2, \ldots, n_k; r)$.

When $r = 2$ there is a useful graph theory representation of Ramsey's theorem. In this case, Ramsey's theorem says that if we partition the 2-subsets of a sufficiently large set into k classes there will be an n_i-subset all of whose 2-subsets are in the i^{th} class of the partition, for some i. This problem is still very difficult to visualize. Suppose we allow the elements of a set V to be represented by vertices. We can then represent a 2-subset by an edge joining the elements of the 2-subset. Hence the 2-subsets of a set V are represented by the complete graph on $|V|$ vertices. The classes of a partition of the 2-subsets can clearly be represented by "colouring" all the edges in a class with the same colour. Therefore, a partition of the 2-subsets of V into k classes can be represented by a *k-edge colouring* of the complete graph on $|V|$ vertices.

If there exists an n_i-subset all of whose 2-subsets are in the i^{th} class of the partition, then in the graph representation there exists a set S of n_i vertices such that all the edges with both ends in S have colour i.

Suppose that each edge of the complete graph K_n is assigned a colour from $\{1, 2, \ldots, k\}$. For $i = 1, 2, \ldots, k$ let G_i be the spanning subgraph of K_n induced by the edges of colour i. Then $(G_1, G_2, \ldots, G_k)$ is called a *k-edge colouring of K_n*.

We now state Ramsey's theorem for $r = 2$ in terms of the graph theory representation:

Theorem 2.3.2 *Let $k \geq 2$ and $n_i \geq 3$ for $i = 1, 2, \ldots, k$. The classical Ramsey number $r(n_1, n_2, \ldots, n_k)$ is the least integer n such that for any k-edge colouring $(G_1, G_2, \ldots, G_k)$ of K_n, there exists $i \in \{1, 2, \ldots, k\}$ such that G_i contains K_{n_i} as a subgraph.*

The most trivial Ramsey number is $r(3, 3) = 6$. It can easily be seen that $r(3, 3) \leq 6$ by considering any vertex v in K_6 and any 2-edge colouring of K_6. There are 5 edges incident to v and therefore by the pigeonhole principle 3 of these edges are of the same colour. In keeping with the usual practice, we will call the two colours red and blue and denote the induced subgraphs by R and B. Without loss of generality there are 3 vertices adjacent to v in R, say x_1, x_2, x_3. Now if there are any edges in $R[\{x_1, x_2, x_3\}]$ then such an edge together with the red edges from v form a red K_3. If there are no edges in $R[\{x_1, x_2, x_3\}]$ then $B[\{x_1, x_2, x_3\}]$ is a blue K_3. Therefore any colouring (R, B) of K_6 contains a K_3 in R or B (or both). To establish that

$r(3,3) = 6$, it must be shown that there exists a colouring (R, B) of K_5 with no K_3 in R or B. Such a colouring can be seen in figure 2.1:

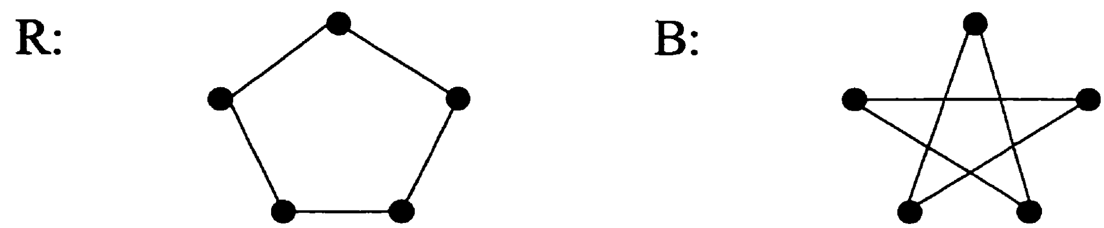

Figure 2.1: A colouring (R, B) of K_5 with no K_3 in R or B

The method used to prove $r(3,3) = 6$ demonstrates the two steps needed to prove the value of any Ramsey number. Firstly, a proof must be given to show $r(n_1, n_2, \ldots, n_k) \leq n$. Then, a k-edge colouring of K_{n-1} must be found in which G_i does not contain K_{n_i} for all i. The Ramsey numbers have proven immensely difficult to evaluate. All known 2-colour Ramsey numbers, $r(l, m)$, are listed in Table 2.1.

l \ m	3	4	5	6	7	8	9
3	6	9	14	18	23	28	36
4		18	25				

Table 2.1: Known 2-colour Ramsey numbers $r(l, m)$

The only other known classical Ramsey number is $r(3,3,3) = 17$, which was found by Greenwood and Gleason [22]. Although very few Ramsey numbers are known, the attempts at evaluation have produced many bounds for the 2-colour Ramsey numbers. A complete table of known bounds with references can be found in [28]. The following theorem is commonly used to obtain an upper bound on a 2-colour Ramsey number:

Theorem 2.3.3 $r(l,m) \le r(l-1,m) + r(l,m-1)$, *with strict inequality when both summands on the right are even.*

Corollary 2.3.4 $r(l,m) \le \begin{pmatrix} l+m-2 \\ l-1 \end{pmatrix}$.

A great deal of work has been done on asymptotic bounds. Theorems 2.3.5 and 2.3.6 are examples of such bounds.

Theorem 2.3.5 *[21] For fixed n and large m, $r(m,n) \le c(m^{n-1} log log m)/log m$, where c depends on n.*

For $n = 3$ and $m \ge 3$ this can be improved to:

Theorem 2.3.6 *[1] $r(m,3) \le cm^2/log m$.*

Ramsey theory has provided beautiful concise proofs for other results. The following theorem can be proved by taking $f(m,n) = r(m+1, n+1) - 1$.

Theorem 2.3.7 *[30] There is a function $f(m,n)$ with the following property: If $x_1, x_2, \ldots, x_N$ is any sequence of distinct real numbers with $N > f(m,n)$, then there is either a monotone increasing sequence of length greater than m, or a monotone decreasing sequence of length greater than n.*

The following geometric fact can also be established using Ramsey theory:

Theorem 2.3.8 *[7] There is a smallest integer $N(n)$ such that any collection of $N \geq N(n)$ points in the plane, no 3 collinear, has a subset of n points forming a convex n-gon.*

The proof of Theorem 2.3.8 involves looking at any $r(n, 5; 4)$ points, and colouring the 4-sets red if they form a convex quadrilateral and blue otherwise.

2.4 Generalized Ramsey Theory

Generalization is one of the most important features of mathematics. We have seen the classical Ramsey numbers defined in terms of cliques, where $r(n_1, n_2, \ldots, n_k)$ gives us the smallest K_n which must have a clique of a particular size in one of its monochromatic subgraphs. An extension of this concept is obtained by replacing a clique with a general graph. Thus the *generalized Ramsey number* $R(F_1, F_2, \ldots, F_k)$ is the smallest n such that for any k-edge colouring $(G_1, G_2, \ldots, G_k)$ of K_n, the graph F_i is a subgraph of G_i for some i. These new numbers certainly do generalize the classical Ramsey numbers in that $R(K_{n_1}, K_{n_2}, \ldots, K_{n_k}) = r(n_1, n_2, \ldots, n_k)$.

A simple generalized Ramsey number result is given in Theorem 2.4.1.

Theorem 2.4.1 *$R(G, K_2) = n$ where $n = |V(G)|$.*

Proof Consider any 2-edge colouring of K_n. If any edge is coloured blue then there exists a K_2 in B. Otherwise, $R = K_n$ and hence contains G as a subgraph. Therefore

$R(G, K_2) \leq n$. Now consider the colouring of K_{n-1} in which all edges are coloured red. There is no G in R as R does not have enough vertices, and B contains no K_2 as B has no edges. Therefore, $R(G, K_2) > n - 1$. Thus $R(G, K_2) = n = |V(G)|$. ∎

Radziszowski's survey paper [28] provides a very thorough summary of known results on generalized Ramsey numbers and contains an enormous listing of references on the subject. A sampling of some of these numbers will be given here.

Theorem 2.4.2 *[28]*

$R(P_n, P_m) = n + \lfloor \frac{m}{2} \rfloor - 1$, *for all* $n \geq m \geq 2$

$R(C_3, C_3) = 6$

$R(C_4, C_4) = 6$

$R(C_4, C_4, C_4) = 11$

$R_k(C_4) \leq k^2 + k + 1$ *for all* $k \geq 1$, *where* $R_k(C_4) = R(\underbrace{C_4, C_4, \ldots, C_4}_{k\ arguments})$

$R_k(C_4) \geq k^2 - k + 2$ *for all* $k - 1$ *a prime power*

$R(G, G) \geq \lfloor (4|V(G)| - 1)/3 \rfloor$ *for any connected graph* G

Chapter 3

CO-irredundant Ramsey Numbers

This chapter will introduce the CO-irredundant Ramsey numbers and show how several of them are calculated.

3.1 Introduction to CO-irredundant Ramsey Numbers

Recall that the classical Ramsey number $r(l, m)$ is the smallest n such that for any colouring (R, B) of the edges of K_n, K_l is a subgraph of R or K_m is a subgraph of B. Notice that R contains a K_l if and only if B contains an independent set of size l, and similarly B contains a K_m if and only if R contains an independent set of size m. Therefore, the definition of the classical Ramsey numbers can be stated in terms of independent sets instead of cliques. Now $r(l, m)$ is the smallest n such that for any colouring (R, B) of the edges of K_n, R contains an independent set of size m or B contains an independent set of size l. Recall now the following facts:

$$X \ \ independent \Longrightarrow X \ \ irredundant \Longrightarrow X \ \ CO - irredundant$$

and

$$COIR(G) \geq IR(G) \geq \beta(G) \qquad (3.1.1)$$

Thus it is natural to generalize Ramsey's theorem in terms of irredundant and CO-iredundant sets.

Let $k \geq 2$ and $n_i \geq 3$ for $i = 1, 2, \ldots, k$. The *irredundant Ramsey number* $s(n_1, \ldots, n_k)$ (*CO-irredundant Ramsey number* $t(n_1, \ldots, n_k)$) is the least integer n such that for any k-edge colouring $(G_1, G_2, \ldots, G_k)$ of K_n, there exists $i \in \{1, 2, \ldots, k\}$ such that $IR(\overline{G_i})$ $(COIR(\overline{G_i})) \geq n_i$.

The existence of the classical Ramsey numbers together with (3.1.1) guarantees the existence of the other two types of Ramsey numbers. Furthermore, (3.1.1) gives

$$t(n_1, \ldots, n_k) \leq s(n_1, \ldots, n_k) \leq r(n_1, \ldots, n_k).$$

We have seen that the classical Ramsey numbers are very difficult to evaluate. Calculation of irredundant Ramsey numbers has also proven to be hard. The known values for $k = 2$ can be seen in Table 3.1. The only other known irredundant Ramsey number is $s(3, 3, 3) = 13$ ([14], [15]).

l \ m	3	4	5	6	7
3	6 [5]	8 [5]	12 [5]	15 [6]	18 [9] [12]
4		13 [11]			

Table 3.1: Known 2-colour irredundant Ramsey numbers $s(l, m)$

Asymptotic estimations on the irredundant Ramsey numbers have been made by Chen, Hattingh and Rousseau [8] and by Erdös and Hattingh [16]. The reader is also referred to the survey article by Mynhardt [26].

As CO-irredundance is a generalization of irredundance, it is reasonable to expect that the CO-irredundant Ramsey numbers will also be challenging to calculate. Theorem 2.2.4 showed that a vertex in a CO-irredundant set must have a private neighbour of one of three types. We now develope some notation relating to these three types of private neighbours.

Let X be a CO-irredundant set. A vertex $u \in PN(v, X)$ is called an XPN of v. If u is an XPN of type (i) or (iii), i.e. a private neighbour of v in X, then u is called an *internal private neighbour of* v (abbreviated iXPN). If v has a private neighbour of type (ii), i.e. if there exists $u \in V - X$ such that $N(u) \cap X = \{v\}$, we say that u is an *external private neighbour of* v (abbreviated eXPN). Furthermore, we will abbreviate "CO-irredundant" to "CO-irr." and denote a CO-irr. set of size m by cm for ease of notation.

The following simple observation will be repeatedly used.

Theorem 3.1.1 *X is a CO–irr. set of G such that each $x \in X$ has an iXPN if and only if $\Delta(G[X]) \leq 1$ (i.e. $G[X] \cong \lambda K_1 \cup \mu K_2$).*

Proof Let $x \in X$. If x is not isolated in X, then x has an $iXPN$, say y. Since y is not adjacent to any vertex in $X - x$, y must have x as its $iXPN$. Therefore, both x and y have degree 1. Therefore $\Delta(G[X]) \leq 1$. ∎

Theorem 2.3.3 states that $r(l,m) \leq r(l-1,m) + r(l,m-1)$ with strict inequality if both summands are even. Analagous theorems hold for the irredundant and CO-irredundant Ramsey numbers and are usually the starting points for finding upper bounds.

Theorem 3.1.2 *$t(l,m) \leq t(l-1,m) + t(l,m-1)$ with strict inequality if both summands are even.*

Proof Consider the complete graph on $t(l-1,m)+t(l,m-1)$ vertices and any 2-edge colouring (R,B). A vertex v is adjacent to either i) $t(l-1,m)$ vertices in R or ii) $t(l,m-1)$ vertices in B. In i), these $t(l-1,m)$ vertices contain either a $c(l-1)$ in B or a *cm* in R. In the second case, there is a *cm* in R. In the first case, the $c(l-1)$ together with v forms a *cl* in B. Similarly for ii).

If $t(l-1,m)$ and $t(l,m-1)$ are both even, consider the complete graph on $t(l-1,m) + t(l,m-1) - 1$ vertices. Since $|V|$ is odd, there exists a vertex v with even degree in R and in B (Theorem 2.1.1). Let $R_v = N_R(v)$ and let $B_v = N_B(v)$. Either $|R_v| \geq t(l-1,m) - 1$ or $|B_v| \geq t(l,m-1) - 1$. Without loss of generality suppose that the former is true. Then $|R_v| \geq t(l-1,m)$ as $|R_v|$ is even. By definition of $t(l-1,m)$, $R[R_v]$ contains a cm or $B[R_v]$ contains a $c(l-1)$. Therefore, either $R[R_v \cup \{v\}]$ contains a cm or $B[R_v \cup \{v\}]$ contains a cl. ∎

Part of the difficulty in evaluating the CO-irredundant Ramsey numbers is that there is no useful characterization of cm's for most values of m. However, theorems have been established which state precisely when a graph contains a $c3$ or a $c4$.

Theorem 3.1.3 *B has a $c3$ if and only if R has P_3 as a subgraph.*

Proof Let R have P_3 as a subgraph and xy, yz be red edges. Then $\Delta(B[\{x, y, z\}]) \leq 1$ and $\{x, y, z\}$ is a blue $c3$ (by Theorem 3.1.1).

Conversely, let $X = \{x, y, z\}$ be a blue $c3$. If say x is a blue XPN of type (i), then x is isolated in $B[\{x, y, z\}]$ and x has red degree at least two as required. Otherwise $B[\{x, y, z\}]$ is P_3 or K_3. In either case at least one vertex say x has a blue eXPN u, which implies that uy, uz are red as required. ∎

Theorem 3.1.4 *B has a $c4$ if and only if R has C_4 as a subgraph.*

Proof If X is the vertex set of a red C_4, then $B[X]$ has maximum degree one which implies that X is a blue $c4$ (Theorem 3.1.1).

Conversely suppose that $X = \{1, 2, 3, 4\}$ is a blue $c4$. If the maximum degree $\Delta(B[X]) \leq 1$, then $R[X]$ contains a C_4. Otherwise without loss of generality 12 and 13 are blue. If 4 is isolated in $B[X]$, then at least two of 1, 2, 3 have blue eXPNs. If 4 is not isolated in $B[X]$, then at most two vertices of X have iXPNs and so again at least two vertices have blue eXPNs. With suitable relabelling, if 1, 2 have blue eXPNs 5, 6 respectively, then 3, 5, 4, 6 is the vertex sequence of a red C_4. ∎

No theorem has been found which shows precisely when a graph contains a $c5$. The following theorem relates to graphs with a $c5$. Note that the graph $K_5 - 2K_2$ is simply the graph obtained by removing two nonadjacent edges from K_5.

Theorem 3.1.5 *B has a $c5$ in which at least three vertices have an internal private neighbour if and only if R has a $K_5 - 2K_2$.*

Proof

($\Leftarrow$) Suppose R contains a $K_5 - 2K_2$. Then B contains a set of 5 vertices which induce a graph with ≤ 2 (nonadjacent) edges. These 5 vertices are a $c5$ in which all the vertices have an internal private neighbour.

($\Rightarrow$) Assume B has a $c5$, $X = \{1, 2, 3, 4, 5\}$, and vertices 3, 4, 5 all have iXPN's. There are 3 cases: *i*) 1 and 2 have iXPN's, *ii*) 2 has an iXPN but 1 does not, or *iii*) neither 1 nor 2 has an iXPN.

i) Since all vertices in X have an iXPN, $B[X]$ contains at most 2 (nonadjacent) edges. Then $R[X] \supseteq K_5 - 2K_2$.

ii) Without loss of generality 1 is adjacent to 2, so 2 must be adjacent to some other vertex, as 1 has no iXPN. Say 2 is adjacent to 3. Now 3 has an iXPN which is not 1, 2 or 3. Without loss of generality 3 has private neighbour 4. Now 4 must have private neighbour 5 and hence 5 is not adjacent to 1, 2 or 3. Thus 5 has no iXPN, which contradicts the assumption.

iii) Let 1 and 2 have eXPN's x and y respectively. At least one of $3, 4, 5$ has its iXPN in $\{3, 4, 5\}$. Say 3 has a private neighbour in $\{3, 4, 5\}$. If $B[\{3, 4, 5\}]$ has ≤ 1 edge then $R[\{x, y, 3, 4, 5\}] \supseteq K_5 - 2K_2$. Otherwise, $B[\{3, 4, 5\}]$ is the path 435 and the iXPN of 3 (which is 4 or 5) has no internal private neighbour. contradicting the assumption. ∎

3.2 Calculation of $t(3, m)$, $t(3, 3, m)$, and $t(n_1, \ldots, n_k)$ where $n_i \in \{3, 4\}$

In this section we will calculate $t(3, m)$, $t(3, 3, m)$, and some values of $t(n_1, \ldots, n_k)$ where $n_i \in \{3, 4\}$. Theorems 3.1.3 and 3.1.4 will be frequently used.

Theorem 3.2.1 *For any $m \geq 3$, $t(3, m) = m$.*

Proof Let $B = K_{m-1}$, $R = \overline{K}_{m-1}$ and consider the 2-edge colouring (R, B) of K_{m-1}. Then B has no $c3$, R has no cm and so $t(3, m) > m - 1$. Now let (R, B) be any 2-edge colouring of K_m (vertex set V). If $\Delta(R) \geq 2$, then B has a $c3$ by Theorem 3.1.3. Otherwise $\Delta(R) \leq 1$ and V is a red cm by Theorem 3.1.1. ∎

Theorem 3.2.2

(i) For odd $m \geq 3$, $t(3, 3, m) = 2m - 1$.

(ii) For even $m \geq 4$, $t(3, 3, m) = 2m - 2$.

Proof

Lower bounds

As in the earlier work, for example, 12 denotes the edge joining vertices 1 and 2. If variables are involved in vertex labels, the edge joining vertices a and b will be denoted by (a, b). Let $\{1, \ldots, n\}$ be the vertex set of K_n where $n \equiv 0 \pmod 4$. Define

$$\begin{aligned} B_n^* &= \{12, 34, \ldots, (n-1, n)\} \\ \text{and} \qquad R_n^* &= \{13, 24, 57, 68, \ldots, (n-3, n-1), (n-2, n)\}\ . \end{aligned}$$

If m is odd, then $2m - 2 \equiv 0 \pmod 4$. Let (R, B, G) be the 3–edge colouring of K_{2m-2} where the edge sets of R, B are R_{2m-2}^* and B_{2m-2}^* respectively. Then R and B have maximum degree one and so neither $\overline{R}$ nor $\overline{B}$ has a $c3$ (Theorem 3.1.3). Moreover $\overline{G} = R \cup B \cong (\frac{m-1}{2})C_4$ which has no cm. Hence $t(3, 3, m) > 2m - 2$.

If m is even, then $2m - 4 \equiv 0 \pmod 4$. Let (R, B, G) be the 3–edge colouring of K_{2m-3} (vertex set $\{1, \ldots, 2m-3\}$) where edge sets of R, B are R_{2m-4}^* and B_{2m-4}^* respectively. As above neither $\overline{R}$ nor $\overline{B}$ has a $c3$. Further $\overline{G} \cong (\frac{m-2}{2})C_4 \cup K_1$ which has no cm. Hence $t(3, 3, m) > 2m - 3$.

Upper bounds

To establish the upper bounds suppose to the contrary that for m odd (even), (R, B, G) is a 3-edge colouring of K_{2m-1} (K_{2m-2}) with no $c3$ in $\overline{R}$ or $\overline{B}$ and no cm in $\overline{G}$.

Then $\Delta(R)$ and $\Delta(B)$ are at most one (Theorem 3.1.3) and so $\Delta(\overline{G}) = \Delta(R \cup B) \leq 2$. Thus components of $\overline{G}$ are paths, cycles or isolated vertices. Each such component X of $\overline{G}$ with t vertices has a CO-irr. set of size at least $\frac{t}{2}$ and if $X \not\cong C_4$, then X has a CO-irr. set of size at least $\frac{t+1}{2}$.

If m is odd, the union of these CO-irr. sets is a CO-irr. set of $\overline{G}$ of size at least $\frac{2m-1}{2}$, i.e. $\overline{G}$ has a *cm*.

If m is even, then $2m - 2 \equiv 2 \pmod 4$. Hence not all components are C_4's. Therefore, in this case also, $\overline{G}$ has a CO-irr. set of size at least $\frac{2m-1}{2}$ and $\overline{G}$ has a *cm*.

Therefore for m odd (even), $t(3,3,m) \leq 2m-1$ $(2m-2)$ as required. ∎

Some values of $t(n_1, \ldots, n_k)$ where $n_i \in \{3,4\}$ may be obtained from Theorems 3.1.3, 3.1.4, and the generalized Ramsey numbers listed in Section 2.4.

Theorem 3.2.3 *For $i = 1, \ldots, k$ let $n_i \in \{3,4\}$ and $F_i = P_3$ (C_4) if $n_i = 3\,(4)$. Then $t(n_1, \ldots, n_k) = R(F_1, \ldots, F_k)$.*

Proof By Theorem 3.1.3 and Theorem 3.1.4, for any k-edge colouring $(G_1, \ldots, G_k)$ of K_n, G_i contains F_i as a subgraph if and only if $\overline{G_i}$ has a cn_i. ∎

From Theorem 3.2.3 we immediately obtain the following results. References to the work on the corresponding generalized Ramsey numbers may be found in [28].

Theorem 3.2.4

(i) $t(4,4) = 6$.

(ii) $t(4,4,4) = 11$.

(iii) $t(3, 3, \ldots, 3)$ (*k arguments*) $= \begin{cases} k+2 & \textit{if } k \textit{ is odd} \\ k+1 & \textit{if } k \textit{ is even.} \end{cases}$

(iv) $t(3, 3, 4) = 6$.

(v) $t(3, 4, 4) = 8$.

(vi) $t(4, 4, 4, 4) \geq 18$.

(vii) $t(4, 4, 4, 4, 4) \geq 25$.

(viii) $t(4, \ldots, 4)$ (*k arguments*) $\leq k^2 + k + 1$.

(ix) $t(4, \ldots, 4)$ (*k arguments*) $\geq k^2 - k + 2$, *if $k - 1$ is a prime power.*

3.3 Calculation of $t(4, m)$ for $m = 5, 6$, and 7

In this section we evaluate the CO-irredundant Ramsey numbers $t(4, 5)$, $t(4, 6)$, and $t(4, 7)$. For each of these values, a proof will be given to establish $t(4, m) \leq n$. Then, a 2-edge colouring (R, B) of K_{n-1} will be given which contains no *cm* in R and no $c4$ in B, proving that $t(4, m) = n$. An edge colouring (R, B) of K_n with no *cl* in B and no *cm* in R will be referred to as a $t(l, m)$ *Ramsey colouring of* K_n.

3.3.1 $t(4, 5) = 8$

The first theorem of this section will be used in the calculation of all three numbers $t(4, 5)$, $t(4, 6)$, and $t(4, 7)$.

Theorem 3.3.1 *Let (R, B) be a $t(l, m)$ Ramsey colouring of K_n and consider an arbitrary vertex v. Then*

$$n - t(l, m-1) \leq deg_R(v) \leq t(l-1, m) - 1.$$

Proof Let $R_v = N_R(v)$. Then $deg_R(v) = |R_v|$. Suppose firstly that $|R_v| \geq t(l-1, m)$. If $B[R_v]$ contains a $c(l-1)$, X, then since all edges from v to R_v are red, $X \cup \{v\}$ is a *cl* in B, a contradiction. But then by the Ramsey property, $R[R_v]$ contains a *cm*, also a contradiction and thus the upper bound holds.

Let $B_v = N_B(v)$. If $|R_v| \leq n - t(l, m-1) - 1$, then $|B_v| \geq t(l, m-1)$. Since $B[B_v]$ does not contain a *cl*, it follows that $R[B_v]$ contains a $c(m-1)$ which, together with v, forms a *cm* in R, a contradiction. ∎

Theorem 3.3.2 $t(4, 5) = 8$.

Proof Let (R, B) be the 2-edge colouring of K_7 where $R \cong C_7$. Then R has no C_4, hence (by Theorem 3.1.4) B has no $c4$. Moreover R has no $c5$ and we conclude that $t(4, 5) > 7$.

In order to prove that $t(4, 5) \leq 8$, suppose to the contrary that (R, B) is a 2-edge colouring of K_8 with no blue $c4$ and no red $c5$. We establish a sequence of lemmas leading to contradictions. Let $V = \{1, \ldots, 8\}$

Lemma 3.3.3 *For any vertex v, $2 \leq \deg_R(v) \leq 3$.*

Proof of Lemma 3.3.3. By Theorem 3.3.1 $\delta(R) \geq 2$.

Next suppose that contrary to Lemma 3.3.3 the edges 12, 13, 14, 15 are all red. Then to avoid a C_4 in $R[\{1,\ldots,5\}]$, without loss of generality 2, 3, 4, 5 is the vertex sequence of a blue C_4.

If at most one of 24, 35 is red, then, say, 2 is isolated in $R[\{2,3,4,5\}]$ and since $\deg_R(2) \geq 2$, say $26 \in R$. Any vertex of $\{6,7,8\}$ sends at most one red edge to $\{2,3,4,5\}$ (avoid C_4 in R). Hence $R[\{2,3,4,5,6\}]$ has maximum degree at most one and $\{2,3,4,5,6\}$ is a red $c5$. We conclude that 24, 35 are red.

If, say, 6 sends no red edge to $\{2,3,4,5\}$, then $\{2,3,4,5,6\}$ is a red $c5$. Hence each of 6, 7, 8 send exactly one red edge to $\{2,3,4,5\}$.

Suppose, say, both 6 and 7 send their red edge to 2. Then $\{3,4,5,6,7\}$ is a red $c5$. Hence without loss of generality 26, 37, 48 are the only red edges between $\{6,7,8\}$ and $\{2,3,4,5\}$.

To avoid red C_4's 68, 16, 17, 18 are all blue and since $\delta(R) \geq 2$, 67 and 78 are red. There are no additional red edges i.e. R is completely specified. But $\{2,3,5,6,8\}$ is a $c5$ in R, a contradiction which establishes Lemma 3.3.3.

A vertex of R will now be called *saturated* when its degree in R is three (i.e. the maximum degree given by Lemma 3.3.3).

Lemma 3.3.4 *If* $1,\ldots,5$ *is the vertex sequence of a red* C_5*, then each vertex of* $Y = \{6,7,8\}$ *sends at most one red edge to* $X = \{1,\ldots,5\}$.

Proof of Lemma 3.3.4.

If Lemma 3.3.4 is false, then to avoid red C_4's without loss of generality 61,62 are red and 1, 2 are saturated. We have two cases to consider.

Case 1. 6 is isolated in $R[Y]$.

Since $\delta(R) \geq 2$ (by Lemma 3.3.3), 7 and 8 each send a red edge to $\{3, 4, 5\}$. At most three red edges join $\{3, 4, 5\}$ to $\{7, 8\}$ (saturation), hence to make $\delta(R) \geq 2$, $78 \in R$. To avoid C_4's in R, without loss of generality 73 and 85 are in R which implies that 74, 84, 83, 75 are all blue (avoid red C_4's). But now $\{1, 6, 7, 8, 4\}$ is a red $c5$.

Case 2. $67 \in R$.

Then 73, 75 are blue (avoid red C_4's). If $78 \in B$, then to ensure $\deg_R(7) \geq 2$, $74 \in R$. The degree requirement of 8 implies that 83 and 85 are red which forms a red C_4, a contradiction which shows that $78 \in R$.

Now $\{1, 2, 5, 6, 7\}$ is a red $c5$ unless 74 or 85 is red. If $74 \in R$, then 83 or 85 is red and a red C_4 is formed in each case. If 85 is red, then 74 and 83 are blue (avoid red C_4's). Now $\{1, 2, 3, 6, 7\}$ is a red $c5$ irrespective of the colour of 84.

Lemma 3.3.5 $\Delta(R) = 2$.

Proof of Lemma 3.3.5.

Suppose to the contrary that R has vertex 1 of degree three and 12, 13, 14 are red. To avoid red C_4's, $R[\{2, 3, 4\}]$ has at most one edge and any vertex of $\{5, 6, 7, 8\}$ sends at most one red edge to $\{2, 3, 4\}$.

Case 1. $\{2, 3, 4\}$ is independent.

Since $\delta(R) \geq 2$, each vertex of $\{2, 3, 4\}$ sends a red edge to $\{5, 6, 7, 8\}$ and (to avoid red C_4's) without loss of generality we may assume that 25, 36, 47 are all red. To avoid the red $c6$ $\{2, 5, 3, 6, 4, 7\}$ without loss of generality $56 \in R$ and then $\{2, 5, 3, 4, 7\}$ is a red $c5$ unless $57 \in R$. Lemma 3.3.4 now implies that $67 \in B$ and hence $\{2, 3, 6, 4, 7\}$ is a red $c5$.

Case 2. $23 \in R$.

If say 5 and 6 do not send red edges to $\{2, 3, 4\}$, then $\{2, 3, 4, 5, 6\}$ is a red $c5$. Hence one of the following subcases occur.

Subcase (i). 25, 36 and 47 are red.

Then $56 \in B$ (no red C_4) and 57, 67 are blue by Lemma 3.3.4. In order to make red degrees of 5, 6 and 7 at least two, we have that 8 has red neighbours 5, 6, 7, and this situation is impossible by Case 1.

Subcase (ii). 25, 46 and 47 are red.

Then 56 and 57 are blue (by Lemma 3.3.4) and so $58 \in R$ (degree of 5). Without loss of generality $86 \in R$ (degree of 8) and now $67 \in R$ (degree of 7). No further red edges are possible and $\{1, 3, 6, 7, 5\}$ is a red $c5$, a contradiction which completes the proof of Lemma 3.3.5.

By Lemmas 3.3.3 and 3.3.5, R is regular of degree two. Since there is no red C_4, $R \cong C_3 \cup C_5$ and contains a $c5$. This completes the proof of Theorem 3.3.2. ∎

3.3.2 $t(4,6) = 11$

Theorem 3.3.6 $t(4,6) = 11$.

Proof We first show that $t(4,6) > 10$. Let R' be the graph with $V = \{0, 1, \ldots, 9\}$ and edges so that 1, 3, 5, 7, 9 is the vertex sequence of a C_5 and 123, 345, 567, 789, 901 are C_3's. R' has no C_4 and hence $\overline{R'}$ has no $c4$ (Theorem 3.1.4). Suppose that X is a $c6$ of R'.

If X is independent, then $|X \cap \{1, \ldots, 5\}| \leq 2$ and $|X \cap \{6, \ldots, 0\}| \leq 3$. Hence $|X| \leq 5$, a contradiction.

Suppose that D is the vertex set of a component of $R'[X]$. If $|D| = 2$, then without loss of generality $D = \{1,2\}$ or $D = \{1,3\}$. If $D = \{1,2\}$, then $X - \{1,2\} \subseteq V - N[\{1,2\}] = \{4,5,6,7,8\}$ and it is easy to check that X is not a $c6$. If $D = \{1,3\}$, then $X - \{1,3\} \subseteq V - N[\{1,3\}] = \{6,7,8\}$ and $|X| \leq 5$, a contradiction.

Hence there exists D such that $|D| \geq 3$. Since $R'[D]$ contains no K_3 (there cannot exist an eXPN for the vertex of degree two), without loss of generality D contains $\{2,3,4\}$, $\{1,3,4\}$ or $\{1,3,5\}$. If $\{2,3,4\} \subseteq D$, then (since $G[D]$ contains no K_3) $D = \{2,3,4\}$ and 2 has no XPN. If $\{1,3,4\} \subseteq D$, then $5 \notin D$ and 4 has no XPN. If $\{1,3,5\} \subseteq D$, then neither 2 nor 4 are XPNs for 3 hence without loss of generality 1 is an XPN of 3. Hence 1 has degree one in $R'[X]$ and $X \cap \{2,4,0,9\} = \emptyset$. However $\{5,6,7\}$ is not contained in X and so $|X| \leq 5$, the final contradiction which proves that R' has no $c6$.

Therefore $(R', \overline{R'})$ is the required 2–edge colouring of K_{10} which shows that $t(4,6) > 10$.

In order to prove that $t(4,6) \leq 11$, suppose to the contrary that (R, B) is a 2-edge colouring of K_{11} with neither blue $c4$ nor red $c6$. By Thoerem 3.1.4, R has no C_4. We establish two properties, Lemma 3.3.7 and Lemma 3.3.8, of the graph R.

Lemma 3.3.7 *R has 8 vertices of degree three and 3 vertices of degree 4.*

Proof of Lemma 3.3.7. By Theorem 3.3.1, $\delta(R) \geq 3$. If R has at least four vertices of degree four or more then the number of edges in R is at least $\frac{1}{2}(4\times 4+7\times 3) = 18\frac{1}{2}$. However the Turan number $T(11, C_4)$ (i.e., the greatest number of edges in an 11–vertex graph with no C_4) is 18 [3], a contradiction. Hence R has at least 8 vertices of degree three.

Let R_v be the set of vertices joined by red edges to vertex v where $r = |R_v| \geq 5$. Let $B_v = V - (R_v \cup \{v\})$ and observe that each $u \in B_v$ sends at most one red edge to R_v (to avoid red C_4's). Hence the number of edges in $R[R_v]$ is at least

$$\lceil \tfrac{1}{2}(3r - |B_v| - r) \rceil = \lceil \tfrac{1}{2}(3r - (10 - r) - r) \rceil = \lceil \tfrac{3r}{2} - 5 \rceil > \tfrac{r}{2}\ .$$

for $r \geq 5$. Hence $R[R_v]$ contains a P_3 and so $R[R_v \cup \{v\}]$ has a C_4, a contradiction which proves $\Delta(R) \leq 4$.

Now R has either 8 or 10 vertices of degree 3. It remains to show that R cannot have ten vertices of degree three and one of degree four. Suppose to the contrary that $V = \{v, 1, 2, \ldots, 9, 0\}$, where $v1$, $v2$, $v3$, $v4$ are red while $1, \ldots, 9, 0$ all have degree three. Let $R_v = \{1, \ldots, 4\}$ and $B_v = \{5, \ldots, 9, 0\}$. Since $t(4,4) = 6$ and B has no $c4$. $R[B_v]$ has a $c4$ say W. If some $u \in R_v$ sent no red edge to B_v, then $W \cup \{u, v\}$ is a red $c6$ and we conclude that each $u \in R_v$ sends at least one red edge to B_v. Furthermore to avoid C_4's no $u \in B_v$ sends more than one red edge to R_v. Hence without loss of generality 15, 26, 37, 48 are red. At most two additional red edges (from 9, 0) link R_v to B_v. Therefore the number of red edges in $R[R_v]$ is at least $\frac{1}{2}[4 \times 3 - 10] = 1$. To avoid C_4's $R[R_v]$ has at most two (indepenent) edges. Suppose that $12 \in R$. If 34 is also in R then no $u \in R_v$ is adjacent (in R) to $\{9, 0\}$ ($\deg_R(u) = 3$) and so $R_v \cup \{9, 0\}$ is a red $c6$. Thus 12 is the only edge of $R[R_v]$ and without loss of generality $39 \in R$ ($\deg_R(3) = 3$). Now $R_v \cup \{8, 9\}$ is a red $c6$, unless $89 \in R$ and $R_v \cup \{7, 8\}$ is a red $c6$ unless $78 \in R$. Therefore 89 and 78 are red which produces the red C_4 3, 7, 8, 9, a contradiction which completes the proof of Lemma 3.3.7.

Lemma 3.3.8 *Vertices of degree four in R are adjacent.*

Proof of Lemma 3.3.8.

Let $V = \{\alpha, \beta, 1, \ldots, 9\}$ and suppose contrary to the statement that α and β have red degree four but $\alpha\beta \in B$.

Firstly assume that α and β have no common neighbour. Specifically let all edges from α to $\{1, 2, 3, 4\}$ and from β to $\{5, 6, 7, 8\}$ be red. Then vertex 9 sends three red edges to $\{1, \ldots, 8\}$ and hence at least two to $\{1, 2, 3, 4\}$ or to $\{5, 6, 7, 8\}$. Thus a C_4 is formed, a contradiction.

Secondly suppose that α and β have the common neighbour 4 in fact α, β send red edges to $\{1,2,3,4\}$ and $\{4,5,6,7\}$ respectively.

Each of 8, 9 send at most one red edge to $\{1,2,3,4\}$ and to $\{4,5,6,7\}$ (to avoid C_4's). Hence 84 and 94 are blue. Also both 8 and 9 send at least two red edges to $\{1,\ldots,7\}$ ($\delta(R) \geq 3$). We conclude:

- $deg_R(8) = deg_R(9) = 3$
- $89 \in R$
- each of 8, 9 sends precisely one red edge

  to $\{1,2,3\}$ and to $\{5,6,7\}$ (3.3.1)

Hence exactly 12 red edges join $\{1,\ldots,7\}$ to $\{\alpha,\beta,8,9\}$ and so the number of edges in $R[\{1,\ldots,7\}] = \frac{1}{2}[(4\times 1)+(6\times 3)-12] = 5$. Moreover to avoid C_4's both $R[\{1,2,3,4\}]$ and $R[\{4,5,6,7\}]$ have at most two edges.

Therefore without losing generality $26 \in R$ and since $\deg_R(4) \geq 3$, say $43 \in R$ ($42 \in B$ to avoid red C_4). The C_4-free property now also implies that 16, 25, 27, 13, 14, 23, 24, 35, 36, 37 and 46 are all blue. There are now two cases.

Case 1. 26 is the only edge in R from $\{1,2,3\}$ to $\{5,6,7\}$.

Then $R[\{1,2,3,4\}] \cong R[\{5,6,7,8\}] \cong 2K_2$ (to avoid C_4's and to achieve 5 edges in $R[\{1,\ldots,7\}]$). Hence $12 \in R$ and (recall Lemma 3.3.7) 4 is the third vertex of degree four in R. Since $46 \in B$, $57 \in B$ and hence without loss of generality 45 and

67 are red while 47, 56 are blue. By (3.3.1) without loss of generality 85 and 97 are red. Therefore in order to satisfy (3.3.1) and to avoid C_4's, 81 and 93 are in R. This completes R which has the $c6$ $\{1, 2, 4, 5, 7, 9\}$.

Case 2. There exists a second edge in R from $\{1, 2, 3\}$ to $\{5, 6, 7\}$.

Without loss of generality this second red edge is 15 which implies that 17 and 45 are blue (to avoid red C_4's). Since $\deg_R(7) \geq 3$ and the degree of 7 in $R[\{4, 5, 6, 7\}]$ is at most one, we may assume that $79 \in R$. The possibilities for the remaining two edges to make up the five of $R[\{1, \ldots, 7\}]$ are 12, 56, 75, 76, 74. Since 12 and 56 are not both red (red C_4), without loss of generality 76 or 74 is a red edge.

If $76 \in R$, then 75, 74, 65 are all blue (avoid red C_4's). The edge 12 is the only remaining possibility for the fifth edge of $R[\{1, \ldots, 7\}]$ which is now completely defined and has the $c6$ $\{6, 7, 1, 5, 4, 3\}$.

If $74 \in R$, then 76, 75 are blue (avoid C_4's in R) and 4 is the third vertex of degree four in R. Hence each of 5, 6 and 7 have red degree three. The two remaining candidates for the fifth edge of $R[\{1, \ldots, 7\}]$ are 12 and 56. If $56 \in R$, then 5, 6 and 7 are all saturated in R and 8 cannot send a red edge to $\{5, 6, 7\}$, a contradiction with (3.3.1). Therefore $12 \in R$ which saturates 1 and 2. Now only one of 8, 9 can send a red edge to $\{1, 2, 3\}$, again contradicting (3.3.1). This completes the proof of Case 2 and of Lemma 3.3.8.

By Lemma 3.3.8, the three vertices α, β, γ of red degree four (Lemma 3.3.7) form a red triangle. To avoid red C_4's, no pair from $\{\alpha, \beta, \gamma\}$ has a second common neighbour. Let 1, 2 (resp. 3, 4 and 5, 6) be the other two red neighbours of α (resp. β and γ). To avoid red C_4's the only possible edges in $R[\{1, \ldots, 6\}]$ are 12, 34 and 56. Then $\{1, \ldots, 6\}$ is a red $c6$ by Theorem 3.2.4. This final contradiction completes the proof of Theorem 3.3.6. ∎

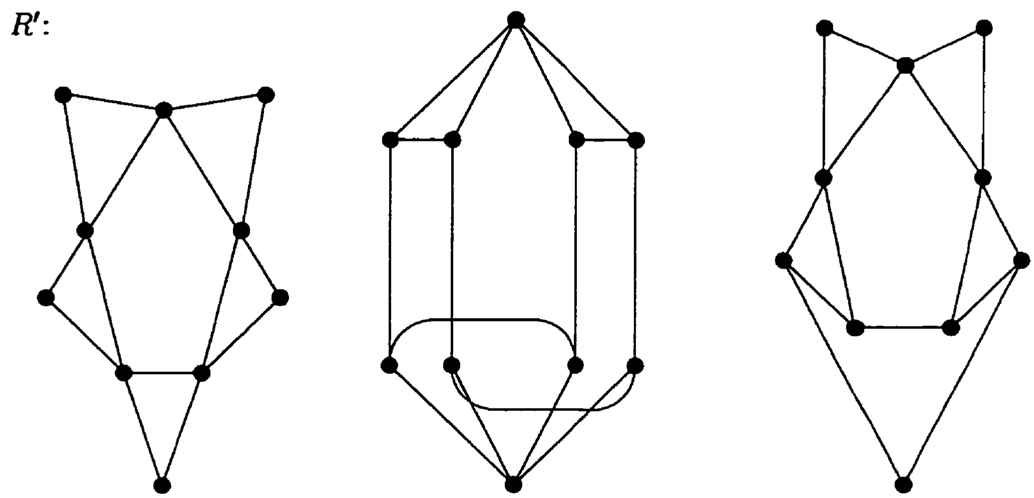

Figure 3.1: Three $t(4, 6)$-critical graphs

A $t(n_1, \ldots, n_k)$ Ramsey colouring of K_n is called $t(n_1, \ldots, n_k)$*-critical* if $n = t(n_1, \ldots, n_k) - 1$.

Analogous critical colourings for the 2-colour classical Ramsey numbers have been well-studied [28]. For example it is well known that the only $r(3, 3)$–critical colouring is $(C_5, \overline{C}_5)$.

Work on such critical colourings will appear elsewhere but preliminary investigations indicate that there are only three $t(4,6)$–critical colourings (R, B) with $\Delta(R) = 4$. The three graphs R are depicted in Figure 3.1. The graph R' is that used in the proof of Theorem 3.3.6 and criticality for all three cases was checked by a computer program written by G. MacGillivray (Appendix A).

3.3.3 $t(4,7) = 14$

The following additional notation will simplify the proof that $t(4,7) = 14$:

Given a 2-edge colouring (R, B) of K_n, each vertex v and its neighbours in R and B, respectively, induce a partition $(\{v\}, R_v, B_v)$ of $V(K_n)$ where

$$R_v = N_R(v)$$

$$B_v = N_B(v).$$

For any $x \in R_v$, define

$$S_{x,v} = \{u \in B_v : ux \in E(R)\}.$$

Note that $S_{x,v} = N_R(x) - R_v - \{v\}$. In addition, define

$$T_v = B_v - \bigcup_{x \in R_v} S_{x,v} = \{u \in B_v : ux \in E(B) forall x \in R_v\}.$$

Our evaluation uses the following theorem which contains many facts that were used in the proofs of earlier theorems without being formally stated.

Theorem 3.3.9 *Let $m \geq 4$. Consider a $t(4, m)$ Ramsey colouring (R, B) of K_n and let $v \in V(K_n)$ be arbitrary.*

(i) *Each vertex in B_v is adjacent (in R) to at most one vertex in R_v.*

(ii) $\Delta(R[R_v]) \leq 1$.

(iii) $|R_v| \leq m - 1$.

(iv) *For each $x \in R_v, |S_{x,v}| \leq m - |R_v|$.*

(v) *For each $x, y \in R_v$ with $xy \in E(R), |S_{x,v}| + |S_{y,v}| \leq m - |R_v| + 1$.*

Proof

i) If $u \in B_v$ is adjacent to $x, y \in R_v$ with $x \neq y$, then $uxvy$ is a C_4, contradicting Theorem 3.1.4.

ii) If $\Delta(R[R_v]) \geq 2$, then $R[R_v]$ contains P_3 as a subgraph, which forms a C_4 with v in R, again contradicting Theorem 3.1.4.

iii) Follows from Theorem 3.3.1.

iv) Suppose $|S_{x,v}| > m - |R_v|$ for some $x \in R_v$. Note that $\Delta(R[S_{x,v} \cup R_v - \{x\}]) \leq 1$ and $|S_{x,v} \cup R_v - \{x\}| = |S_{x,v}| + |R_v| - 1 \geq m$, a contradiction.

v) Suppose $x, y \in R_v$ with $xy \in E(R)$ and $|S_{x,v}| + |S_{y,v}| > m - |R_v| + 1$. By (*i*), $S_{x,v} \cap S_{y,v} = \emptyset$. Further, to avoid a C_4 in R containing x and y, there is no red edge between $S_{x,v}$ and $S_{y,v}$. Hence $\Delta(R[S_{x,v} \cup S_{y,v}]) \leq 1$, and if $X = S_{x,v} \cup S_{y,v} \cup R_v - \{x, y\}$, then $\Delta(R[X]) \leq 1$ and $|X| = |S_{x,v}| + |S_{y,v}| + |R_v| - 2 \geq m$, a contradiction. ∎

Theorem 3.3.10 $t(4, 7) = 14$

Proof We establish that $t(4, 7) \geq 14$ by constructing a graph R on 13 vertices which has no $c7$ and no C_4. Such a graph is given in Figure 3.2. Computer verification (Appendix A) confirms that (R, B), where R is the graph of Figure 3.2, is a $t(4, 7)$ Ramsey colouring of K_{13}.

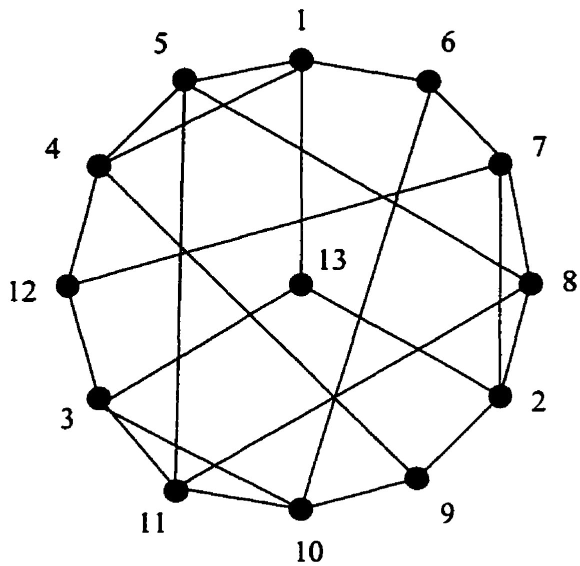

Figure 3.2: A graph on 13 vertices with no $c7$ and no C_4

It remains to be shown that $t(4, 7) \leq 14$. Suppose to the contrary that (R, B) is a $t(4, 7)$ Ramsey colouring of K_{14}. By Theorem 3.3.1, $3 \leq |R_v| \leq 6$ for each vertex $v \in V$. However, if there is a vertex v with $|R_v| = 6$, then by Theorem 3.3.9 (*iv*), $|S_{x,v}| \leq 1$ for each $x \in R_v$. Thus there is a vertex $u \in T_v$ and it follows from Theorem 3.3.9 (*ii*) that $R_v \cup \{u\}$ is a $c7$, a contradiction. Hence $3 \leq |R_v| \leq 5$ for each vertex $v \in V$. We now prove a series of lemmas.

Lemma 3.3.11 *R contains no adjacent vertices u and v of degree three and R contains no adjacent vertices u of degree four and v of degree three such that u and v lie on a common K_3.*

Proof In each case $|V(K_{14}) - N[\{u,v\}]| \geq 8$. But then $V(K_{14}) - N[\{u,v\}]$ contains a $c5$, S, as $t(4,5) = 8$. Thus $S \cup \{u,v\}$ is a $c7$, a contradiction.

Lemma 3.3.12 *For each vertex v, $3 \leq |R_v| \leq 4$.*

Proof Suppose $|R_v| = 5$. Since the maximum degree in $R_v \leq 1$ (to avoid C_4's), $|S_{x,v}| \geq 1$ for each $x \in R_v$. Since $|B_v| = 8$, $|S_{x_i,v}| = 1$ for at least two vertices x_i. These vertices are not isolated in $R[R_v]$ and by Lemma 3.3.11 are not adjacent. Therefore they are both adjacent to vertices y_1 and y_2 in R_v with $y_1 \neq y_2$ such that $|S_{y_i,v}| \geq 3$. But then $|B_v| \geq 8 + 2$ and $|V(R)| > 14$, a contradiction.

Lemma 3.3.13 *R is not 4-regular.*

Proof Since there are more than 9 vertices under discussion, we will now represent the edge uv by $u - v$ for clarity. Suppose R is 4-regular and consider an arbitrary vertex v. The 4-regularity of R and a counting argument show that $|T_v| = 1$, $|S_{x,v}| = 2$ for each $x \in R_v$ and $R[R_v] \cong 2K_2$. Let $T_v = \{u\}$, $R_v = \{1,2,3,4\}$ with 1-2 and 3-4 red, $S_{1,v} = \{5,6\}$, $S_{2,v} = \{7,8\}$, $S_{3,v} = \{9,10\}$ and $S_{4,v} = \{11,12\}$. Since $|R_v| = 4$ and to avoid C_4's, u is adjacent to at most one vertex in each $S_{i,v}$, $i \in R_v$, it follows that u is adjacent to exactly one vertex in each $S_{i,v}$. By symmetry we may assume that

u-6, u-8, u-10, and u-12 are red. By the above argument for $(\{u\}, R_u, B_u)$ it follows that $R[\{6, 8, 10, 12\}] \cong 2K_2$ and since 6-8 and 10-12 are blue (to avoid C_4's), we may assume without loss of generality that 6-12 and 8-10 are red. By also repeating the argument for $(\{6\}, R_6, B_6)$ we see that 5-6 and similarly 7-8, 9-10 and 11-12 are red. Consider vertex 5. Since $|R_5| = 4$, 5 is adjacent in R to two vertices in $\{7, 9, 11\}$. But 5-7 is blue (to avoid the red C_4 5-7-2-1) and 5-11 is blue (to avoid 5-11-12-6), a contradiction.

By the above lemmas R consists of vertices of degree three and four. We next show that R has a vertex of degree three which lies on a K_3.

Lemma 3.3.14 *R has a vertex v with $R[R_v] \cong K_1 \cup K_2$.*

Proof Suppose this is not the case. By Theorem 3.3.9 (ii) and Lemma 3.3.13 there exists a vertex v with $R[R_v] \cong \overline{K_3}$. By Lemma 3.3.11, $|S_{x,v}| = 3$ for each $x \in R_v$ and hence $|T_v| = 1$. Say $T_v = \{u\}$, $R_v = \{1, 2, 3\}$, $S_{1,v} = \{4, 5, 6\}$, $S_{2,v} = \{7, 8, 9\}$ and $S_{3,v} = \{10, 11, 12\}$. Since $3 \leq |R_u| \leq 4$ and u is adjacent to at most one vertex in $S_{i,v}$ for each $i \in \{1, 2, 3\}$, it follows that $|R_u| = 3$. Without loss of generality say $R_u = \{5, 8, 11\}$.

Consider the three edges 1-6, 2-9, and 3-12 and note that the only possible further red edges between these six vertices are edges in $R[\{6, 9, 12\}]$. To avoid the $c7$ $\{1, 6, 2, 9, 12, u\}$, at least one of these three edges is red; without loss of generality say 6-9 is red. Then 6-7 is blue to avoid a C_4. Now consider 1-6 and 2-7 and note that 7-x is red for at most one $x \in \{10, 12\}$. By symmetry we may assume that 7-10 is blue. To avoid the red $c7$ $\{1, 6, 2, 7, 3, 10, u\}$, 6-10 is red and thus 6-12 is blue. Then 7-12 is red to avoid the $c7$ $\{1, 6, 2, 7, 3, 12, u\}$. Considering 1-4, 2-7 and 3-10, we find similarly that 4-10 is blue since 6-10 is red, and so 4-7 is red. Now, 4-7 and 7-12 red implies 4-9 and 9-12 blue, respectively. Thus, to avoid the $c7\{1, 4, 2, 9, 3, 12, u\}$, 4-12 is red. Similarly $\{1, 4, 2, 9, 3, 10, u\}$ shows that 9-10 is red.

The set $\{4, 7, 6, 10, 8, u, v\}$ and the edge colouring described above now imply that 7-8 or 4-6 is red. But if 7-8 is red, then 8-9 is blue and so $\{4, 12, 6, 9, 8, u, v\}$ shows that 4-6 is red anyway. Similarly, 7-9 and 10-12 are red, but then we have the C_4's 4-6-9-7 and 4-6-10-12, a contradiction which completes the proof of Lemma 3.3.14.

To complete the proof that $t(4, 7) \leq 14$, let v be a vertex with $R[R_v] = K_1 \cup K_2$; say $R_v = \{1, 2, 3\}$, where 1-3 is red. Then 1 (and 3) can not have degree 3 (as v has degree 3) and can not have degree 4 (as it is in a K_3 with a vertex of degree 3). This contradicts Lemma 3.3.12 and completes the proof. ∎

3.4 Bounds on t(5,5)

The best known bounds for $t(5, 5)$ are given in our last result.

Theorem 3.4.1 $14 \leq t(5, 5) \leq 15$.

Proof The upper bound follows immediately from Theorem 3.1.2 since $t(4, 5) = t(5, 4) = 8$. The lower bound can be established with the following edge colouring of K_{13}. Let the vertices of K_{13} be labelled $0, 1, 2, \ldots, 12$ and (R, B) be the edge colouring of K_{13} in which each vertex v is adjacent in R to $v+1, v+3, v+4, v+9, v+10, v+12$ where addition is modulo 13. The computer program of the appendix verified that neither R nor B has a $c5$ and so $t(5,5) \geq 14$. ∎

In fact the graph R of Theorem 3.4.1 (depicted in Figure 3.3) is a self complementary circulant graph. It is easily checked that $f : v \to 2v$ is an isomorphism from R to B: For example, $(v, v+10)$ is an edge of R and $(f(v), f(v+10)) = (2v, 2v+20) = (2v, 2v+7)$ is an edge of B. The circulant structure and the self complementary property permit the lower bound to be established analytically.

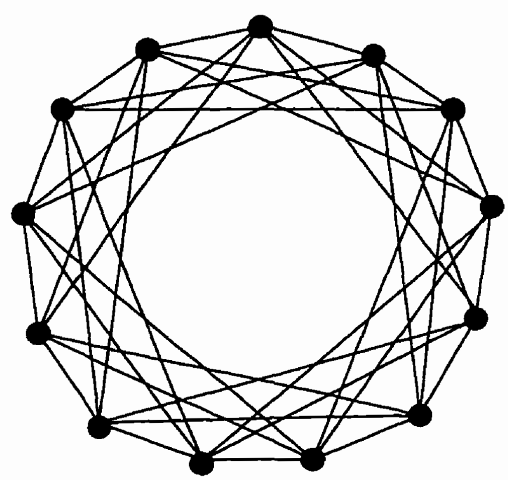

Figure 3.3: A self-complementary graph on 13 vertices with no $c5$

In view of Theorem 3.4.1, the value of $t(5,5)$ depends on the existence or non-existence of a 2-edge colouring (R, B) of K_{14} with no $c5$ in either R or B. Such a colouring must have the following properties. Firstly, Theorem 3.3.1 shows that for any vertex v, $6 \leq deg(v) \leq 7$. Hence all vertices must have degree 6 or 7 in both R

and B. Secondly, it is known that the generalized Ramsey number $R(K_5-2K_2, K_5-2K_2)=15$. Thus there exists a set X of 2-edge colourings of K_{14} in which neither colour has a K_5-2K_2. Because of Theorem 3.1.5, any colouring not in X contains a $c5$ in R or B. So far we have been unable to find a colouring in X without a $c5$ in at least one colour.

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Appendix A

Program For Finding CO-irr. Sets

```
program CoIR (input, output);

 const
  max_nu = 18;

 type
  vertex = integer;
  adjacency_matrix = array[1..max_nu, 1..max_nu] of vertex;
  vertex_list = array[0..max_nu] of integer;
  vertex_set = array[1..max_nu] of integer;

 var
  nu: integer;
  A: adjacency_matrix;
  x, y: vertex;
  co_ir_size: integer;
  co_ir_size_comp: integer;
  S: vertex_list;
  lastsubset: boolean;
  co_ir_found: boolean;

 procedure initialize_adjacency_matrix (var A: adjacency_matrix;
var nu: integer);
  var
   i, j: integer;
```

```
    x: vertex;
  begin
   for i := 1 to nu do
    for j := 1 to nu do
     A[i, j] := 0;
   for i := 1 to nu do
    begin
     while (not eoln(input)) do
      begin
       read(x);
       if (x <> i)  and (x >= 1) and (x <= nu) then
        begin
        A[i, x] := 1;
        A[x, i] := 1;
        end;
      end;
     readln;
    end;
   writeln;
   writeln;
   writeln('The adjacency matrix of your graph.');
   writeln;
   for i := 1 to nu do
    begin
     for j := 1 to nu do
      write(A[i, j] : 2);
     writeln;
    end;
  end;

  procedure complement_adjacency_matrix(var A: adjacency_matrix;
   var nu: integer);
 var
    i, j: integer;
 begin
    for i := 1 to nu do
       for j := i+1 to nu do begin
          A[i,j] := 1 - A[i,j];
```

```
        A[j,i] := 1 - A[j,i];
      end;
end;

 procedure first_kset (n, k: integer; var S: vertex_list;
var lastsubset: boolean);
{}
{Initialization for generation of all k-subsets of 1..n}
{in lexicographic order.}
{The k-sets are stored in S. The algorithm is from Reingold,}
{Neivergelt and Deo}
{Combinatorial Algorithms, page 181.}
{}
  var
   i: integer;
 begin
  for i := 0 to k do
   S[i] := i;
  for i := k + 1 to max_nu do
   S[i] := 0;
  lastsubset := false;
 end; { first_kset}


 procedure next_kset (n, k: integer; var S: vertex_list;
var lastsubset: boolean);
{}
{Generate the nextk-subsets of 1..n in lexicographic order and}
{return it in S}
{The algorithm is from Reingold,Neivergelt and Deo}
{Combinatorial Algorithms, page 181.}
{}
  var
   i, j: integer;

 begin
  lastsubset := (S[1] = n - k + 1);
  if not lastsubset then
```

```
   begin
    j := k;
    while (S[j] = n - k + j) do
     j := j - 1;
    S[j] := S[j] + 1;
    for i := j + 1 to k do
     S[i] := S[i - 1] + 1;
   end;
 end; { next_kset }


 procedure print_subset (var S: vertex_list; k: integer);
  var
   i: integer;

 begin
  for i := 1 to k do
   write(S[i] : 3);
  writeln;
 end; { print_subset }


 function co_irredundent (var S: vertex_list; k: integer; var A:
adjacency_matrix; nu: integer): boolean;
  var
   Nv, NS_minus_v: vertex_set;
   i, j, m, x, v: integer;
   v_has_pn: boolean;
   diffs_all_non_empty: boolean;

 begin
    diffs_all_non_empty := (k > 0);
    for i := 1 to k do
    begin
       v := S[i];
       for j := 1 to nu do
          Nv[j] := A[v,j];
       Nv[v] := 1;
```

```
        for m := 1 to nu do
           NS_minus_v[m] := 0;
        v_has_pn := false;
        for m := 1 to k do
        begin
           x := S[m];
           if (x <> v) then
              for j := 1 to nu do
                 if A[x, j] = 1 then
                    NS_minus_v[j] := 1;
        end;
        for j := 1 to nu do
           v_has_pn := v_has_pn or ((Nv[j] = 1) and (NS_minus_v[j] = 0));
        diffs_all_non_empty := diffs_all_non_empty and v_has_pn;
     end;
     co_irredundent := diffs_all_non_empty;
  end;


begin
 readln(nu);
 writeln('Number of vertices in the graph: ', nu:1);
 initialize_adjacency_matrix(A, nu);
 readln(co_ir_size, co_ir_size_comp);
 writeln;
 writeln('Size of the co-irredundent set to check for in G: ',
  co_ir_size:1);
 writeln('Size of the co-irredundant set to check for in G complement: ',
  co_ir_size_comp:1);
 first_kset(nu, co_ir_size, S, lastsubset);
 while (not lastsubset) and (not co_ir_found) do
  begin
   co_ir_found := co_irredundent(S, co_ir_size, A, nu);
   if (not co_ir_found) then
    next_kset(nu, co_ir_size, S, lastsubset);
  end;
 writeln;
 if co_ir_found then
```

```
  begin
   writeln('A co-irredundent set S of size ', co_ir_size : 1, ' was
    found:');
   write('S:');
   print_subset(S, co_ir_size);
  end
 else begin
   writeln('No co-irredundant set of size ', co_ir_size:1, ' was
    found in G.');
   complement_adjacency_matrix(A, nu);
   first_kset(nu, co_ir_size_comp, S, lastsubset);
   while(not lastsubset) and (not co_ir_found) do
   begin
      co_ir_found := co_irredundent(S, co_ir_size_comp, A, nu);
      if (not co_ir_found) then
         next_kset(nu, co_ir_size_comp, S, lastsubset);
   end;
   if co_ir_found then
   begin
      writeln('A co-irredundant set of size ',co_ir_size_comp:1,
       'was found in G complement');
      write('S: ');
      print_subset(S, co_ir_size_comp);
   end else
      writeln('No co-irredundent set S of size ', co_ir_size_comp :
       1, ' was found in G complement.');
   end;
end.
```

IMAGE EVALUATION TEST TARGET (QA-3)

150mm

6"

APPLIED IMAGE, Inc
1653 East Main Street
Rochester, NY 14609 USA
Phone: 716/482-0300
Fax: 716/288-5989