LAPLACE TRANSFORMS, PROBABILITIES AND QUEUES

by

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Abstract

In this thesis, we discuss a probabilistic interpretation of the Laplace transform of probability density functions (p.d.f.) for waiting times in queues. We interpret the Laplace transform of a p.d.f. as the probability that the corresponding random variable wins a race against (i.e., is less than) an exponential random variable. This interpretation is used to compute Laplace transforms of some p.d.f.'s, interpret some properties of the Laplace transform and prove some results for $M/G/1$ queues. In addition, we explore probabilistic interpretations of the z-transform (probability generating function) and its relationship to the Laplace transform.

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1 Introduction

The Laplace transform is an often used integral transform that is employed in many diverse fields of mathematics. It is particularly well known for its use in solving linear differential equations with constant coefficients. The study of stochastic processes also utilizes Laplace transforms in areas such as risk theory, renewal theory and queueing theory. In fact, many well-known results for $M/G/1$ queues are stated in terms of Laplace transforms.

We will restrict our study of Laplace transforms to queueing applications. We are, therefore, concerned with transforms of probability density functions (p.d.f.'s) corresponding to waiting times in queues. In this case, there is a probabilistic interpretation of the Laplace transform. The Laplace transform of a p.d.f. is the probability that the corresponding random variable is smaller than an exponential random variable with a particular rate. This interpretation can be employed to compute transforms of certain p.d.f.'s and prove relationships between quantities of interest in queueing theory without the standard computational and integration techniques.

The probabilistic interpretation of the Laplace transform was first introduced in the literature in 1949 by van Dantzig [21] whose original purpose was to give an interpretation of the z-transform (probability generating function). Van Dantzig's interpretation (which he called "the theory of collective marks") and its associated techniques were described by Runnenburg [19], [20]. In these papers, applications to queueing theory were emphasized. Råde also utilized these interpretations to solve problems in applied probability from a practical point of view, that would be understandable by both the technician and the theoretician [15].

Recently, Cong has completed a dissertation [4] and published articles [2, 3] on queueing theory and collective marks. In these papers, Cong derives results for queueing systems with complicated restrictions. Cong's results are more general and have shorter, more efficient proofs, than previous results regarding the same queueing models.

It is worth noting that van Dantzig, Runnenburg, Råde and Cong are all associated with the University of Amsterdam. While the probabilistic interpretation of Laplace transforms is known outside of Amsterdam, it does not seem to be well known and is definitely under-utilized as a tool in the analysis of queues. For instance, Lipsky [13] mentions the interpretation of Laplace transforms and Haight [9] notes the collective marks interpretation of the z-transform, but they do not use these insights to prove any results.

Kleinrock [12] also notes the interpretation and derives some renewal theory results, but fails to utilize it in situations where the proofs could be made more efficient and intuitive. Most standard queueing texts ignore this subject completely.

For the reasons above, the focus of this thesis to bring attention to the probabilistic interpretation of Laplace transforms and build upon this interpretation to provide a framework for the analysis of queues.

This thesis begins with relevant definitions in chapter 2, and a general discussion of Laplace transforms, probability distributions and random variables in chapter 3. Chapter 4 introduces the probabilistic interpretation of the Laplace transform of certain probability density functions and gives an intuitive interpretation of some of the properties of the Laplace transform. We compute transforms of several p.d.f.'s in chapter 5. Chapters 6 and 7 parallel chapters 4 and 5, this time giving interpretations of the z-transform and using these interpretations to calculate transforms of discrete distributions. Chapter 6 also discusses the close relationship between the Laplace transform and the z-transform. The results from chapters 4 through 7 are then applied to queues to produce results for $M/G/1$ systems in chapter 8. Finally, we make some concluding remarks and discuss some issues for future investigation in chapter 9, including some ideas on possible methods for inverting Laplace transforms.

The contributions of this thesis are the new proofs of theorems 4.3, 8.1, 8.3, the new proofs of properties 5.6, 7.1, 7.2, 7.3, and the introduction of corollaries 5.2 and 8.3 and property 5.3. The other significant contribution of this thesis is that it provides a collection of ideas concerning the probabilistic interpretation of transforms of probability distributions. The presentation of these results gives an alternate method for dealing with transforms in stochastic processes.

2 Definitions

In this chapter, we give the definitions, along with some discussion, of the continuous and discrete distributions that will be used in this thesis.

2.1 Continuous Distributions

Since we are focusing on waiting times, we will consider probability density functions, $f(x)$, with non-negative support, i.e., $f(x) \geq 0$ for $x \geq 0$ and $f(x) = 0$ for $x < 0$. Some densities of particular interest are the exponential, Erlang, generalized Erlang, hyperexponential and phase-type.

Definition 2. 1 *X is an exponential random variable with parameter $\lambda > 0$ (denoted $X \sim \mathrm{ex}(\lambda)$) if the p.d.f. of X is*

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x > 0 \\ 0 & \text{else.} \end{cases}$$

The exponential distribution plays a prominent role in queueing theory because of its "memoryless" property. This property, along with others, will be discussed in chapter 3.

Definition 2. 2 *X is an Erlang random variable with parameters (n, λ), $\lambda > 0$, n a positive integer ($X \sim \mathrm{Er}(n, \lambda)$), if the p.d.f. of X is*

$$f(x) = \begin{cases} \frac{\lambda^n}{(n-1)!} x^{n-1} e^{-\lambda x} & \text{for } x > 0 \\ 0 & \text{else.} \end{cases}$$

Note that the Erlang distribution is a special case of the gamma distribution $f(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}, x > 0$ where $\alpha = n$ and $\beta = \frac{1}{\lambda}$.

Definition 2. 3 *X is a generalized Erlang random variable ($X \sim genEr(\lambda_1, \lambda_2, \ldots, \lambda_n)$) if $X = \sum_{i=1}^{n} X_i$ where $X_i \sim \mathrm{ex}(\lambda_i)$ and the X_i's are mutually independent.*

The generalized Erlang distribution was originally designed to model non-exponential distributions by requiring that items pass through n (possibly fictitious) stages where the time spent at stage i is exponential with rate λ_i. The standard Erlang random variable is a special case of the generalized Erlang random variable with $\lambda_i = \lambda, i = 1, 2, \ldots, n$.

Definition 2.4 *X is a hyperexponential random variable with parameters $(\lambda_1, \ldots, \lambda_n, a_1, \ldots, a_n)$, $\lambda_i > 0, i = 1, 2, \ldots, n$, $a_i \geq 0, i = 1, 2, \ldots, n$ and $\sum_{i=1}^{n} a_i = 1$ ($X \sim$ hyperex$(\lambda_1, \ldots, \lambda_n, a_1, \ldots, a_n)$), if the p.d.f. of X is*

$$f(x) = \begin{cases} \sum_{i=1}^{n} a_i \lambda_i e^{-\lambda_i x} & \text{for } x > 0 \\ 0 & \text{else.} \end{cases}$$

The hyperexponential (also called the mixed exponential) distribution is used in queueing networks to model situations where there is uncertainty as to which of n parallel service nodes will be entered. The interpretation is that a customer will enter service node i with probability a_i and, upon entry, the service time will be exponentially distributed with rate λ_i.

Definition 2.5 *X is a phase-type random variable with parameters $(\alpha, \mathbf{T})$ ($X \sim \mathrm{PH}(\alpha, \mathbf{T})$) if the cumulative distribution function (c.d.f.) of X is*

$$F(x) = \begin{cases} 1 - \alpha e^{\mathbf{T}x}\mathbf{e} & \text{for } x \geq 0 \\ 0 & \text{else.} \end{cases}$$

where $\mathbf{e}$ is an $(m+1) \times 1$ vector of ones and α and $\mathbf{T}$ are defined below.

The phase-type distribution is characterized as the time until absorption for a continuous-time Markov process with rate matrix (infinitesimal generator)

$$Q = \begin{bmatrix} \mathbf{T} & \mathbf{T^0} \\ \mathbf{0} & 0 \end{bmatrix}$$

where there are m transient states and a single absorbing state, labeled $m+1$. We take $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ to be the initial probability vector and $\alpha_{m+1} = 1 - \sum_{i=1}^{m} \alpha_i$ is the probability of starting in state $m+1$. Here, $\mathbf{T}$ is $m \times m$, $\mathbf{T^0}$ is $m \times 1$ and $\mathbf{0}$ is a $1 \times m$ vector of zeros. Here, t_{ij} $(i \neq j)$, represents the rate at which we move to state j given that we are in state i. It is also worth noting that each row of Q sums to 0 (i.e., $\sum_{j=1}^{m+1} q_{ij} = 0, i = 1, \ldots, m+1$).

The phase-type distribution is extremely flexible with its choice of many parameters and can be used to model many distributions for stochastic processes. In fact, a result from Cox and Smith [5] (page 116) shows that any waiting time density can be approximated arbitrarily close by a phase-type density.

It is also worth noting that each of the probability densities defined in this section is a special case of the phase-type density [14].

2.2 Discrete Distributions

We consider discrete distributions for random variables that arise from two different situations. We are interested in variables that correspond to counts, for example, the number of customers arriving in a specified period or the number of customers in a queue. The other type of variables that we are interested in are variables that represent the number of steps until the occurrence of an event of interest. The particular variables we will be interested in are the geometric, Poisson and discrete phase-type.

Definition 2. 6 *N is a geometric random variable with parameter p ($N \sim geom(p)$) if the probability mass function for N is*

$$p_n = q^{n-1}p \qquad \text{for } n = 1, 2, 3, \ldots$$

where $0 < p < 1$ and $p + q = 1$.

The geometric random variable describes the number of independent trials until the first "success" where, on each trial, the probability of success is p.

Definition 2.7 *N is a Poisson random variable with parameter $\lambda > 0$ ($N \sim Poisson(\lambda)$) if the probability mass function for N is*

$$p_n = \frac{\lambda^n e^{-\lambda}}{n!} \qquad \textit{for } n = 0, 1, 2, \ldots .$$

The Poisson random variable plays an important role in variables corresponding to counts and, in particular, in several discrete aspects of queueing systems. The Poisson process is intimately related to the exponential distribution and has many interesting and useful properties which will be described in chapter 3.

Continuous phase-type densities have a discrete phase-type analog. A discrete random variable is of phase-type if it represents the number of steps until absorption in a Markov chain with transition matrix

$$P = \begin{bmatrix} \mathbf{T} & \mathbf{T^0} \\ \mathbf{0} & 1 \end{bmatrix}.$$

Again, there are m transient states and a single absorbing state, state $m+1$. We take $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ to be the initial probability vector and $\alpha_{m+1} = 1 - \sum_{i=1}^{m} \alpha_i$ is the probability of starting in state $m+1$. As before, $\mathbf{T}$ is $m \times m$, $\mathbf{T^0}$ is $m \times 1$ and $\mathbf{0}$ is a $1 \times m$ vector of zeros. Here, t_{ij} is the probability of moving to state j on the next step given that the system is in state i. We also note that each row of P sums to 1 (i.e., $\sum_{j=1}^{m+1} p_{ij} = 1, i = 1 \ldots, m+1$).

Definition 2. 8 *N is a discrete phase-type random variable with parameter $(\alpha, \mathbf{T})$ ($N \sim PH(\alpha, \mathbf{T})$) if the probability mass function for N is*

$$p_n = \begin{cases} \alpha_{m+1} & \text{for } n = 0 \\ \alpha \mathbf{T}^{n-1}\mathbf{T^0} & \text{for } n = 1, 2, 3, \ldots . \end{cases}$$

The events that correspond to absorption on step n, are starting in state i, moving from state i to state j in $n-1$ steps and finally moving from state j to state $m+1$, the absorbing state, on the n^{th} step. Clearly, $N = 0$ only if we start in state $m+1$ which occurs with probability α_{m+1}. Summing over all other starting states, we have, for $n = 1, 2, \ldots$,

$$\begin{aligned} & P(N = n) \\ = \ & \sum_{i=1}^{m} P(\text{start in } i) \sum_{j=1}^{m} P(i \rightarrow j \text{ in } n-1 \text{ steps}) P(j \rightarrow m+1 \text{ on } n^{th} \text{ step}) \end{aligned}$$

$$= \alpha \mathbf{T}^{n-1}\mathbf{T}^0.$$

3 Laplace Transforms, Random Variables and Probabilities

In this chapter, we discuss some general concepts regarding Laplace transforms and some particular results for Laplace transforms of probability density functions. In addition, we give some important results regarding the exponential and Poisson distributions.

3.1 Laplace Transforms

To begin with, we define the Laplace transform in the standard way [16], [22].

Definition 3. 1 *The Laplace transform of a function $f(x)$ is denoted by $f^*(s)$ and is given by*

$$f^*(s) = \int_0^\infty e^{-sx} f(x)\, dx,$$

where $s > 0$.

We will also be interested in the Laplace-Stieltjes transform.

Definition 3. 2 *The Laplace-Stieltjes transform of a function $f(x)$ is denoted by $f^*_{LS}(s)$ and is given by*

$$f^*_{LS}(s) = \int_0^\infty e^{-sx}\, dF(x).$$

The Laplace-Stieltjes transform is used to transform functions which possess both discrete and continuous parts and reduces to the standard Laplace transform in the fully continuous case.

Since we will be focusing on Laplace transforms of p.d.f.'s with non-negative support, we will discuss the convergence of the Laplace transform in this case.

Property 3. 1 *If $f(x)$ is a p.d.f. with non-negative support, then the improper integral*

$$\int_0^\infty e^{-sx} f(x)\, dx$$

converges uniformly for all $s \geq 0$.

Proof

Since $f(x) \geq 0$ for $x \geq 0$ and $0 \leq e^{-sx} \leq 1$ for all $x \geq 0$ and $s \geq 0$, the integrand $e^{-sx} f(x) \geq 0$ for $x \geq 0$ and $s \geq 0$. Also, $\int_0^t e^{-sx} f(x) dx \leq \int_0^t f(x) dx$.

for all $t \geq 0$. It follows that $\int_0^t e^{-sx} f(x)\, dx$ is increasing in t and bounded by 1. Therefore, the integral converges uniformly for all $s \geq 0$.

This result shows that the Laplace transform of all p.d.f.'s with non-negative support exists for all $s \geq 0$. Further, it must be the case that

$$0 \leq f^*(s) \leq 1 \text{ for all } s \geq 0.$$

This sets the stage for viewing Laplace transform as a probability.

Given a function $f^*(s)$, we wish to know whether or not $f^*(s)$ could be the transform of a density function with non-negative support. Widder [22] gives necessary and sufficient conditions for this determination as stated in property 3.2.

Property 3. 2 *The integral $f^*(s) = \int_0^\infty e^{-sx} dF(x)$, where $F(x)$ is a bounded non-decreasing function of x, converges for all s if and only if $f^*(s)$ is completely monotonic. That is,*

$$(-1)^k \frac{d^k}{ds^k} f^*(s) \geq 0 \textit{ for all } k \geq 0 \textit{ and } s > 0. \qquad (1)$$

If $f^*(s)$ is to be the Laplace transform of a p.d.f., say $f(x)$, then $F(x)$ is the corresponding c.d.f. Therefore, if we can confirm that $f^*(s)$ is completely monotonic and we consider $F(x) = \int_0^x f(t)\, dt$, we can conclude, from the above property, that $f(x)$ is positive and that $F(x)$ has a limit as $x \rightarrow \infty$ which we can scale to be 1. So $f(x)$ is non-negative for $x \geq 0$ and $\int_0^\infty f(t)dt = 1$ and thus, $f(x)$ can represent the p.d.f. of a waiting time. Note that an sufficient condition for (1) which is given by Apostol [1] is

$$\int_0^\infty x^n f(x)\, dx \text{ exists for } n = 1, 2, 3, \ldots .$$

This result will also give the uniform convergence of $\int_0^\infty e^{-sx} f(x)\, dx$ for all $s > 0$.

If any two continuous functions have the same Laplace transform, then those functions must be identical [17]. In this sense, all the important information regarding a density is contained within its Laplace transform. For example, all moments of a density function may be obtained directly from its Laplace transform.

Property 3.3 *If m_i denotes the i^{th} moment of X where the p.d.f. of X is $f(x)$, then*

$$m_i = (-1)^i \frac{d^i}{ds^i} f^*(s) \mid_{s=0} .$$

Proof

$$(-1)^i \frac{d^i}{ds^i} f^*(s) \mid_{s=0} \quad = \quad (-1)^i \frac{d^i}{ds^i} \int_0^\infty e^{-sx} f(x)\, dx \mid_{s=0}$$

$$
\begin{aligned}
&= (-1)^i \int_0^\infty \left(\frac{\partial^i}{\partial s^i} e^{-sx} \mid_{s=0}\right) f(x)\, dx \\
&= (-1)^i \int_0^\infty \left((-x)^i e^{-sx} \mid_{s=0}\right) f(x)\, dx \\
&= \int_0^\infty x^i f(x)\, dx \\
&= m_i. \blacksquare
\end{aligned}
$$

This result is related to the fact that the Laplace transform of a p.d.f. is its moment generating function, $M(t)$, evaluated at $t = -s$. For further discussion of Laplace transforms, the reader is referred to Rainville [16] or Widder [22].

3.2 Probabilities and Random Variables

In our interpretation, we will be interested in the probability that one random variable, say Y, exceeds another random variable, X, which we denote $P(Y > X)$. For continuous variables, this probability is defined by Hogg and Craig [10] to be

$$P(Y > X) = \int_0^\infty \int_x^\infty f(x, y)\, dy\, dx$$

where $f(x, y)$ is the joint p.d.f. of X and Y. Note that we restrict our random variables to those which are independent with non-negative support.

A more intuitive way to view this probability is

$$P(Y > X) = \int_0^\infty P(Y > x) f(x)\, dx \quad \text{(independence)}$$

where $f(x)$ is the p.d.f. of X. We read this expression as the probability that Y exceeds a specific value of x taken as a weighted average with respect to $f(x)$ over all possible values for X.

The discrete analog of this probability for discrete random variables M and N is

$$P(M > N) = \sum_{n=0}^{\infty} \sum_{m=n+1}^{\infty} p_{n,m}$$

where $p_{n,m}$ is the joint probability mass function of N and M. As in the continuous case, we may view this probability as

$$P(M > N) = \sum_{n=0}^{\infty} P(M > n) p_n$$

where p_n is the probability mass function for N and M and N are assumed to be independent.

3.3 Important Properties of the Exponential and Poisson Distributions

In this section, we will list several well-known properties that will be valuable in handling various queueing situations. Most of these results can be found in [18].

Property 3. 4 *The exponential distribution is "memoryless" - that is, if Y is an exponential random variable, then $P(Y > t + s \mid Y > s) = P(Y > t)$.*

Proof

Suppose $Y \sim \mathrm{ex}(\lambda)$. Then

$$\begin{aligned} P(Y > t + s \mid Y > s) &= \frac{P(Y > t + s)}{P(Y > s)} \\ &= \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} \\ &= e^{-\lambda t} \\ &= P(Y > t). \blacksquare \end{aligned}$$

This memoryless property is not confined to specific values of s and t as above, but can be extended to random variables.

Property 3. 5 *If Y is an exponential random variable and X_1 and X_2 are random variables with p.d.f.'s $f_1(x_1)$ and $f_2(x_2)$, respectively, where Y, X_1 and X_2 are all mutually independent, then*

$$P(Y > X_1 + X_2 \mid Y > X_1) = P(Y > X_2).$$

Proof

Suppose $Y \sim \text{ex}(\lambda)$, then

$$\begin{aligned}
& P(Y > X_1 + X_2 \mid Y > X_1) \\
= & \int_0^\infty \int_0^\infty P(Y > x_1 + x_2 \mid Y > x_1) f_1(x_1) f_2(x_2)\, dx_1\, dx_2 \\
= & \int_0^\infty \int_0^\infty P(Y > x_2) f_1(x_1) f_2(x_2)\, dx_1\, dx_2 \quad \text{(property 3.4)} \\
= & \int_0^\infty P(Y > x_2) f_2(x_2)\, dx_2 \\
= & P(Y > X_2). \blacksquare
\end{aligned}$$

It can also be shown that the only continuous density with this property is the exponential distribution [6].

An interpretation of the memoryless property is that the distribution of the time until the next event from a memoryless process is the same regardless of the time that an observer has already waited for the event to occur.

It should also be noted that the only discrete random variable that has this property is the geometric [11].

Property 3. 6 *The number of events in an interval* $(0, t)$ *is Poisson*(λt) *if and only if the time between events is ex*(λ).

Proof

Suppose that $N(t)$ is the number of events in $(0, t)$ and that $N(t) \sim Poisson(\lambda t)$. Let T be the time until the next event (starting at time 0). Then, we define

$$
\begin{aligned}
F_T(t) &= P(T \leq t) \\
&= 1 - P(T > t) \\
&= 1 - P(N(t) = 0) \\
&= 1 - \frac{(\lambda t)^0 e^{-\lambda t}}{0!} \\
&= 1 - e^{-\lambda t}.
\end{aligned}
$$

We recognize this expression as the c.d.f. of an exponential random variable with rate λ and therefore, $T \sim \mathrm{ex}(\lambda)$.

Now, suppose that the time between events, T, is exponential with rate λ. Then, the time of occurrence of the n^{th} event, denoted τ_n, is $\mathrm{Er}(n, \lambda)$.

Therefore,

$$
\begin{aligned}
P(N(t) \leq n) &= P(\tau_{n+1} > t) \\
&= \int_t^{\infty} \frac{\lambda^{n+1}}{n!} x^n e^{-\lambda x}\, dx \\
&= \sum_{i=0}^{n} \frac{-(\lambda x)^i e^{-\lambda x}}{i!} \Big|_{x=t}^{x=\infty} \qquad \text{(integration by parts)} \\
&= \sum_{i=0}^{n} \frac{(\lambda t)^i e^{-\lambda t}}{i!}.
\end{aligned}
$$

We recognize this last summation as the cumulative distribution of a Poisson random variable with parameter λt. Thus, we conclude that $N(t) \sim$ Poisson(λt). ∎

Property 3. 7 *If $X \sim \text{ex}(\lambda_1)$ and $Y \sim \text{ex}(\lambda_2)$ where X and Y are independent, then $min(X, Y) \sim \text{ex}(\lambda_1 + \lambda_2)$.*

Proof

Let $X \sim \text{ex}(\lambda_1)$ and $Y \sim \text{ex}(\lambda_2)$ and $Z = \min(X, Y)$. The cumulative distribution function of Z is

$$
\begin{aligned}
F_Z(z) &= P(Z \leq z) \\
&= P(\min(X, Y) \leq z) \\
&= 1 - P(\min(X, Y) > z)
\end{aligned}
$$

$$
\begin{aligned}
&= 1 - P(X > z, Y > z) \\
&= 1 - P(X > z)P(Y > z) \qquad \text{(by independence)} \\
&= 1 - e^{-\lambda_1 z} e^{-\lambda_2 z} \\
&= 1 - e^{-(\lambda_1 + \lambda_2) z}
\end{aligned}
$$

which we recognize as the cumulative distribution function of an exponential random variable with rate $\lambda_1 + \lambda_2$. ∎

Property 3.8 *If $X \sim \mathrm{ex}(\lambda_1)$, $Y \sim \mathrm{ex}(\lambda_2)$ where X and Y are independent, then $P(Y > X) = \frac{\lambda_1}{\lambda_1 + \lambda_2}$.*

Proof

$$
\begin{aligned}
P(Y > X) &= \int_0^\infty \int_x^\infty \lambda_1 e^{-\lambda_1 x} \lambda_2 e^{-\lambda_2 y} \, dy \, dx \\
&= \int_0^\infty \lambda_1 e^{-(\lambda_1 + \lambda_2) x} dx \\
&= \frac{\lambda_1}{\lambda_1 + \lambda_2}. \ ∎
\end{aligned}
$$

The Poisson process has been regarded as the mathematical model that captures the essence of a truly random process where no readily discernible pattern appears to anyone observing the process. The lack of pattern is largely due to the memoryless property of the exponential inter-event times associated with the Poisson process, and the fact that events are independent of each other.

With this randomness in mind, we have an intuitive explanation for the $P(Y > X)$ where X and Y are exponential random variables. If there are type 1 events occurring with exponential inter-event times at rate λ_1 per unit time and events of type 2 with exponential inter-event times occurring at rate λ_2 per unit time, then all together random events occur at rate $\lambda_1 + \lambda_2$. Now, since truly random events will fall uniformly on any interval, given that a known number of events have occurred in that interval [18], the probability that the first event is of type 1 is simply the proportion of events that are of type 1, namely, $\frac{\lambda_1}{\lambda_1+\lambda_2}$.

Property 3.9 *Let $N(t)$ be a Poisson process with mean λt in which we count two types of events. If an event is of type 1 with probability p and type 2 with probability $1-p$, then the processes $N_1(t)$ and $N_2(t)$ which count type 1 and type 2 events, respectively, are independent with $N_1(t) \sim \text{Poisson}(\lambda tp)$ and $N_2(t) \sim \text{Poisson}(\lambda t(1-p))$.*

Proof

$$P(N_1(t) = n, N_2(t) = m)$$

$$
\begin{aligned}
&= \sum_{k=0}^{\infty} P(N_1(t) = n, N_2(t) = m \mid N(t) = k) P(N(t) = k) \\
&= P(N_1(t) = n, N_2(t) = m \mid N(t) = n + m) P(N(t) = n + m) \\
&= P(N_1(t) = n, N_2(t) = m \mid N(t) = n + m) \frac{(\lambda t)^{n+m} e^{-\lambda t}}{(n+m)!} \\
&= \binom{n+m}{n} p^n (1-p)^m \frac{(\lambda t)^{n+m} e^{-\lambda t}}{(n+m)!} \\
&= \frac{(\lambda tp)^n e^{-\lambda tp}}{n!} \frac{(\lambda t(1-p))^m e^{\lambda t(1-p)}}{m!}.
\end{aligned}
$$

Therefore, the marginals of $N_1(t)$ and $N_2(t)$ are $\frac{(\lambda tp)^n e^{-\lambda tp}}{n!}$ and $\frac{(\lambda t(1-p))^m e^{-\lambda t(1-p)}}{m!}$, respectively, and the result follows. ∎

4 Laplace Transforms and the Catastrophe Process

In this section, we interpret the Laplace transform of probability density functions as the probability that the corresponding random variable "wins a race" against an exponentially distributed catastrophe. We also use this interpretation to give intuitive explanations of some of the properties of the Laplace transform.

Our interest is in a process which generates events where the time until the next event has p.d.f. $f(x)$. To calculate the Laplace transform of $f(x)$, consider an independent process that generates "catastrophes" (a catastrophe is simply another type of event). If the time between catastrophes is distributed as an exponential random variable with rate s then we find that the Laplace transform of the distribution of the time until the next event, $f^*(s)$, is simply the long-term proportion of time that the event occurs before the catastrophe. This result is summarized in the following theorem.

Theorem 4.1 *Let X and Y be independent random variables. Further, suppose that $Y \sim \mathrm{ex}(s)$ and the p.d.f. of X is $f(x)$. Then,*

$$f^*(s) = P(Y > X).$$

Proof

$$\begin{aligned} P(Y > X) &= \int_0^\infty \int_x^\infty f(x) s e^{-sy} dy\, dx \\ &= \int_0^\infty f(x) e^{-sy} \Big|_{y=\infty}^{y=x} dx \\ &= \int_0^\infty f(x) e^{-sx} dx \\ &= f^*(s). \blacksquare \end{aligned}$$

It is worth noting that since the time until the next catastrophe is exponential, the catastrophe process is Poisson and memoryless. These facts will be of great importance in later chapters.

Since we are dealing with joint probability density functions, by Fubini's theorem [7], we may change the order of integration in the proof of Theorem 4.1. As a result, we find an expression for the Laplace transform of the c.d.f. of X that relates it to the probability $P(Y > X)$.

Theorem 4. 2 *Let X and Y satisfy the conditions in Theorem 4.1. Let $F(x)$ be the c.d.f. of X where $f(x) = \frac{d}{dx}F(x)$. Then*

$$F^*(s) = \frac{1}{s}P(Y > X).$$

Proof

$$\begin{aligned} P(Y > X) &= \int_0^\infty \int_0^y f(x) s e^{-sy} dx\, dy \\ &= \int_0^\infty s e^{-sy} \left(\int_0^y f(x) dx \right) dy \\ &= s \int_0^\infty e^{-sy} F(y) dy \\ &= sF^*(s). \blacksquare \end{aligned}$$

Using the result of Theorems 4.1 and 4.2, we obtain a well known result relating the Laplace transform of a function and its derivative (for the special case of the class of probability densities with non-negative support) as described in the following corollary.

Corollary 4. 1 *Let $F(x)$ and $f(x)$ be, respectively, the c.d.f. and the p.d.f. of a random variable X. Then $\frac{d}{dx}F(x) = f(x)$ and*

$$F^*(s) = \frac{1}{s} f^*(s).$$

Proof

From Theorems 4.1 and 4.2, $f^*(s) = P(Y > X) = sF^*(s)$ and the result follows. ∎

A classical result regarding the n-fold convolution of functions may be obtained using Theorem 4.1. The convolution of functions f_1 and f_2 is defined by Hogg and Craig [10] to be

$$f_1 * f_2(x) = \int_{-\infty}^{\infty} f_1(x-t) f_2(t)\, dt.$$

which reduces to the standard definition [17]:

$$f_1 * f_2 = \int_0^x f_1(x-t) f_2(t)\, dt.$$

because $f_1(x)$ and $f_2(x)$ are assumed to have non-negative support.

The convolution operator is associative and we denote the n-fold convolution of $f_1, f_2, \ldots, f_n$ as $f_1 * f_2 * \ldots * f_n$. The classical Laplace transform result for such functions is that the Laplace transform of the convolution of functions is the product of the Laplace transforms of each function in the convolution. That is,

$$f(x) = f_1 * f_2 * \ldots * f_n(x) \Longrightarrow f^*(s) = \prod_{i=1}^{n} f_i^*(s).$$

From distribution theory, we observe that the density function of the sum of n independent random variables is the n-fold convolution of the probability density functions of each random variable in the summation [10]. Thus, using Theorem 4.1, the Laplace transform of this convolution is the probability that the catastrophe happens after all n of the events occur in succession.

Theorem 4.3 *Let $X_1, X_2, \ldots, X_n$ be a sequence of n independent random variables where each X_i has p.d.f. $f_i(x_i)$. If $X = \sum_{i=1}^{n} X_i$ and the p.d.f. of X is $f(x)$ then,*

$$f^*(s) = \prod_{i=1}^{n} f_i^*(s).$$

Proof

Let $Y \sim \text{ex}(s)$. Then

$$
\begin{aligned}
f^*(s) &= P(Y > X) \\
&= P(Y > \sum_{i=1}^{n} X_i) \\
&= P(Y > X_1)P(Y > X_1 + X_2 \mid Y > X_1) \cdots \\
&\qquad \cdots P(Y > X_1 + X_2 + \ldots + X_n \mid Y > X_1 + X_2 + \ldots X_{n-1}) \\
&= P(Y > X_1)P(Y > X_2) \ldots P(Y > X_n) \qquad \text{(property 3.5)} \\
&= f_1^*(s) f_2^*(s) \ldots f_n^*(s). \blacksquare
\end{aligned}
$$

Since the catastrophe process is memoryless, if we are given that k events have occurred before the catastrophe, we simply reset the "race" between the length of time for the remaining $n - k$ events to occur and the catastrophe.

Our probabilistic interpretation also allows us to numerically compute a Laplace transform using simulation (or real data), provided that we are able to simulate (or obtain) random values from the density function in question. To compute $f^*(s)$ for particular values for s, we can simulate a series of exponential values $\{y_1, y_2, \ldots, y_n\}$ at rate s and a series of values from the density in question, $f(x)$, $\{x_1, x_2, \ldots, x_n\}$ and take the proportion of pairs (x_i, y_i) such that $y_i > x_i$. In the limiting case, this is exactly the value of the Laplace transform.

5 Transforms of Continuous Distributions

In this chapter. we compute the Laplace transforms for some important p.d.f.'s of waiting times through the use of the probabilistic interpretation.

Property 5. 1 *If* $X \sim \text{ex}(\lambda)$. *then*

$$f^*(s) = \frac{\lambda}{\lambda + s}.$$

Proof

$$\begin{aligned} f^*(s) &= P(Y > X) \\ &= \frac{\lambda}{\lambda + s} \qquad \text{(Property 3.8).} \blacksquare \end{aligned}$$

Calculating the Laplace transform using the probabilistic interpretation this way requires no integration and since we have an intuitive feeling for $P(Y > X)$ when X and Y are exponential it is quite natural for us to derive the Laplace transform in this way.

Property 5. 2 *If* $X \sim \text{genEr}(\lambda_1 \lambda_2, \ldots, \lambda_n)$, *then*

$$f^*(s) = \prod_{i=1}^{n} \frac{\lambda_i}{\lambda_i + s}.$$

Proof

Since $X \sim \text{genEr}(\lambda_1 \lambda_2, \ldots, \lambda_n)$, $X = \sum_{i=1}^{n} X_i$ where $X_i \sim \text{ex}(\lambda_i)$ and thus,

$$\begin{aligned} f^*(s) &= P(Y > X) \\ &= P(Y > X_1 + X_2 + \ldots + X_n) \\ &= \prod_{i=1}^{n} f_i^*(s) \qquad \text{(by Theorem 4.3)} \\ &= \prod_{i=1}^{n} \frac{\lambda_i}{\lambda_i + s}. \blacksquare \end{aligned}$$

Corollary 5. 1 *If* $X \sim \text{Er}(n, \lambda)$, *then*

$$f^*(s) = (\frac{\lambda}{\lambda + s})^n.$$

Proof

This result follows directly from property 5.2 using the standard Erlang random variable with $\lambda_i = \lambda$ for $i = 1, 2, \ldots, n$. $\blacksquare$

Property 5. 3 *If* $X_{(k)}$ *is the* k^{th} *order statistic of a random sample,* $S_n = \{X_1, X_2, \ldots, X_n\}$, *of size* n, *where each* $X_i \sim \text{ex}(\lambda)$ *and we denote the density function for* $X_{(k)}$ *as* $f_k(x)$, *then*

$$f_k^*(s) = \prod_{i=0}^{k-1} \frac{(n-i)\lambda}{(n-i)\lambda + s}.$$

Proof

Let $S_i = \{X_1^{(i)}, X_2^{(i)}, \ldots, X_i^{(i)}\}$ denote a random sample of size i from an exponential distribution with rate λ. Then,

$$\begin{aligned} f_k^*(s) &= P(Y > X_{(k)}) \\ &= P(Y > X_{(1)})P(Y > X_{(2)} \mid Y > X_{(1)}) \cdots P(Y > X_{(k)} \mid Y > X_{(k-1)}). \end{aligned}$$

Now, we have n exponentials and the catastrophe simultaneously running a race. Given that j of the exponentials ($j = 1, \ldots, k-1$) have finished before the catastrophe, we reset the race and require that one of the remaining $n - j$ exponentials occurs before the catastrophe, at which point, we again reset the race. So, given that j events have occurred (finished the race), we need that the catastrophe is greater than the minimum of $n - j$ exponentials. Thus,

$$\begin{aligned} f^*(s) &= P(Y > X_{(1)})P(Y > X_{(2)} \mid Y > X_{(1)}) \cdots P(Y > X_{(k)} \mid Y > X_{(k-1)}) \\ &= P(Y > \min(S_n))P(Y > \min(S_{n-1})) \cdots P(Y > \min(S_{n-k+1})) \\ &= \frac{n\lambda}{n\lambda + s}\frac{(n-1)\lambda}{(n-1)\lambda + s} \cdots \frac{(n-k+1)\lambda}{(n-k+1)\lambda + s} \\ &= \prod_{i=0}^{k-1} \frac{(n-i)\lambda}{(n-i)\lambda + s}. \end{aligned}$$

In the above expression, the sample S_i contains i exponential random variables, each with rate λ and therefore, by property 3.3, $\min(S_i) \sim \text{ex}(i\lambda)$. ∎

We next derive a second expression for the Laplace transform of the k^{th} order statistic of n exponential random variables.

Property 5.4 *If $X_{(k)}$ is the k^{th} order statistic of a random sample, $S_n = \{X_1, X_2, \ldots, X_n\}$, of size n, where each $X_i \sim \mathrm{ex}(\lambda)$ and we denote the density function for $X_{(k)}$ as $f_k(x)$, then*

$$f_k^*(s) = \frac{\lambda n!}{(n-k)!(k-1)!} \sum_{i=0}^{k-1} \binom{k-1}{i} (-1)^{k-1-i} \frac{1}{\lambda(n-i)+s}$$

Proof

Here, we compute Laplace transform directly, using the standard p.d.f. for an order statistic [10].

$$\begin{aligned}
f_k^*(s) &= \int_0^\infty e^{-sx} f_k(x)\, dx \\
&= \int_0^\infty e^{-sx} \frac{n!}{(n-k)!(k-1)!} F(x)^{k-1} f(x)(1-F(x))^{n-k}\, dx \\
&= \int_0^\infty e^{-sx} \frac{n!}{(n-k)!(k-1)!} (1-e^{-\lambda x})^{k-1} \lambda e^{-\lambda x} (e^{-\lambda x})^{n-k} \\
&= \frac{\lambda n!}{(n-k)!(k-1)!} \int_0^\infty (\sum_{i=0}^{k-1} \binom{k-1}{i} (-e^{-\lambda x})^{k-1-i}) e^{-(\lambda(n-k+1)+s)x}\, dx \\
&= \frac{\lambda n!}{(n-k)!(k-1)!} \sum_{i=0}^{k-1} \binom{k-1}{i} (-1)^{k-1-i} \int_0^\infty e^{-(\lambda(n-i)+s)x}\, dx \\
&= \frac{\lambda n!}{(n-k)!(k-1)!} \sum_{i=0}^{k-1} \binom{k-1}{i} (-1)^{k-1-i} \frac{1}{\lambda(n-i)+s}. \blacksquare
\end{aligned}$$

Properties 5.3 and 5.4 suggest one powerful advantage of the probabilistic interpretation of the Laplace transform when we compute a transform in two alternate ways. In the above two properties, we have derived an expression for $P(Y > X) = f_k^*(s)$ from a probabilistic point of view and an expression $f_k^*(s)$ in the standard way, so the resulting expressions must be equal. This gives the following corollary.

Corollary 5. 2

$$\prod_{i=0}^{k-1} \frac{(n-i)\lambda}{(n-i)\lambda + s} = \frac{\lambda n!}{(n-k)!(k-1)!} \sum_{i=0}^{k-1} \binom{k-1}{i} (-1)^{k-1-i} \frac{1}{\lambda(n-i)+s}.$$

The interpretation allows us to relate expressions that would not otherwise be readily computed or perhaps would not even be considered since they may appear to be unrelated. We will use this technique again to obtain some queueing theoretic results in chapter 8. In the structure of many queues, we often find it natural to look at things from many different points of view which enables us to use this technique.

Property 5. 5 *If* $X \sim \text{hyperex}(\lambda_1 \lambda_2, \ldots, \lambda_n, a_1, a_2, \ldots, a_n)$, *then*

$$f^*(s) = \sum_{i=1}^{n} a_i \frac{\lambda_i}{\lambda_i + s}.$$

Proof

Let $Y \sim \mathrm{ex}(s)$ and $X_i \sim \mathrm{ex}(\lambda_i)$. Then

$$\begin{aligned} f^*(s) &= P(Y > X) \\ &= \sum_{i=1}^{n} a_i P(Y > X_i) \\ &= \sum_{i=1}^{n} a_i \frac{\lambda_i}{\lambda_i + s}. \quad \blacksquare \end{aligned}$$

Property 5.6 *The Laplace-Stieltjes transform of the p.d.f. of a phase-type random variable, $X \sim \mathrm{PH}(\alpha, \mathbf{T})$ is $f^*_{LS}(s) = \alpha_{m+1} + \alpha(s\mathbf{I} - \mathbf{T})^{-1}\mathbf{T^0}$.*

Proof

Recall that the phase-type random variable is the time until absorption for a continuous time Markov process. Let q_i be the probability that absorption occurs before the catastrophe given that we start in state i. Clearly, $q_{m+1} = 1$. For all other i, at the time of first transition, either absorption is immediate or we move from state i to state j and the race restarts as if we had started in state j. Let us denote the (i, j) element of T by t_{ij} and the i^{th} element of T^0 as t_i^0. Thus, for $i = 1, 2, \ldots, m$,

$$\begin{aligned} q_i &= \frac{t_i^0}{\sum_{k=1:k\neq i}^{m} t_{ik} + t_i^0 + s} + \sum_{j=1:j\neq i}^{m} \frac{t_{ij}}{\sum_{k=1:k\neq i}^{m} t_{ik} + t_i^0 + s} q_j \\ &= \frac{t_i^0}{\sum_{k=1:k\neq i}^{m} t_{ik} + t_i^0 + s} + \sum_{j=1}^{m} \frac{t_{ij}}{\sum_{k=1:k\neq i}^{m} t_{ik} + t_i^0 + s} q_j - \frac{t_{ii}}{\sum_{k=1:k\neq i}^{m} t_{ik} + t_i^0 + s} q_i. \end{aligned}$$

Solving the above for q_i and using the fact that $\sum_{k=1}^{m} t_{i,k} + t_i^0 = 0$ gives

$$1 + \frac{t_{ii}}{\sum_{k=1:k\neq i}^{m} t_{ik} + t_i^0 + s} q_i = \frac{t_i^0}{\sum_{k=1:k\neq i}^{m} t_{ik} + t_i^0 + s} + \sum_{j=1}^{m} \frac{t_{ij}}{\sum_{k=1:k\neq i}^{m} t_{ik} + t_i^0 + s} q_j.$$

So

$$sq_i = t_i^0 + \sum_{j=1}^{m} t_{ij} q_j.$$

If. $\mathbf{q} = [q_1 . q_2 . \ldots . q_m]'$. we have. in matrix from.

$$s\mathbf{q} = \mathbf{T^0} + \mathbf{T}\mathbf{q}.$$

Solving for $\mathbf{q}$ gives the solution for the probability of absorption before catastrophe conditional on starting state as

$$\mathbf{q} = (s\mathbf{I} - \mathbf{T})^{-1}\mathbf{T^0}.$$

Finally. to solve for the Laplace-Stieltjes transform.

$$\begin{aligned} f_{LS}^*(s) &= P(Y > X) \\ &= \alpha_{m+1} + \sum_{i=1}^{m} \alpha_i q_i \\ &= \alpha_{m+1} + \alpha\mathbf{q} \\ &= \alpha_{m+1} + \alpha(s\mathbf{I} - \mathbf{T})^{-1}\mathbf{T^0}. \ \blacksquare \end{aligned}$$

The existence of $(s\mathbf{I} - \mathbf{T})^{-1}$ is ensured under the assumption that states 1 through m are transient [14]. Note that this result could have been obtained using matrix calculus but we choose to compute this Laplace transform in an intuitive way by exploring the underpinnings of the associated Markov process.

6 The z-transform and its Interpretations

The discrete analog of the Laplace transform is the z-transform. For a probability mass function $\{p_n, n = 0, 1, 2, \ldots\}$ of a discrete random variable, say N, the z-transform is defined to be

$$\tilde{p}(z) = \sum_{n=0}^{\infty} z^n p_n.$$

The function $\tilde{p}(z)$ is also known as the probability generating function.

In the discrete setting, if N represents the number of steps until the first occurrence of an event where $p_n = P(\text{first event occurs on } n^{th} \text{ step})$, we may consider a discrete geometric catastrophe process where the number of steps until the next catastrophe, M, is geometric with parameter (probability of success) $1 - z$. With this in mind, we obtain the following theorem.

Theorem 6.1 *Let* $M \sim \mathrm{geom}(1 - z)$ *and* N *be a discrete random variable with distribution* $\{p_n, n \geq 0\}$, *where* $p_n = P(\textit{first event occurs on } n^{th} \textit{ step})$.

Then.

$$\tilde{p}(z) = P(M > N).$$

Proof

$$\begin{aligned}
P(M > N) &= \sum_{n=0}^{\infty} p_n P(M > n) \\
&= \sum_{n=0}^{\infty} p_n P \left(\begin{array}{l} \text{at least } n \text{ failures from catastrophe process} \\ \\ \text{process before the first success} \end{array} \right) \\
&= \sum_{n=0}^{\infty} p_n z^n \\
&= \tilde{p}(z).
\end{aligned}$$

This z-transform interpretation is the exact parallel of the interpretation the Laplace transform. The Laplace transform of a probability density is the probability that the corresponding variable wins a race against (i.e.. is less than) a memoryless catastrophe process and the z-transform is the probability that the variable of interest wins a race (i.e., in a fewer number of steps) against a discrete memoryless catastrophe process.

An alternate probabilistic interpretation comes from the theory of collective marks [21]. Let $\{p_n, n = 0, 1, 2, \ldots\}$ be the probability mass function for a random counting process on N. Then it is to our advantage to view the z-transform as a different probability than the one which was illustrated in Theorem 6.1. Suppose that we are counting things of interest in some stochastic process which we will call "items". As we count each item in the process, we randomly mark that item with probability z. Then the z-transform of the distribution of N is the probability that all items counted are marked.

Theorem 6. 2 *Let N be a discrete random variable with probability mass function $\{p_n, n = 0, 1, 2, \ldots\}$. If each item is marked with probability z, then $\tilde{p}(z) = P($all items are marked$)$.*

Proof

$$\begin{aligned}
\tilde{p}(z) &= \sum_{n=0}^{\infty} p_n z^n \\
&= \sum_{n=0}^{\infty} P(\text{observe } n \text{ items}) P(n \text{ observed items are marked}) \\
&= P(\text{all items in the process are marked}). \blacksquare
\end{aligned}$$

As with the Laplace transform, we can obtain all moments of a variable N from the $\tilde{p}(z)$ by manipulating the following result.

Property 6. 1 *Consider a discrete random variable N with probability gen-*

erating function $\tilde{p}(z)$, then

$$E(N(N-1)\cdots(N-i+1)) = \frac{d^i}{dz^i}\tilde{p}(z)\mid_{z=1}.$$

In addition, we can also obtain the elements of the actual probability mass function from its generating function.

Property 6. 2 *If $\tilde{p}(z)$ is the generating function for a probability mass function $\{p_n, n = 0, 1, 2, \ldots\}$ then,*

$$p_n = \frac{1}{n!}\frac{d^n}{dz^n}\tilde{p}(z)\mid_{z=0}.$$

7 Transforms of Discrete Distributions

Property 7. 1 *Let $N \sim \mathrm{geom}(p)$ and $q = 1 - p$ then*

$$\tilde{p}(z) = \frac{pz}{1-qz}.$$

Proof

Let $M \sim \mathrm{geom}(1-z)$. Then,

$$\begin{aligned}\tilde{p}(z) &= P(M > N) \\ &= P(\text{success and no catastrophe on step 1})\end{aligned}$$

$$
\begin{aligned}
&+P(\text{failure})P(\text{no catastrophe on step } 1)P(M > N)) \\
&= P(N = 1, M > 1) + P(N > 1)P(M > 1)P(M > N \mid N > 1, M > 1) \\
&= pz + qz\tilde{p}(z)
\end{aligned}
$$

and solving for $\tilde{p}(z)$, the result follows. ∎

The above property illustrates the advantage of the memoryless property of the catastrophe processes. Here, if there is no catastrophe and no success, we simply restart the race. The probability that the event occurs before the catastrophe in the new race must again be $\tilde{p}(z)$. With the quantity we are pursuing, $\tilde{p}(z)$, appearing on both sides of the equation, we simply solve for $\tilde{p}(z)$.

Property 7.2 *Let* $N \sim$ Poisson(λ). *Then*

$$\tilde{p}(z) = e^{-\lambda(1-z)}.$$

Proof

Let the probability of an event being marked be z. Since N is Poisson(λ), the number of marked customers N_m is Poisson(λz) and the number of non-marked customers. N_{nm} is Poisson($\lambda(1-z)$) by property 3.9.

$$\begin{aligned}\tilde{p}(z) &= P(\text{all events marked}) \\ &= P(\text{no non-marked event in the process}) \\ &= P(N_{nm}=0) \\ &= \frac{(\lambda(1-z))^0 e^{-\lambda(1-z)}}{0!} \\ &= e^{-\lambda(1-z)}. \blacksquare\end{aligned}$$

Property 7.3 *Let $N \sim PH(\alpha, \mathbf{T})$ then*

$$\tilde{p}(z) = \alpha_{m+1} + z\alpha(\mathbf{I} - z\mathbf{T})^{-1}\mathbf{T}^0.$$

Proof

Recall that a discrete phase-type random variable is characterized as the number of steps until absorption. Let $M \sim geom(1-z)$ be the number of steps to achieve the next catastrophe. Let q_i be the probability that absorption occurs before the catastrophe given that we start in state i (i.e., $q_i = P(M > N \mid \text{ start in } i)$. Note that $q_{m+1} = 1$. We denote the (i, j) element of T by t_{ij} and the i^{th} element of T^0 as t_i^0.

Now, on the first transition, we either move from i to the absorbing state with probability t_i^0 or we move from state i to state j and restart the race.

That is,

$$q_i = zt_i^0 + z\sum_{j=1}^{m} t_{ij}q_j.$$

If we construct a vector of q_i's, $\mathbf{q} = [q_1, q_2, \ldots, q_m]'$, then

$$\mathbf{q} = z\mathbf{T^0} + z\mathbf{T}\mathbf{q}.$$

Solving for q gives

$$\mathbf{q} = z(\mathbf{I} - z\mathbf{T})^{-1}\mathbf{T^0}.$$

Again, we know $(\mathbf{I} - z\mathbf{T})^{-1}$ exists under the assumption that states 1 through m are transient [14].

Having solved for the probability of absorption before catastrophe, conditional on starting in state i, finding the probability that absorption occurs before the catastrophe is as follows.

$$\begin{aligned}
\tilde{p}(z) &= P(M > N) \\
&= \alpha_{m+1}q_{m+1} + \sum_{i=1}^{m} \alpha_i q_i \\
&= \alpha_{m+1} + \alpha\mathbf{q} \\
&= \alpha_{m+1} + z\alpha(\mathbf{I} - z\mathbf{T})^{-1}\mathbf{T^0}. \blacksquare
\end{aligned}$$

8 Results Related to Queueing Theory

In this chapter we will use the interpretations presented in chapters 4-7 to prove some results in queueing theory.

8.1 Laplace Transforms and the Busy Period

The busy period is the length of time from the beginning of service of the first customer to the first time when there are no customers in the system. In our analysis, we will require a specific characterization of the busy period that is outlined below.

The length of the busy period is independent of the order of service since as long as there is any work to be done, the server is still busy. Thus, it is to our advantage to consider the busy period under a Last-Come-First-Served (LCFS) discipline.

Under LCFS discipline, customer 1 arrives to begin the busy period, then some random number of customers, N, arrives during the service of customer 1. After service of customer 1 is completed, we then place the N^{th} customer who arrived during the service of customer 1 into service as though it had just arrived. Now, before we return to begin service on the $(N-1)^{st}$ customer that arrived during service of customer 1, we must serve all customers in a "personal" busy period associated with customer N. We conclude that the total busy period is the service time of customer 1 plus the sum of the busy periods associated with each customer who arrives during service of customer 1. Further, the distribution of the "personal" busy period associated with customers arriving during service of customer 1 is the same as the distribution of the entire busy period.

The distribution of the length of the busy period is generally difficult to compute, but for an $M/G/1$ queue, we can relate Laplace transforms of the distributions of the length of the busy period and the service time in a functional equation.

Theorem 8. 1 *For an $M/G/1$ queue, if the p.d.f. of the service time is $b(x)$ and the p.d.f. of the busy period is $g(x)$, then*

$$g^*(s) = b^*(s + \lambda(1 - g^*(s))).$$

Proof

$g^*(s)$ represents the probability that a busy period ends before the catastrophe. Now, each customer who arrives during service of customer 1 has a busy period associated with it and the probability that this particular customer's personal busy period will not end before (i.e., be interrupted by) the catastrophe is $1 - g^*(s)$. As customers arrive during the service of customer 1, we attach a mark to each customer designating whether or not that customer's busy period will be interrupted when it comes to run the race against the catastrophe. So we mark "catastrophic customers" (those whose busy period will be interrupted) with probability $1 - g^*(s)$ and "good customers" (who win the race) with probability $g^*(s)$. Now, the interarrival time of catastrophic customers denoted Y_c is $\text{ex}(\lambda(1 - g^*(s)))$. Let $Y \sim \text{ex}(s)$ be the time until the next catastrophe and X be the service time with p.d.f. $b(x)$.

$$
\begin{aligned}
g^*(s) &= P(\text{entire busy period ends before catastrophe occurs}) \\
&= P(\text{service time of customer 1 ends before the catastrophe} \\
&\qquad\qquad \text{and before the arrival of a catastrophic customer}) \\
&= P(min(Y, Y_c) > X) \\
&= b^*(s + \lambda(1 - g^*(s)))
\end{aligned}
$$

The busy period will only be interrupted by the catastrophe if the service time of customer 1 is interrupted or one of the personal busy periods is interrupted. Thus, racing the catastrophe against the busy period is equivalent to racing service time against the occurrence of the actual catastrophe and the arrival of a catastrophic customer.

8.2 Poisson Process and Random Intervals

Using both the catastrophe interpretation of Laplace transforms and the collective marking interpretation for probability generating functions, we obtain a series of classical results relating the z-transform of the number of occurrences in a random interval and the Laplace transform of the p.d.f. of the length of the interval where the events are governed by a Poisson process.

Theorem 8. 2 *Assume the number of events, N(t), occurring in* $(0, t)$ *is Poisson with* $p_n(t) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$. *Let* X *be the length of a random interval with p.d.f.* $f(x)$. *Let* $\tilde{p}(z)$ *be the z-transform of* $N(X)$. *Then,*

$$\tilde{p}(z) = f^*(\lambda(1 - z)).$$

Proof

If we mark the events in our Poisson stream with probability z then the "thinned" process which generates non-marked events is Poisson with rate $\lambda(1 - z)$. Let Y be the time between successive unmarked occurrences. Then

$Y \sim \text{ex}(\lambda(1-z))$. Therefore.

$$\begin{aligned}
\tilde{p}(z) &= P(\text{all events in random period are marked}) \\
&= P(\text{period ends before an unmarked event occurs}) \\
&= P(Y > X) \\
&= f^*(\lambda(1-z)). \blacksquare
\end{aligned}$$

In this situation. the "catastrophe" is the arrival of an unmarked event. Since N is distributed according to a Poisson process. the time between such catastrophes is exponential with rate $\lambda(1-z)$ and so we are racing the length of the interval against the $\text{ex}(\lambda(1-z))$ which we recognize. due to our interpretation. as a Laplace transform.

From Theorem 8.2. we obtain several specific results for an $M/G/1$ queueing system. In this system. since the arrival process is Poisson. we may apply Theorem 8.2 where $N(t)$ is the number of customers arriving in $(0. t)$ and the interval in question can be interpreted as any particular interval of interest.

Corollary 8. 1 *If U is the number of arrivals during the busy period and $g(x)$ is the p.d.f. of the length of the busy period then $\tilde{u}(z) = g^*(\lambda(1-z))$.*

Corollary 8. 2 *If V is the number of arrivals during a service period and $b(x)$ is the p.d.f. for service time then $\tilde{v}(z) = b^*(\lambda(1-z))$.*

Corollary 8. 3 *If Q is the number of customers arriving during a particular customer's total system time, then*

$$\tilde{q}(z) = w^*(\lambda(1-z)),$$

where $w(x)$ is the p.d.f. of a customer's total system time.

These results follow directly from Theorem 8.2.

It is interesting to note that the number of arrivals during a customer's system time is exactly the number of customers in the system at the time of service completion of that customer. This number of customers in the system after a service completion forms the standard Markov chain associated with an $M/G/1$ queue [8, 12, 18].

Corollary 8.3 also produces Little's formula in the case of an $M/G/1$ queue.

$$\begin{aligned}
\tilde{q}(z) &= w^*(\lambda(1-z)) \\
\Longrightarrow \frac{d}{dz}\tilde{q}(z)\mid_{z=1} &= \frac{d}{dz}w^*(\lambda(1-z))\mid_{z=1} \\
\Longrightarrow \tilde{q}'(1) &= w^{*\prime}(0)\lambda \\
\Longrightarrow E(Q) &= \lambda E(W).
\end{aligned}$$

Since Q is the number of customers in the system after a service completion. which has been shown to represent the number in the system at any point in time [8. 12]. $E(Q)$ is really the expected queue length which is usually denoted $E(L)$. Therefore. our result becomes

$$E(L) = \lambda E(W).$$

8.3 Number Served in the Busy Period

Using collective marks. we establish a functional relationship for the z-transform of the number of customers served in the busy period in terms of the service distribution.

Theorem 8. 3 *If R is the number of customers served during the busy period for an $M/G/1$ queue. then*

$$\tilde{r}(z) = zb^*(\lambda(1 - \tilde{r}(z))$$

where $b(x)$ is the p.d.f. of the service time. X.

Proof

As in Theorem 8.1. we will split the stream of customers arriving during the service of customer 1 into two independent streams. those whose personal busy period will contain an unmarked customer which is Poisson($\lambda(1-\tilde{r}(z))$) and those whose busy period contains no unmarked customers which is Poisson($\lambda\tilde{r}(z)$). If we denote the time until the next arrival of a customer whose personal busy period contains at least one unmarked customer as Y, then $Y \sim \mathrm{ex}(\lambda(1-\tilde{r}(z)))$. Now,

$$\begin{aligned}
\tilde{r}(z) &= P(\text{all those served in busy period are marked}) \\
&= P\left(\begin{array}{l}\text{customer 1 is marked and all others arriving} \\ \text{during busy period are marked}\end{array}\right) \\
&= zP\left(\begin{array}{l}\text{personal busy periods associated with each customer} \\ \text{arriving during service of customer 1 contains} \\ \text{no unmarked customers}\end{array}\right) \\
&= zP\left(\begin{array}{l}\text{service of customer 1 ends before the arrival} \\ \text{of a customer whose personal busy period} \\ \text{contains an unmarked customer}\end{array}\right) \\
&= zP(Y > X) \\
&= zb^*(\lambda(1-\tilde{r}(z))\,).
\end{aligned}$$

The proof of Theorems 8.1 and 8.3 are an improvement in length over the traditional proofs and force us to acquire a better understanding of the structure of an $M/G/1$ queue.

9 Conclusions

We have presented several results regarding the transforms of distribution which are common in queueing theory. We computed these results using probabilistic arguments, rather than the standard calculus techniques. We have also, using our interpretations, obtained new proofs of classical results for $M/G/1$ queues.

The application of the probabilistic interpretation of transforms certainly does not end here. There are many other situations that can be explored using the techniques outlined in this thesis. Areas that require further study include $GI/M/m$ and $GI/G/m$ queues. We wish to see how our interpretation may be used to obtain classical results for these queueing models. Not only should we be seeking new proofs of old results, but the probabilistic interpretation should allow us to find new results that only become clear with this new probabilistic perspective. We can also explore queueing systems with other constraints such as bulk service or arrival, balking, vacations and priorities as in the work started by Cong [4, 2, 3]. Outside of queueing theory, there is also the possibility to use this interpretation in the study of risk, renewal processes and other stochastic processes.

Another issue is the problem of inverting the transform. We have found ways of obtaining Laplace transforms and z-transforms in certain queueing situations, but have said nothing about how to invert the transform, which is often difficult or impossible. We are interested to see if we can use our probabilistic interpretation to create some method for inverting Laplace transforms. Given a Laplace transform or a set of points from a Laplace transform, perhaps we could somehow fit a Laplace transform of a known distribution through these data and come up with an approximation for the original density. Another avenue may be, for a specific value of s, to simulate random values from a exponential distribution at rate s and using the known Laplace transform, see if we can approximate a random sample from the original distribution.

The probabilistic interpretation of the Laplace transform and the z-transform give us some new insight into related problems in stochastic processes and will definitely provide a rich source of research material for the future.

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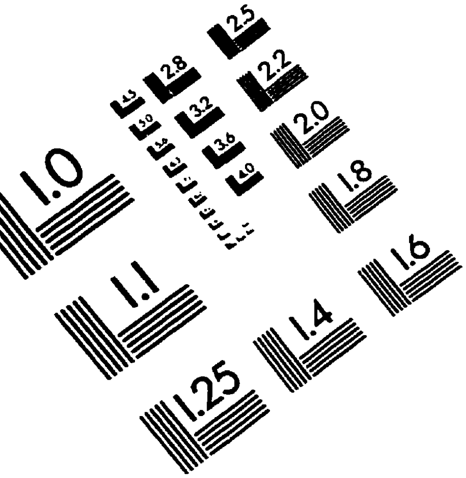
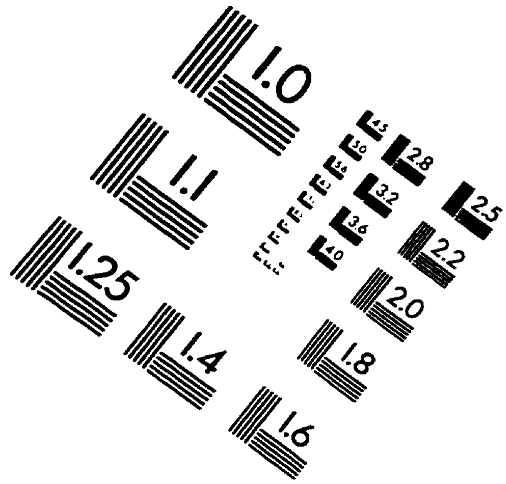
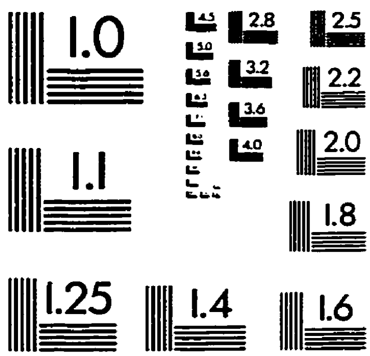
IMAGE EVALUATION TEST TARGET (QA-3)







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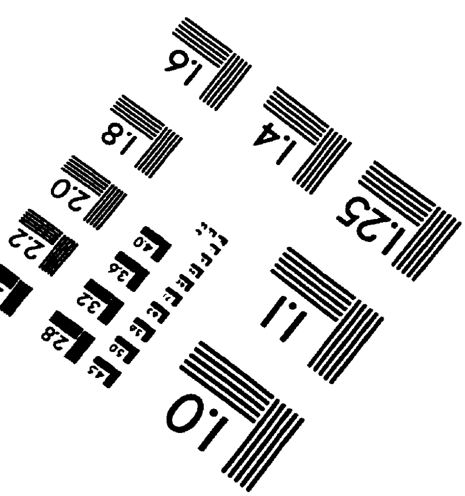

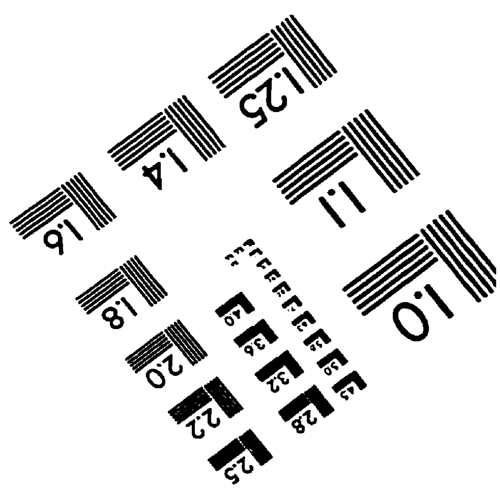

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