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Mixed Norm Control of a Helicopter

by

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ABSTRACT

This thesis presents the results of a study into the design and analysis of robust full authority flight control laws to improve the handling qualities for a fly-by-wire actuated Bell 205 helicopter using mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -optimization. The emphasis in design is to meet stringent US Army handling qualities specifications against the constraint of robust stability. Use of mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -optimization promises to address an outstanding problem in helicopter flight control which has been the unrealistic mapping of multiple design objectives into a single norm.

A full authority flight control system is expected to improve helicopter handling qualities and remove the pilot from the feedback loop. Control laws are designed to provide fast, reliable, repeatable responses to pilot command inputs for vertical velocity, pitch and roll attitudes, and yaw rate in an optimal fashion. The Bell 205 does not naturally exhibit good handling qualities, as the dynamic characteristics of the helicopter do not comply with military specifications.

A possible solution to robust flight controller design to improve the handling qualities of the aircraft is to incorporate optimal nominal $\mathcal{H}_2$ model following performance and $\mathcal{H}_\infty$ robust stability. This is formulated using a mixed $\mathcal{H}_2/\mathcal{H}_\infty$ controller synthesis where optimal nominal performance is represented as a model following design optimization of an ideal target model based on military specifications for the various aircraft modes. $\mathcal{H}_\infty$ -optimization is addressed via the incorporation of a structured uncertainty description of the aircraft model. Extensive effort has been expended to describe the error dynamics between the response of the model and the actual helicopter from both experimental data and pilot experience to reduce the trade-off between performance and robustness.

The mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem addresses genuine system requirements and results show effective robust controllers designed for both a benchmark problem and for the Bell 205 helicopter. Structured singular value analysis of system robustness is used to identify

acceptable variation in aircraft dynamics for which stability is assured. These variations agree well with simulation testing of robust stability limits. The $\mathcal{H}_2/\mathcal{H}_\infty$ controller on the other hand was found to predict very conservative limits.

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DEDICATION

The challenges of this world, I wish to face together with her. The accomplishment of challenges met, I wish to share with her. It is with these thoughts that I would like to acknowledge the inestimable support of Kimberly in the completion of this dissertation. I hope to offer her the same immeasurable support and I dedicate this work to her.

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LIST OF SYMBOLS

$\mathcal{R}$	field of real numbers
$\mathbb{C}$	field of complex numbers
$\in$	belong to
$\subset$	subset
$\not\subset$	not a subset
$\cap$	intersection
I_i	$i \times i$ identity matrix
$0_{i \times j}$	$i \times j$ zero matrix
M^T	transpose of M
M^*	complex conjugate transpose of M
M^{-1}	inverse of M
M^+	Moore-Penrose pseudoinverse of the matrix M
$\det(M)$	determinant of M
$\text{trace}(M)$	trace of M
$\lambda(M)$	eigenvalue of M
$\rho(M)$	spectral radius of M
$\bar{\sigma}(M)$	largest singular value of M
$\mu(M)$	structured singular value of M
$\text{im} M$	image (or range space) of M
$\mathfrak{N}_-(M)$	stable invariant subspace of M
$\mathfrak{N}_+(M)$	antistable invariant subspace of M
$D^\perp$	orthogonal complement of D
$S^\perp$	orthogonal complement of subspace S

$\ e\ $	norm of e (e is a vector, matrix, signal, or system)
$Re(\alpha)$	real part of $\alpha \in \mathbb{C}$
$Ric(H)$	Riccati operator
$r^{(d)} = dr^d / dt^d$	differentiation
$F_l(M, K)$	lower linear fractional transformation
$F_u(M, \Delta)$	upper linear fractional transformation
$\mathcal{H}_2$	square-integrable functions on the imaginary axis analytic in $\text{Re}(s) > 0$
$\mathcal{H}_\infty$	bounded functions analytic in $\text{Re}(s) > 0$
prefix $\mathcal{R}$	real rational, i.e. $\mathcal{RH}_\infty$
s	Laplacian domain
$\begin{bmatrix} A & B \\ C & D \end{bmatrix}$	shorthand of state space realization $C(sI - A)^{-1}B + D$
(A, B)	continuous time state transition and input matrices
K	controller
G	plant model or system matrix
$\delta(t)$	unit impulse
$g(t)$	impulse response matrix
$g*f$	convolution of g and f
Δ	perturbation matrix
w	external input
z	external or performance output
u	control signal
y	measured variables available for feedback

U	forward velocity
W	vertical velocity
Q	pitch rate
V	lateral velocity
P	roll rate
R	yaw rate
θ	pitch attitude
ϕ	roll attitude
r_W	vertical velocity command
r_θ	pitch attitude command
r_ϕ	roll attitude command
r_R	yaw rate command
τ	time constant
ω	natural frequency
ζ	damping ratio

ACRONYMS

ACAH	attitude command attitude hold
ACT	active control technology
ARE	algebraic Riccati equation
CGT	command generator technique
FBW	fly-by-wire
FCS	flight control system
FDLTI	finite dimensional linear time invariant
EMF	explicit model following
GCC	general control configuration
IMF	implicit model following

LFT	linear fractional transformation
LQ	linear quadratic
LQG	linear quadratic Gaussian
LQR	linear quadratic regulator
LTI	linear time invariant
LTR	loop transfer recovery
MIMO	multi-input multi-output
RCHH	rate command heading hold
SAS	stability augmentation system
SCAS	stability and control augmentation system
SISO	single-input single-output
SSV	structured singular value (μ)
UCE	user cue environment

CHAPTER 1

INTRODUCTION

1.1 Motivation

Motivation for this thesis extends from the following problem statement: how can robust controllers be designed to improve the handling qualities of a Bell 205 helicopter operating in a steady hover condition? The **HELICOPTER PROBLEM** is defined below and formulated to address the real world engineering control problem that is the primary motivation for this thesis.

HELICOPTER PROBLEM:

The helicopter control problem is to design robust flight controllers for a Bell 205 helicopter operating in a steady hover condition to meet known handling qualities specifications. The helicopter is fully instrumented so that all relevant aircraft states are measurable and configured to allow full authority fly-by-wire actuation. The real world control problem is formulated as a robust stability and tracker problem with model following behaviour for the nominal system as the performance criterion. Robust stability to plant perturbations is based on experimental and analytical models of the uncertainty dynamics.

The manual piloting of helicopters is challenging and mentally taxing. This is due, in part, to the inherent instabilities and highly nonlinear, cross-coupled nature of the

helicopter. It is difficult to design and build helicopters that naturally exhibit good handling qualities. Despite the shortcomings in handling qualities, pilots need to fly and perform critical missions effectively in helicopters. In particular, the inherent cross-coupling effects between the modes controlled by the pilot cause difficulty in handling and limit the desired flight path. This is of paramount concern when considering the tasks involved in such critical missions as precision hovering during search and rescue.

The design of flight control systems (FCS) for fixed-wing aircraft is a mature discipline [STE92]. In contrast, the application of control theory to helicopter flight control systems is much less well-developed and more complex [PIE96]. In the past low authority stability and control augmentation systems (SCAS) have been developed to provide artificial stability to ensure the aircraft would not wander off when the attention of the pilot was divided with other tasks in critical missions as the example cited above. The requirement then, on any helicopter FCS is to stabilize the aircraft and decouple the controlled inputs, thus reducing pilot workload, but still to allow fast and tailored response to pilot inputs.

Aircraft flight control design has been dominated by classical control techniques. Recent years have seen a growing interest in application of robust, nonlinear, and adaptive control theory. This development has been motivated in part by new possibilities offered by Active Control Technology (ACT). ACT applications include 'fly-by-wire' (FBW) and 'fly-by-light' technologies. The most significant change affecting handling qualities of fixed-wing aircraft has been the adoption of FBW control systems. In an effort to reduce pilot workload and provide the desired response to pilot inputs, handling qualities are to be improved through the use of ACT. The term 'active control' is used to signify that a digital computer is directly and dynamically involved in flight control. A full authority helicopter FCS with FBW is expected to improve the helicopter handling qualities providing the pilot with the aforementioned benefits. Full authority implies that the pilot is removed from the feedback loop with the FCS providing fast, reliable, repeatable responses to control inputs in an optimal fashion.

Generally speaking, the objective of control system design is to make a selected output of a real physical system behave in a desired way by manipulating the input based on a representative mathematical model. An important motivation for the use of feedback in control systems is the ability to provide satisfactory performance not only for operations under known conditions but also in the presence of uncertainties. It is important to realize at the outset, no mathematical model can precisely model a real physical system. There inevitably exists uncertainty in the plant model, which may have several origins including unmodelled high frequency dynamics, uncertainty in parameter values and linearization. It is therefore important that any controller designed to give a feedback system certain desirable properties does so not only to the nominal plant but also for every plant in the uncertainty set including the “worst-case” uncertainty. A controller that possesses this property is termed *robust*.

The term *robustness* as used in this thesis will refer to the extent to which a model of an open loop system may be changed from the nominal design model without destabilizing the overall closed loop system. In particular this thesis will only consider stability robustness where the closed loop system is guaranteed stable despite perturbations. It is assumed the control design uses linearized dynamics to synthesize a fixed dynamic compensator. Control design methodologies that directly address nonlinear system dynamics lead to *nonlinear* control systems; real-time changes in compensator structure result in *adaptive* control systems. These topics will not be addressed in this thesis.

It should also be appreciated that model uncertainty is not the only concern when it comes to robustness. If a control design is based on an optimization, then robustness problems may also be caused by the mathematical objective function not properly describing the real control problem. This is often overlooked in control system design but is a key and underlying concern in this thesis development. The aim of this thesis is to use modern control theory of feedback system analysis, design and synthesis to optimize the performance and robustness of control systems. This is in contrast with

traditional optimal control methods, such as the Linear Quadratic (LQ) theory, which optimize performance but not robustness.

In general, it is not practical to assess system robustness by experimentation alone and therefore robustness analysis tests will be presented and used determine the extent to which the design model can be perturbed without losing its desirable properties.

1.2 Previous Work and Related Literature

By the 1950s, classical frequency response methods (Nyquist diagram, Bode plot, Root locus, etc) for single-input single-output (SISO) systems, had developed into powerful design tools. The essential tool in development being complex analysis. The focus during this period of time was on loop shaping of SISO systems for stability, sensitivity reduction, noise suppression, etc. These tools provided practising engineers with procedures that ensure robust stability and certain desired performance levels are achieved. Design techniques, such as sequential loop closing, the characteristic locus method and inverse Nyquist array method, are generalizations of the classical SISO techniques. There are several reasons for the continued success of these methods for dealing with single-loop and multi-loop problems arising from some multi-input multi-output (MIMO) plant. Unfortunately these classical techniques do not readily extend to the multivariable case. Although these contributions succeeded in extending many of the classical frequency domain concepts to the multivariable case [ROS72], they failed to satisfactorily address the issue of robustness. Moreover these classical techniques can falter on MIMO problems that contain a high degree of cross-coupling between the controlled and measured variables [GRE95].

In the past, helicopter SCAS have usually used classical SISO techniques to design a series of single-loop designs one control loop at a time. Each axis is tuned for the desired response without introducing a pilot-induced oscillation (PIO) or other undesirable behaviour associated with high gain feedback [BAI89]. These single-loop design gains

were then reduced to prevent problems induced by multivariable SCAS operation in flight. Cross-couplings were reduced by using *ad hoc* cross-feed gains based on open loop aircraft models.

A completely different approach to frequency response design emerged from the work of Wiener on prediction theory for stochastic processes. He demonstrated that certain design problems involving quadratic integral performance indices may be solved analytically. The first success with control optimization came in the 1950s with the introduction of the Wiener-Hopf-Kalman theory of optimal control [YOU76], now most widely known as Linear Quadratic Gaussian (LQG) optimal control. The name originates from the assumption that the known plant dynamics are *linear* where the objective is to minimize *quadratic* integral performance indices subject to *Gaussian* white noise and disturbances. At roughly the same time the United States and the Soviet Union were funding a massive research program into the guidance and manoeuvring of space vehicles. As it turned out, the then new optimal control theory was well suited to many of the control problems that arose from the space program. There were two main reasons for this. The first being the underlying assumption of LQG theory is the plant has a known linear description, and the exogenous noise and disturbances acting on the feedback system are stochastic in nature, but have known statistical properties. Since space vehicles have dynamics that are essentially ballistic in character, it is possible to develop accurate mathematical models of their behaviour. In addition the description for external disturbances based on white noise are often appropriate in aerospace applications.

In the 1960s, aerospace engineers were interested in minimum fuel consumption such as minimizing the use of retrorockets. Performance criteria of this type are easily embedded in the LQG framework that was especially developed to minimize quadratic performance indices. Another revolutionary feature of the theory is that it offers a true synthesis procedure. It was widely believed the control system designer had finally been relieved of the burdensome task of designing by trial and error [LEH81]. Despite the wide spread

success of LQG optimal control in the aerospace sector, successful application in other sectors have been sporadic and few. It soon became apparent there was a serious mismatch between the underlying assumptions of the LQG theory and industrial control problems [GRE95].

The next major period in control system theory was the period between 1960 and 1975, during which the *state space* became the popular design space based in the time domain. Because the n dimensional state space does not lend itself to plotting, the graphical methods made popular in the frequency domain played less of a role in this period of control development. Rather, the essential tools used here were optimization, the calculus of variations, ordinary differential equations and Hilbert spaces. In the early 1960s Kalman introduced a number of key state variable concepts, i.e., controllability, observability, optimal linear quadratic state feedback, optimal state estimation (Kalman filtering), etc. Unfortunately, with a few notable exceptions, the problem of plant uncertainty was largely ignored during this period of time.

In the late 1970s a new interest in the problem of plant uncertainty began to appear. This interest at the beginning of the modern robust control period (1975 – present) would come from two papers written in the early 1960s, the time when plant uncertainty was largely ignored. The first was a paper by Zames written in 1963 introducing the concept of the “small gain” principle, which plays a key role in robust stability criteria [ZAM63]. The other paper written by Kalman in 1964 demonstrated that for SISO systems optimal LQ state feedback control laws had some very strong robustness properties [KAL64]. In 1977 Safanov and Athans demonstrated how these robustness properties extended to MIMO systems [SAF77]. It became apparent that LQG optimal controllers initially thought to have very attractive robustness properties could instead exhibit poor stability robustness. When state estimate feedback is used instead of state feedback these desirable robustness properties vanish. Doyle brought this to the attention of the control community in 1978 who showed, by example, there exist LQG combinations with arbitrarily small gain margins [DOY78]. LQG optimality does not automatically ensure

stability robustness and so the robustness of an LQG design must be analyzed *a posteriori*. Doyle and Stein were able to show, however, the desirable loop return difference properties of the optimal LQ state feedback control law could be asymptotically recovered by suitable design of the Kalman filter in the feedback loop [DOY79]. As a result of the considerable design experience with optimal LQ state feedback and LQG design during the state variable period, there was a strong interest in extending their robustness results. A multivariable robust design philosophy emerged, which was identified as the LQG/LTR (linear quadratic Gaussian loop transfer recovery) approach [DOY81]. Unfortunately, this method is restricted to minimum phase systems and therefore has limited applicability to a large class of engineering control problems [SKO96].

In the 1980s a modern version of complex analysis extended the classical notions of stability margins to MIMO systems in a systematic way. A number of papers on the robust control of multivariable systems appeared in the February 1981 issue of the IEEE TRANSACTIONS ON AUTOMATIC CONTROL on *Linear Multivariable Control Systems*. The main focus of these papers was the use of the singular values in the design of robust multivariable systems in the frequency domain. Basically, this represented an extension of the classical Bode design approach to multivariable systems. Unfortunately, stability and performance measures based on singular values are generally too conservative for structured plant variations [DOY81]. In 1982, Doyle introduced the concept of structured singular value (μ) to deal with this problem [DOY82]. The method of μ -analysis gives an effective tool for assessing robustness of the closed loop system in the presence of structured uncertainty. The structured singular value of a matrix M reduces to the singular value of the matrix M when perturbations are unstructured.

Motivated by the shortcomings of LQG control there was a significant shift towards $\mathcal{H}_\infty$ -optimization for the development of robust control synthesis procedures. One of the motivations for the original introduction of $\mathcal{H}_\infty$ methods by Zames [ZAM81] was to bring plant uncertainty, specified in the frequency domain, back to the forefront as it had been

in classical control in contrast to analytical methods such as LQG. In particular the $\mathcal{H}_\infty$ approach is more effective than the linear quadratic approach in characterizing robustness with respect to plant uncertainties [ZAM81], [ZAM83]. Zames argued the poor robustness properties of LQG could be attributed to the integral criterion in terms of the $\mathcal{H}_2$ -norm and also criticized the representation of uncertain disturbances by white noise processes as often unrealistic. While the quadratic performance index may be a meaningful cost function when the inputs and disturbances are known precisely, in many situations this is not possible in which case it is more meaningful to minimize the maximum error that can occur over the particular class of signals.

The important property of the $\mathcal{H}_\infty$ -norm comes from the application of the small gain theorem. In this case the $\mathcal{H}_\infty$ -norm is a very natural norm in evaluating stability margins, which guarantees stability for a feedback system if the product of the $\mathcal{H}_\infty$ -norms of the frequency response matrices in the feedback loop is less than one. The $\mathcal{H}_\infty$ -norm of a transfer function matrix being the maximum over all frequencies of its largest singular value. The loop shaping required in order to achieve a robust design is also easier to perform using $\mathcal{H}_\infty$ -optimization than with LQ optimal control. The synthesis of controllers that achieve an $\mathcal{H}_\infty$ -norm specification hence gives a well-defined mathematical problem with simple state space $\mathcal{H}_\infty$ controller formulae given in [DOY89]. The $\mathcal{H}_\infty$ control problem can be easily solved and can represent specifications for both performance and robustness. The main disadvantage was that no solution existed until 1989 and then it only addressed disturbance rejection and robust stability but not nominal performance.

The control of a helicopter is a genuinely multivariable problem in which one usually considers four inputs and four outputs with a significant degree of inter-axis coupling. Furthermore, a controller must drive multiple actuators based on information from multiple sensors and command inputs. In the case of such multivariable problems, the only effective and efficient means of obtaining a solution is to use multivariable control

design techniques. Classical control techniques that close a single-loop at a time are often inappropriate for designing control systems for MIMO plants and strong coupling between the loops. Further, classical techniques do not address robustness in multivariable systems adequately, as they only address stability margins in single-loops and neglect simultaneous perturbations in several loops. True multivariable analysis and design for helicopter flight control promises improved performance in the on-axis loops while offering better decoupling than previously experienced [PAR85], [MAN90]. Moreover, the use of modern multivariable control analysis tools allows a more rigorous analysis and consequently improved stability robustness of the closed loop system [LOW91]. The design of rotorcraft FCS, for robustness and performance, has been studied over a number of years using a variety of multivariable methods including: eigenstructure assignment [GAR90], [SAM90], [MAN92]; [POS92]; sliding mode control [PIE95]; $\mathcal{H}_2$ design [TAK93], [PIE96]; and $\mathcal{H}_\infty$ -optimization [YUE90].

Several approaches are available for the control system designer to meet handling qualities requirements. Possibly the most common is the use of eigenstructure assignment. Usually the designer specifies a relatively fast eigenvalue and an associated eigenvector direction that allows decoupling of the major axes in the aircraft. A method for specifying the eigenstructure of the closed loop system so that the speed of response (or bandwidth) is satisfactory with respect to the desired handling qualities specifications is shown in [GAR90].

A second method for meeting handling qualities requirements in rotorcraft is through the so-called loopshaping techniques. These are commonly referred to as the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control design techniques or the Youla parameterization methods. Essentially, the frequency domain shape of the (weighted) closed loop transfer function is forced to lie in a region of the gain plot for the transfer function. Typically, in order to meet bandwidth requirements, the region has a lower bound for small frequencies and an upper bound at high frequencies for noise and disturbance rejection.

Helicopter control is a genuinely multivariable problem in which previous efforts have considered the application of well-established techniques such as LQG/LTR and $\mathcal{H}_\infty$. The outstanding problem has been the inability to map realistic design objectives into a single norm, which the synthesis methods then optimize. Recently, there has been a great deal of interest in formulating a mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control methodology. To serve this purpose, it has been popular recently to optimize the $\mathcal{H}_2$ performance subject to an upper bound on the $\mathcal{H}_\infty$ -norm. The $\mathcal{H}_\infty$ -norm bound delivers a certain kind of stability margin and the $\mathcal{H}_2$ -norm represents a (scalar) performance measure, related to the energy behaviour of the output signal. Early approaches included solving the problem for one input/one output, one input/two outputs and two inputs/one output [BER89], [ZHO90], [DOY89], [YEH88], [MUS89], [MUS90], [KHA91]. Another alternative is mixed $\mathcal{H}_2/\mu$ which accounts for the structure of the uncertainty at the expense of a significant increase in the complexity of the synthesis procedure [HAL93]. A mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -norm problem with two exogenous input vectors and two controlled output vectors has been developed for full state feedback [ROT91]. The mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control is ever more realistic than $\mathcal{H}_2$ or $\mathcal{H}_\infty$ control alone. It can potentially take the best parts of $\mathcal{H}_2$ and $\mathcal{H}_\infty$ separately and piece them together for the **HELICOPTER PROBLEM** [TRE98A], [TRE98B].

One of the most attractive features of the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ design method is the ability to address both $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance objectives simultaneously. Although the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ -norms may often be genuine design objectives, to combine all the genuine objectives using a single norm as a cost function requires compromise. As with LQ and $\mathcal{H}_\infty$ synthesis, the norms do not directly include all of the design objectives. For example, LQ synthesis results in a controller that is adept at providing optimal nominal performance with potentially weak robustness characteristics. Similarly, an $\mathcal{H}_\infty$ controller results in a highly robust system with potentially poor performance. Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ design eliminates the need to compromise genuine design objectives and establishes a link between $\mathcal{H}_2$ and $\mathcal{H}_\infty$ -optimizations. Through this method it is possible to synthesize, in

an optimal fashion, flight controllers that directly addresses both the issue of $\mathcal{H}_\infty$ robust stability and optimal nominal $\mathcal{H}_2$ performance.

1.3 Contributions of the Thesis

This thesis is an extension of research efforts conducted at the Flight Research Laboratory of the Institute for Aerospace Research, a division of the National Research Council of Canada directed at the design and analysis of modern multivariable flight controllers on a full authority fly-by-wire actuated Bell 205 helicopter. It presents a critical assessment of the use of mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control in the design of robust flight controllers for the Bell 205 operating in a steady hover condition. The emphasis in the design is to meet stringent US Army handling qualities specifications against the constraint of robust stability to plant perturbations.

The **HELICOPTER PROBLEM**, formulated as a robust stability problem with optimal nominal performance, is solved using a mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control design. The optimal nominal performance is formulated as a model following design optimization of an ideal target model. Military handling qualities specifications for the various aircraft modes provide the basis for an ideal model. It is a designer goal to reproduce this ideal performance in the actual helicopter and synthesize a control system that ensures this behaviour [TRE96] with the additional constraint of robustness to plant uncertainty not included in the handling qualities specifications. Robustness to plant uncertainty is based on experimental and analytical uncertainty models that result in a linear fractional transformational description of the uncertain system dynamics. Structured singular value analysis of system robustness is used to identify acceptable variation in aircraft dynamics within a well-defined uncertainty structure for which stability is assured.

The primary contribution of this thesis is formulating realistic control problems including the **HELICOPTER PROBLEM** within the framework of known $\mathcal{H}_2/\mathcal{H}_\infty$ control problems. Specifically, genuine system requirements are translated using appropriately

selected norms into the necessary mathematical framework that properly describe the real control problem. An example application is given in terms of an accepted benchmark problem involving a mechanical system with large parameter uncertainty. Extensive effort has been expended to formulate the error dynamics between the response of the model and the actual helicopter from both experimental data and pilot experience. Secondly, the nontrivial control problem is solved and a control law synthesized to provide robust flight controllers that ensure $\mathcal{H}_\infty$ robust stability and optimal nominal $\mathcal{H}_2$ model following performance. The control problem is by far nontrivial due to the computational difficulty of the problem solution resulting from the high-order dynamics of the **HELICOPTER PROBLEM**. Lastly, robustness analysis of results by use of structured singular value is extended to the **HELICOPTER PROBLEM** as well as other qualitative and quantitative robust analysis tools.

This thesis advances the use of modern robust control theory in application to helicopter flight control through formulation of the **HELICOPTER PROBLEM** into a mathematical framework that more accurately describes genuine system requirements. A detailed study of the associated uncertainty error dynamics was performed and included in design to provide a structured description of the error dynamics between the mathematical model and the actual aircraft. The result is flight controllers designed to improve the handling qualities of the Bell 205 helicopter to meet US Army specifications with the additional constraint of robustness to model variation.

1.4 Organization

This thesis is organized into 7 chapters of which this chapter is the first. Chapter 2 explains the basic control system of helicopters, the complexities of piloting a helicopter, the limitations of existing control systems and the type of augmentation needed to improve handling qualities. Handling qualities specifications for the helicopter are also presented and used to formulate an ideal target model.

Chapter 3 demonstrates how to use the linear fractional transformation (LFT) to map realistic plant uncertainties into the appropriate mathematical framework for robust control. Experimental frequency domain responses of the actual helicopter provide a clear and reliable picture of the uncertainty between the nominal model and actual helicopter. Stability robustness tests of closed loop feedback systems for unstructured and structured uncertainty including μ -analysis are also presented.

Chapter 4 reviews the general $\mathcal{H}_2$ and $\mathcal{H}_\infty$ -optimal synthesis theory and motivates the use of mixed norm control. It presents the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -optimal synthesis theory and explores the effectiveness of the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -optimal control design in application to an accepted benchmark problem for general application. It is expected that the mixed norm control synthesis would be conservative yet effective in general application.

Chapter 5 presents model following control system design for the Bell 205 helicopter in hover. Two fundamentally different types of model following control designs, explicit and implicit, are discussed. Results of a study comparing the relative merits of controllers using both explicit and implicit model following techniques are used to select a controller most suited to the **HELICOPTER PROBLEM**.

Chapter 6 presents application of mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -optimization for the design of helicopter flight control systems. It is shown that controllers designed using mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -optimization improve the handling qualities of the aircraft that closely match US Army handling qualities specifications against the constraint of robust stability for the aircraft not included in the specifications.

Chapter 7 contains conclusions and suggestions for further research.

CHAPTER 2

HELICOPTER FLIGHT CONTROL

2.1 Introduction

This chapter provides a description of helicopter flight control including a brief overview of how helicopters are flown, the complexities of piloting helicopters, and the limitations of existing control systems for helicopter flight control. Active control technology is introduced as a means of improving the handling qualities of the helicopter. Handling qualities specifications for the helicopter are also presented and used to formulate an ideal target model.

In hovering flight longitudinal motion resolves into an uncoupled vertical velocity mode and an oscillatory mode coupling forward velocity and pitch attitude. Similarly, lateral motion reduces into an uncoupled yaw mode and an oscillatory mode coupling lateral velocity and roll attitude. Each of these coupled modes are dynamically unstable which implies that if the helicopter is perturbed in the longitudinal and lateral axes the amplitude of the pitch and roll attitudes would increase steadily if left uncorrected by the pilot. The task of the pilot is to stop these oscillations from developing using the basic helicopter control system described in §2.2. Flight control systems for the helicopter are also discussed including low authority stability augmentation systems, stability and control augmentation systems and the use of active control technology.

Handling qualities requirements for the Bell 205 helicopter are presented in §2.3 to meet the accepted standard USAAVSCOM 87-A-4 specifications. The handling qualities requirements are presented in the time and frequency domains and quite often in a form not quickly amenable for control system design. The aim of this section is to recast the requirements in a form suitable for the design and analysis of closed loop control systems. It is common for this goal to be achieved by setting the frequency domain requirements as specified bandwidths on the closed loop system while maintaining some degree of performance and stability robustness.

A state variable realization of an ideal target model is developed in §2.4 according to the handling qualities specifications. The desired closed loop system is to be low order, thus the target model is chosen as having low order dynamics. Moreover, the handling qualities specification requests limited cross-coupling behaviour thus the model will be selected with zero cross-coupling. The ideal target model represents behaviour desired in the actual aircraft.

2.2 Helicopter Control

The Bell 205 is a single-turbine powered aircraft with a two-bladed teetering main rotor and an anti-torque tail rotor. Helicopter control is achieved by manipulating the main and tail rotors to produce changes in the rotor blade pitch angle, either collectively to increase or decrease rotor thrust, or cyclically to tilt the rotor tip-path plane and with it the thrust vector. In effect there exist four control surfaces available to the pilot; main rotor and tail rotor collective for heave (vertical) and yaw control respectively, and longitudinal and lateral main rotor cyclic for pitch and roll control respectively. The control system of the main and tail rotors used to control these four surfaces consists of three sub-systems: main rotor collective, main rotor cyclic and tail rotor collective. Most single-rotor helicopters like the Bell 205 use the basic control system in Fig. 2.1 [PRO89].

Figure 2.1: Basic Helicopter Control System.

The reader should not be confused by the workings of the basic control system with the addition of blades to the main rotor as in the four-bladed teetering main rotor illustrated in Fig. 2.1. The function of the main rotor collective is to vary the amount of thrust the main rotor produces. The pilot is able to simultaneously change the pitch of all the blades by the same amount in response to an upward motion of the collective stick in the left hand of the pilot.

The cyclic system controls the direction of thrust by causing the rotor disc to tilt. The tilt results from the shaft responding to fore-and-aft or side-to-side motion of the cyclic stick in the right hand of the pilot. The control linkages act on the swashplate when the stick is moved. The swashplate is then free to tilt in any direction about a large spherical bearing. The swashplate consists of two parts. A non-rotating lower half connected to the control linkages from the cockpit and a rotating upper half that follows the motion of the non-rotating swashplate through a set of bearings. Motion from the rotating swashplate goes up to a pitch horn on each of the two blades resulting in the blade pitch to be changed around a feathering bearing. When the stick is pushed forward, the blade

on the right side has its pitch reduced, whereas the blade on the left side has its pitch increased by the same amount. The pitch of each blade thus changes once every revolution or *cyclically*. The function of cyclic pitch is to change the angle of attack on the two sides of the rotor deliberately unbalancing the lift causing the rotor disc to tilt and redirect the direction of thrust.

The tail rotor control system is a collective system just as on the main rotor used to control the thrust of the tail rotor. Control is by a set of 'rudder pedals' as in a conventional aeroplane to produce a right turn when the right pedal is depressed and a left turn when the left pedal is depressed.

In the past, helicopters have used the basic helicopter control system illustrated in Fig. 2.1 with mechanical links leading from the collective stick, cyclic stick, and rudder pedals to the mechanical actuators that move the rotor blades. This implies that with the left hand on the collective controlling rotor thrust, the right hand on the cyclic stick directing the thrust, and both feet on the rudder pedals controlling anti-torque tail rotor thrust, the pilot is completely occupied flying the aircraft. The manual piloting of helicopters is challenging and mentally taxing due largely in part to the inherent instabilities and highly nonlinear, cross-coupled nature of the helicopter. For example, at low speed without mechanical feedback, the single-rotor helicopter is naturally unstable. The coupled pitch/roll, so-called, 'pendulum' instability is a product of the flapping rotors response to velocity perturbations. Constant corrective control inputs are necessary to maintain a desired hover position and requires considerable attention of the pilot. This is clearly of concern when the attention of the pilot is divided with other critical tasks such as radio communication or equipment deployment. Furthermore, if the outside world visual cues degrade, so that the pilot has difficulty perceiving attitude and velocity relative to the ground, then the hover task becomes increasingly difficult [PAD96].

Without some form of compensation the flying of helicopters is fairly tiring, especially in degraded conditions, or when trying to perform precision tasks. In an effort to reduce pilot workload, and achieve acceptable handling qualities for the helicopter some form of control system is needed. The flight control system (FCS) may also carry out manoeuvres that the pilot is unable to perform due to the accuracy required, the length of time over which it is necessary to carry out the task and/or the lack of visual cues. An example could be a night time rescue where it is necessary to hold the aircraft in a precision hover for an extended period of time over water.

Stability augmentation systems (SAS) have been used in the past to improve handling qualities by damping the pitch and roll motions caused by wind gusts. The SAS makes the helicopter more stable and relieves the pilot from making constant control inputs to retain the desired attitude. This is accomplished using rate-based gyros that generate electric signals proportional to pitch and roll rates. These signals are then proportionally fed back to control hydraulic or electric actuators to provide longitudinal or lateral cyclic inputs that tilt the rotor to resist the motion of the helicopter.

Rate and other more complex feedback systems provide stability augmentation but do nothing for control augmentation. In fact control responses are reduced using SAS as it counteracts both disturbances as well as pilot inputs. The result is a control system that fights pilot inputs creating a helicopter with more sluggish response characteristics. Stability and control augmentation systems (SCAS) provide stability augmentation and control augmentation by feeding forward the control signal of the pilot into the SCAS, applying shaping functions or disabling the stability augmentation during manoeuvres. More modern systems accomplish the same task with greater sophistication, but modern SCAS designs that interface with mechanical control systems will always be limited in their potential to protect against failures [PAD96].

The term limited authority means that most of the available control power remains in the hands of the pilot, passing through mechanical/hydraulic linkages to the rotor actuators.

The feedback controller has limited authority of about 10%, which is just enough to provide stabilization for most aircraft. In the limit, increasing the authority of the control system leads to active control technology (ACT) where the control inputs of the pilot (inceptors) are combined with multiple sensor data in a digital computer to provide tailored response characteristics. In this thesis the concern is with the application of ACT to the **HELICOPTER PROBLEM** such that the pilot is removed from the feedback loop and the controller has 100% authority over the blade actuators. The use of ACT converts commands from the pilot into the appropriate blade control demands to provide the desired response and improve helicopter handling qualities.

2.3 Handling Qualities Requirements

In the 1980s considerable effort had been expended in revising the handling qualities specification for military rotorcraft. This effort was supported by the United States Army and other agencies who contracted the leading authorities in the field of rotorcraft handling qualities assessment to incorporate the perceived heavy pilot workload into a level of handling qualities for safe operation of the aircraft.

The purpose of next section is to state some of the handling qualities specifications for the Bell 205 helicopter operating in various environmental conditions at various pilot workload requirements according to USAAVSCOM 87-A-4 and ADS-33C [HOH88]. A control system design would attempt to achieve the desired handling qualities. The underlying idea is to recast the handling qualities specifications of the published reports into parameters that can be readily used in a control system synthesis algorithm by proposing an appropriate design problem. The scope of this report in control system analysis will be limited to small amplitude manoeuvring with the helicopter operating near a constant trim condition in an ACAH configuration. This specifies an ACAH in pitch and roll and a rate command heading hold (RCHH) in all other channels. These and other response type configurations are the discussed in the following subsection.

2.3.1 Response Types

Most unaugmented helicopters in flight at present exhibit what is commonly referred to as commanded rate response. This is indicated by the observation that for a given step command input by the pilot in one of the command axes (i.e. collective, lateral, longitudinal or tail rotor collective) the aircraft will respond with an angular rate behaviour in that axis, with associated cross-coupling behaviour as well. This is a desirable response type under some circumstances in that it is very flexible for pilot manoeuvring. In clear flying conditions, under minimal adverse effects, pilots tend to prefer flying aircraft exhibiting Rate type response. This is commonly referred in the industry as flying an integrator since a step command results in a ramping time function for angular displacement. Thus in flying conditions that are considered good, a Rate controller will be sufficient for most tasks.

Attitude hold implies that the aircraft will maintain the current angular attitudes in all axes when zero, or nominal operating point, pilot commands are given. The bandwidth augmentation is necessary in that a number of current aircraft do not respond with sufficient energy for high frequency inputs. Under hover or low speed flight conditions, an aircraft configured to operate in an ACAH type of response will be suitable for all standard tasks (stationary and translating) at low speed in a user cue environment of one. With a user cue environment of two, the ACAH format will be suitable for stationary mission task elements such as precision hover manoeuvres, vertical takeoff and landing and bobbing and turning tasks. The ACAH response will provide only level two handling qualities when the user cue environment is three. This implies that the Cooper-Harper scale on handling qualities is between four and six [COO69]. On the other hand, with a user cue environment of three, a translational rate command response will be required for level one handling qualities (Cooper-Harper ratings from one to three). Summaries of the various types of control responses that are relevant are contained in Table 2.1.

Control Type	Name	Description
Rate	Rate Command Attitude Hold	Stick command actuation is proportional to angular rate for aircraft. Null command results in constant attitude and heading, that is zero rate.
TC	Turn Coordination	Lateral acceleration and sideslip are both zero. Command is for bank angle to induce turn in aircraft. Yaw axis commands unnecessary for forward flight.
ACAH	Attitude Command Attitude Hold	Stick command is proportional to attitude or heading changes for aircraft. Null commands result in holding of current attitude.
RCHH	Vertical Rate Command Heading Hold	Vertical direction (collective input) rate command response with null input implying hold of altitude. Null command implying zero rate in heading change.
TRC	Translational Rate Command	Rate command inputs in lateral and forward velocity directions. Translational rates are commanded directly.

Table 2.1: Summary of the various types of control responses.

A complete specification for flight controls will state a given pilot task to be performed in a given operating environment. This operating environment includes the external weather conditions as well as instrumentation and visual aids available to the pilot in the aircraft. From these two conditions, task and environment, an assessment is made as to which type of control scheme is necessary. In general, more harsh external conditions

and more difficult tasks require stronger automatic control authority. This is implied in the sense that as conditions worsen the dynamics of the manoeuvring flight become more constrained by the control actions.

Goodness of flying conditions or the environment conditions in which the aircraft is in flight are often measured via User Cue Environment (UCE) indices. The lower the indexes on a 1 to 3 scale the better the flying condition. The UCE scale is evaluated from the external operating conditions such as weather environmental conditions and other available visual aids for use by the pilot including special instrumentation for visualization. This assessment is found in a score called the Cooper-Harper index [COO69]. The resolution of the Cooper Harper index is higher than the UCE, being a scale of 1 to 10. However, ranges of the Cooper Harper index are used in evaluating the ease of operation of a given aircraft thus the Cooper-Harper index is of the same order of resolution as the UCE index. As the flying condition (UCE) index degrades from 1 to 2, the Rate controllers will not provide sufficient stabilization in the closed loop of pilot/controller/aircraft. Typically, the vertical and yaw directions still maintain a RCHH or height hold feature while the other Euler directions of pitch and roll require attitude command pilot inputs. Finally if the flying environment degrades further to a UCE of 3, the pilot will require TRC controllers in the forward and lateral directions to maintain an appropriate level of stability margin.

2.3.2 General Handling Qualities Requirements

A detailed description of handling qualities expected for rotorcraft in use by NASA has been compiled by [HOH88]. The specifications contained therein relate to helicopter closed loop command behaviour. Many of the required qualities are qualitative in nature or related to reasonableness specifications. An attempt is made here to quantify expectations of an automatic control system used in augmentation of the pilot tasks. The level of handling qualities actually attained by the control system can then be used in assessing a measure of expected performance of the aircraft in various manoeuvres in

given operating environments. It was judged that good performance of the rotorcraft would result if, among other requirements, certain bandwidths were met by the control system linking pilot commands to aircraft response. These bandwidth requirement reflect the perception that pilot workloads would be reduced if pilot commands at a variety of frequency inputs were followed by the aircraft with nearly the same gain for each frequency up to the bandwidth. That is, the frequency domain closed loop response to commands would be flat up to the bandwidth. Due to noise and disturbance rejection requirements, it is also desirable that the frequency response roll off with sufficient attenuation after the bandwidth frequency.

This section will summarize the findings of [HOH88] known as USAAVSCOM Technical Report 87-A-4. The original report contains an exhaustive survey of handling qualities requirements for rotorcraft for use in military operations. The survey includes specifications for a wide variety of response types in numerous mission task elements. This summary will focus on ACAH response type operating so as to achieve level one handling qualities (Cooper-Harper scale indices from one to three) in mild or limited to aggressive manoeuvring, but not as applied to combat elements. Level one handling qualities are required, according to the report, to maintain the aircraft within the operational flight envelope.

Handling qualities requirements for each axis are usually divided into different manoeuvring regimes. These are small amplitude manoeuvres, moderate amplitude manoeuvres and large amplitude manoeuvres. In addition, small amplitude responses are subdivided into short- and mid-term responses implying behaviour in the time domain. Off-axis coupling limitations are also presented. The aircraft is generally considered to be operating in one of two operating conditions: hover or low speed flight and forward flight. The hover or low speed condition implies forward velocities of magnitudes less than 45 knots. Strict hovering behaviour is considered when forward velocity magnitudes of less than 15 knots are present. Forward flight consists of all other operating conditions.

The small amplitude task concerns fine corrections by the pilot operating near trim. These are specified by a bandwidth and equivalent phase delay parameters. Essentially, near the operating point it is desired that the aircraft be insensitive to exogenous disturbances, track the command signal precisely, and exhibit good stability robustness. This last condition is measured by the equivalent phase delay, which quantifies a degree of lag in the closed loop system. If the pilot experiences too much lagging behaviour, a pilot-induced oscillation may result.

The null command response of the aircraft should be stable, although in certain circumstances unstable behaviour with long doubling times are acceptable. A general requirement of the control system is that residual oscillations at a steady trim condition be damped out. Sustained residual oscillations resulting in 0.5 degrees attitude change or 0.05 g accelerations shall be considered objectionable. Pilot-induced oscillations should be damped out with no tendency spontaneous generation of this behaviour due to pilot corrections. Roughly speaking, this behaviour will tend to be eliminated if there is not objectionably large phase delays in the closed loop frequency responses.

2.3.3 Bandwidth and Phase Delay

Two common measures of performance in a number of axes are bandwidth and phase delay. Closed loop rotorcraft system bandwidth, for use in assessing aircraft performance is defined as the frequency at which the system gain is 6 dB higher than the gain at which the phase exhibits a lag of -180 degrees or that frequency at which the phase is -135 degrees. If the bandwidth is gain limited, there may be a tendency for pilot-induced oscillations. That the bandwidth is gain limited implies that the frequency of -135 degrees phase lag is higher than the 6 dB gain margin frequency. The phase delay is defined as the ratio of system phase lag exceeding -180 degrees at twice the frequency where -180 degrees phase lag was first observed, measured in radians, divided by this frequency. Thus denoting as ω_{180} that frequency at which that phase first passes through

-180 degrees and Φ as the phase lag (measured in radians) at the frequency $2\omega_{180}$, then the phase delay, τ_p will be

$$\tau_p = \frac{\Phi - \pi}{2\omega_{180}}. \quad (2.1)$$

Moderate amplitude manoeuvres are typically measured for performance by minimum requirements in the equivalent transient time constant for this axis. That is the parameter τ_r that roughly satisfies that differential equation

$$\tau_r \dot{x} = -x + u_x. \quad (2.2)$$

in some axis, x , for the input driving function u_x . This parameter is, for practical purposes measured as the ratio

$$\tau_r = \frac{q_{peak}}{\Delta\theta_{peak}}. \quad (2.3)$$

for the pitch axis and similarly for the roll axis. In the equation, q_{peak} is the maximum magnitude or peak value of pitch angular rate during the manoeuvre. The value $\Delta\theta_{peak}$ is the total attitude pitch change during the manoeuvre.

Off-axis coupling is an important aspect in rotorcraft control system design and analysis. However, due to subjectivity of individual pilot performance, a standard for off-axis response is difficult to quantify and so is still a subject for debate. Essentially, the requirement in the closed loop system shall be that the off-axis responses due to coupling shall not be objectionable to pilot comfort and safety. Somewhat more specific limits shall also be presented in the specific requirements.

2.3.4 Detailed Handling Qualities Specifications: Pitch and Roll Axes Manoeuvres

First considered will be the small amplitude pitch and roll responses. In general it is sufficient to obtain level one handling qualities in pitch and roll manoeuvres if the system bandwidth exceeds 3 rad/s while at the same time the equivalent phase delay is less than 0.2 s. However, it is important to note that in aggressive tracking manoeuvres, it will be necessary for level one handling qualities to have the system bandwidth meet a minimum of 4 rad/s with the same phase delay. For smaller bandwidths, the handling qualities drop to level two and worse. System operating in this regime shall exhibit damping ratios of at least 0.35. It is noted in [HOH88] that for fully attended operation (that is under condition where the pilot is not distracted from control of the aircraft) it is possible to allow smaller damping ratios in complex pole pairs of natural frequency less than 0.5 rad/s, even to the extent of slightly unstable poles. However, in control systems design, all attempts will be made to satisfy the strict damping ratio requirement.

The minimum achievable angular rate response for rate response tasks, in pitch and roll is expressly lower bounded for large amplitude manoeuvres. In the roll response for all but limited manoeuvring, the minimum response rate magnitude is 50 deg/s or roughly 1 rad/s. Limited manoeuvres require approximately half of this value. Aggressive pitch manoeuvres require minimum pitch rate capability of 30 deg/s. This relaxes to 13 deg/s for limited manoeuvring. On the other hand, for attitude tasks the previously defined lower bounds as expressed in degrees as opposed to degrees per second shall apply.

Roll to pitch and pitch to roll inter-axis cross-coupling are roughly limited by the ratio of maximum excursion from trim in the off-axis to command magnitude in the commanded axis. For level one handling qualities, this cross-coupling magnitude ratio shall be less than 0.25.

2.3.5 Detailed Handling Qualities Specifications: Yaw and Vertical Axes Manoeuvres

Attention is now turned to tasks involving movement in the yaw axis. Small amplitude manoeuvres for level one handling qualities shall have a bandwidth in the yaw axis if not less than 3 rad/s while maintaining a phase delay of not more than 0.2 s for all mission task elements except precision tracking where a bandwidth of 4 rad/s is required. As in the pitch and roll axes, damping ratios of 0.35 shall be maintained for good stability characteristics.

In the yaw axis it is desirable to limit the exogenous disturbance response, or in particular the maximum rate response to disturbing gust inputs is limited. This upper bound is expressed as the ratio of yaw rate to maximum gust velocity (for gusts not exceeding 35 knots). The ratio is upper bounded by 0.66 deg/s/m/s for precise tracking while 0.99 deg/s/m/s will suffice for all other mission task elements. An important cross-coupling in rotorcraft is the collective to yaw rate transfer function. In short, as collective is increased from steady state, the main rotor attempts to cause the helicopter to lift by grabbing more air with an increased pitch on the blade. This increased lift in the rotor is accompanied by an increased drag, which is in turn counteracted by an increased torque from the main turbine, which will cause the aircraft main body to rotate or yaw. In order that such yawing motion is not objectionable, the standard presents some *ad hoc* guidelines so that the off-axis coupling is not too large. In order that these measures can be practically incorporated into a control system design algorithm and test facility, a simplified and more strict or conservative estimate is taken. This estimate is taken as the maximum magnitude of the ratio of maximum yaw rate experienced divided by the maximum vertical translation rate achieved. This ratio should be kept less than 0.15 for agreeable performance.

The vertical axis response to collective inputs is roughly modelled by a first order transfer function with added deadtime from collective to vertical translational rate. Level one handling qualities are achieved if the approximate first order transfer function time

constant is less than 5 s while the approximate pure delay is less than 0.2 s. For a reasonably responsive system, the aircraft engine power should be capable of allowing vertical accelerations exceeding 0.54 m/s/s for level one handling qualities. The collective to rotational (pitch and roll) coupling shall be such that the maximum ratio of attitude to vertical normal acceleration is less than 3 deg/m/s/s for small inputs and 0.75 deg/m/s/s for large inputs.

2.3.6 Handling Qualities Specifications Summary

For the **HELICOPTER PROBLEM**, the controller will attempt to achieve the following specifications. The small amplitude pitch and roll responses are to be of an ACAH type with a bandwidth of at least 3.0 rad/s. The bandwidth for these purposes is defined as the frequency, ω_B , which gives a 45 degree phase stability margin. In moderate amplitude manoeuvring, for pitch and roll attitude excursions of roughly 30 degrees, the relevant handling qualities specifications changes to a time constant. For level one handling qualities, the maximum on-axis rate divided by the maximum attitude excursion, or the attitude quickness, shall have at least a rate of 2.0. Cross-couplings and inter-axis response in this regime are limited by the specification that the maximum ratio of off-axis to on-axis command is to be less than 0.25. The yaw axis response specifications are similar to those for pitch and roll except that yaw rate as opposed to attitude is the command output. For small amplitude manoeuvres for level one handling qualities, the bandwidth is to be at least 3 rad/s. The moderate amplitude manoeuvre time constant ratio is to be at least 1.5 s⁻¹. The cross-coupling collective to yaw response shall be below 15 %.

The vertical rate response will appear roughly as a closed loop first order response, with additional high frequency phase lag due to high order dynamics. For level one handling qualities, the time constant of this first order response shall not be less than 0.2 s⁻¹.

2.4 Ideal Target Model

In this section a state variable realization of an ideal target model is developed. According to the handling qualities specification, the desired closed loop system is to be low order, thus the target model is chosen as having low order dynamics. Moreover, the handling qualities specification requests limited cross-coupling behaviour thus the model will be selected with zero cross-coupling.

Finally, the low order on-axis responses are specified. The two horizontal plane translational velocities (forward and lateral) are not directly controlled in an ACAH system and so are given arbitrarily low weight in the model following design. These responses are driven by the angular attitude in a first order fashion as

$$\dot{U} = -\lambda_U U - \lambda_{U\theta} \theta, \quad \dot{V} = -\lambda_V V + \lambda_{V\phi} \phi \quad (2.4)$$

where λ_U and $\lambda_{U\theta}$ are the desired pole locations for these directions. The pitch attitude enters in a negative sense, as negative or nose-down pitch should induce a positive forward velocity. The yaw and vertical axis responses should follow similar first order responses except that the driving inputs should be tail rotor collective and collective respectively viz.

$$\dot{R} = -\lambda_R R + \lambda_{Rr_R} r_R, \quad \dot{W} = -\lambda_W W + \lambda_{Wr_W} r_W \quad (2.5)$$

for the yaw and vertical directions respectively.

From physical arguments, the pitch and roll attitude responses must take into account the pitch and roll rates thus giving second order response characteristics. These general second order responses can be written as

$$\dot{Q} = -2\zeta_{\theta}\omega_{\theta}Q - \omega_{\theta}^2\theta - \omega_{\theta}^2r_{\theta} \quad (2.6)$$

$$\dot{P} = -2\zeta_{\phi}\omega_{\phi}P - \omega_{\phi}^2\phi - \omega_{\phi}^2r_{\phi} \quad (2.7)$$

for the pitch and roll directions. Combining these equations into a state space model yields the ideal target model

$$\dot{\underline{x}} = \underline{A}\underline{x} + \underline{B}r \quad (2.8)$$

with

$$\underline{x} = \begin{bmatrix} U \\ W \\ Q \\ V \\ P \\ R \\ \theta \\ \phi \end{bmatrix} = \begin{bmatrix} \text{forward velocity} \\ \text{vertical velocity} \\ \text{pitch rate} \\ \text{lateral velocity} \\ \text{roll rate} \\ \text{yaw rate} \\ \text{pitch attitude} \\ \text{roll attitude} \end{bmatrix} \quad (2.9)$$

driven by the reference input

$$r = \begin{bmatrix} r_W \\ r_{\theta} \\ r_{\phi} \\ r_R \end{bmatrix} = \begin{bmatrix} \text{vertical velocity command} \\ \text{pitch attitude command} \\ \text{roll attitude command} \\ \text{yaw rate command} \end{bmatrix}. \quad (2.10)$$

The state transition and input matrices ($\underline{A}$, $\underline{B}$) for the closed loop system are defined as follows

$$\underline{A} = \begin{bmatrix} -\lambda_U & 0 & 0 & 0 & 0 & 0 & -\lambda_{I^*} & 0 \\ 0 & -\lambda_W & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2\zeta_\theta \omega_\theta & 0 & 0 & 0 & -\omega_\theta^2 & 0 \\ 0 & 0 & 0 & -\lambda_{I^*} & 0 & 0 & 0 & \lambda_{I^*} \\ 0 & 0 & 0 & 0 & -2\zeta_\phi \omega_\phi & 0 & 0 & -\omega_\phi^2 \\ 0 & 0 & 0 & 0 & 0 & -\lambda_R & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \quad (2.11)$$

$$\underline{B} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -\lambda_W & 0 & 0 & 0 \\ 0 & -\omega_\theta^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \omega_\phi^2 & 0 \\ 0 & 0 & 0 & \lambda_R \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \quad (2.12)$$

Angular rates are in units of radians per second; translational rates in meters per second; and attitudes in radians. The units of control input are in terms of pilot stick movement measured in centimetres. Values for the model parameters based on the handling qualities specifications are:

$$\begin{aligned} \lambda_U &= 4.0, & \lambda_{I^*} &= 4.0 \\ \lambda_W &= 3.0 \\ \lambda_R &= 5.0 \\ \omega_\theta &= 4.0, & \omega_\phi &= 4.0 \\ \zeta_\theta &= 0.7, & \zeta_\phi &= 0.7 \end{aligned} \quad (2.13)$$

with values often exceed those required by the specification.

2.5 Summary

This chapter describes how current helicopters are flown including the complexities of piloting helicopters. The limitations of existing control systems are discussed and used to motivate the use of active control technology to improve the handling qualities of the helicopter. Handling qualities specifications for the helicopter are also presented and used to formulate an ideal target model.

CHAPTER 3

MODEL UNCERTAINTY AND ROBUSTNESS ANALYSIS

3.1 Introduction

The purpose of this chapter is to demonstrate how to map realistic plant uncertainties into an appropriate mathematical framework for robust control of uncertain dynamical systems. An explanation of how to measure the stability robustness of multivariable feedback systems using structured singular values of certain frequency response matrices is also presented. It is assumed throughout that the models are finite dimensional linear time invariant (FDLTI) systems.

The starting point for linear robust control theory is a linearized plant model, usually in state space form, plus a reliable description of its uncertainty. No mathematical system can precisely model a real physical system. Model uncertainty, introduced in §3.2, will always exist in any real system and in §3.2.1 classes of uncertainty are discussed to include dynamical and parametric uncertainty. The former results from missing or neglected dynamics and the latter from the imperfect knowledge of physical parameter values or from variations of these parameters during operation.

The key to applying robust control methods successfully is to map realistic plant uncertainties into an appropriate mathematical framework. The linear fractional transformation (LFT) is introduced in §3.2.2 as a natural way to model many types of uncertainty in linear systems, including state space parameter uncertainty. There are two

sides to robust control theory: the original synthesis of robust control laws and the analysis of their robustness once designed. Robustness analysis is presented in §3.4 with the $\mathcal{H}_\infty$ -norm defined in §3.4.1 central to the robust control theory. The structured singular value and μ -analysis are introduced in §3.4.3 as a tool to determine the robust stability characteristics of the closed loop feedback system. The combination of LFT modelling and μ -analysis provides a powerful approach for worst-case stability analysis of control systems.

The **HELICOPTER PROBLEM** has been posed as a robust stability problem with optimal nominal performance as the performance criterion. The uncertainty of available low-order models presented in §3.5.1 is seen as high-order unmodelled dynamics. Experimental frequency domain responses of the actual helicopter provide a clear and reliable picture of the uncertainty between the nominal model and actual physical plant. This experimental discrepancy description is discussed in §3.5.2 and a higher-order model developed in §3.5.3 to include these error dynamics providing a more accurate description of the actual helicopter. It is unreasonable to assume that uncertainty can be completely removed from the higher-order nominal model and so §3.5.4 discusses the uncertainty surrounding the resulting linear design model.

3.2 Model Uncertainty

The notion of uncertain dynamical systems is central to robust control theory. It is important to realize at the outset that no mathematical system can precisely model a real physical system. For control design purposes, linear models of relatively low-order complexity must approximate the possibly complex behaviour of nonlinear dynamical systems. The gap between such models and the true physical system is called the *model uncertainty*.

Another cause of uncertainty is the imperfect knowledge of some components of the system, or the alteration of their behaviour due to changes in operating conditions,

ageing, etc. Finally, uncertainty also stems from physical parameters whose value is only approximately known or varies in time. Note that model uncertainty as used here should be distinguished from exogenous actions such as disturbances or measurement noise.

When piecing together the uncertainty description with the nominal model, two fundamental principles must be remembered. The first being that uncertainty should be small where high performance is desired (trade-off between performance and robustness). In other words, the linear model should be sufficiently accurate in the control bandwidth. Lastly, the more information you have about the uncertainty (phase, structure, time invariance, etc.), the higher the achievable performance will be.

3.2.1 Classes of Uncertainty

There are two major classes of uncertainty. They are dynamical and parametric uncertainty. Dynamical uncertainty consists of dynamical components neglected in the linear model. Here the linear model is in error because of missing dynamics, usually at high frequencies, either through deliberate neglect or because of a lack of understanding of the physical process or variation in the dynamical behaviour of the physical system during operation. Parametric uncertainty stems from imperfect knowledge of the physical parameter values, or from variations of these parameters during operation. Here the structure of the model is known, but some of the physical parameters are uncertain such as stiffness and damping coefficients in mechanical systems, capacitors and inductors in electric circuits, etc. Model uncertainty is generally a combination of dynamical and parametric uncertainty. For example, there may be dynamical uncertainty on the system actuators and parametric uncertainty on some sensor coefficients.

3.2.2 Linear Fractional Models of Uncertainty

Factors that influence the choice of uncertainty representation include the available model (state space equations, frequency domain models, etc) and the analysis and

synthesis tools that will be used in the control system design. These include uncertain state space model representations such as polytopic and affine parameter-dependent models. To be successful, a robust control method must map realistic design objectives and plant uncertainties into the linear system mathematical framework. A powerful tool is available to accomplish such a task: the linear fractional transformation (LFT). The LFT approach is to model plant variations as variable linear gains in a general feedback structure. Consider first the linear system M , which can either be a constant matrix or a dynamic system with a state space realization, relating vectors r and v .

$$v = Mr \quad (3.1)$$

If r and v are partitioned into a top and bottom part, then we can draw the relationship in more detail, explicitly showing the partitioned matrix M

$$\begin{aligned} v_1 &= M_{11}r_1 + M_{12}r_2 \\ v_2 &= M_{21}r_1 + M_{22}r_2 \end{aligned} \quad (3.2)$$

Suppose a matrix K relates v_2 to r_2 , as

$$r_2 = Kv_2 \quad (3.3)$$

The LFT of M by K interconnects these two elements, as follows:

Eliminating v_2 and r_2 , leaving the relationship between r_1 and v_1

$$\begin{aligned} v_1 &= [M_{11} + M_{12}K(I - M_{22}K)^{-1}M_{21}] r_1 \\ &= F_l(M, K) r_1 \end{aligned} \quad (3.4)$$

The matrix $F_l(M, K)$ is called the *lower linear fractional transformation*, or lower LFT, for the system M . In robust controller synthesis presented in a later chapter, the lower LFT will indicate the closed loop plant response with control law K connected. Standard optimal control problems can be cast into this form, so that the objective is to find a control law K that minimizes the closed loop system $F_l(M, K)$ according to some prescribed measure.

A dual upper LFT is also in common use. If the upper loop of M is closed with Δ , then:

where

$$\begin{aligned} v_2 &= [M_{22} + M_{21}\Delta(I - M_{11}\Delta)^{-1}M_{12}] r_2 \\ &= F_u(M, \Delta) r_2 \end{aligned} \quad (3.5)$$

The matrix $F_u(M, \Delta)$ is called the *upper linear fractional transformation*, or upper LFT, for the system M . In robust synthesis and analysis the matrix Δ represents the uncertain model variations.

An example illustrating how to represent an uncertain parameter using the LFT may help to make the concept clearer. Suppose c is a parameter, and it is known to take on values

$$2.0 \leq c \leq 2.8$$

Write this as $c = 2.4 + 0.4\delta_c$ where $\delta_c \in [-1, 1]$. This is a linear fractional transformation. Indeed check that

$$\begin{aligned} c &= F_l \left(\begin{bmatrix} 0 & 1 \\ 0.4 & 2.4 \end{bmatrix}, \delta_c \right) \\ &= 2.4 + 0.4\delta_c (1 - 0 \cdot \delta_c)^{-1} 1 \\ &= 2.4 + 0.4\delta_c \end{aligned}$$

So, everywhere

appears in a block diagram, replace with

This basic approach can be extended to model the effects of multiple parameter variations on much more complicated systems with state space realizations.

3.3 General Control Configuration

There are many ways in which plant uncertainty descriptions and feedback design problems can be cast as optimization problems. It is very useful therefore to have a general framework into which any particular problem can be manipulated. All systems considered in this thesis are modelled as FDLTI operating in continuous time with a state space and transfer function representation in the Laplacian s given by:

$$G(s) = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \quad G(s) = C(sI - A)^{-1} B + D$$

where $A \in \mathcal{R}^{n \times n}$, $B \in \mathcal{R}^{n \times m}$, $C \in \mathcal{R}^{l \times n}$ and $D \in \mathcal{R}^{l \times m}$, where $\mathcal{R}^{i \times j}$ represents a real matrix of dimension $i \times j$. The system G is said to be stable if the state space matrix A has no eigenvalues in the closed right half-plane. The general framework to be used in this thesis is depicted in Fig. 3.1 and will be referred to as the General Control Configuration (GCC).

Figure 3.1: General Control Configuration.

The objective is to design a controller K , for the plant G such that the input/output transfer characteristics from the external input vector w to the external output vector z are desirable according to some engineering specifications. The exogenous input w , typically consists of command signals, disturbances and sensor noise; u is the control signal; the output to be controlled z , which customarily consists of the so-called “error” signals

which are to be minimized in some sense to meet control objectives; and y the measured output. The system of Fig. 3.1 is described by

$$\begin{bmatrix} z \\ y \end{bmatrix} = G(s) \begin{bmatrix} w \\ u \end{bmatrix} = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix} \quad (3.6)$$

$$u = K(s)y \quad (3.7)$$

with a state space realization of the generalized plant G given by

$$G = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{bmatrix} \quad (3.8)$$

where $G_{ij} = C_i(sI - A)^{-1}B_j + D_{ij}$.

3.4 Robustness Analysis

This section considers the stability tests of a nominally stable system under both unstructured and structured perturbations. The focus is on analysis methods; therefore the controller may be viewed as just another system component and absorbed into the interconnection structure. The closed loop transfer function matrix from w to z is given by the lower LFT

$$z = F_l(G, K)w \quad (3.9)$$

where

$$F_l(G, K) = G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21} \quad (3.10)$$

denoted by $M(s) = F_r(G, K)$. The perturbation matrix Δ connecting z to w is itself assumed to be FDLTI systems which are norm bounded as input-output operators resulting in the robustness analysis framework depicted in Fig. 3.2.

Figure 3.2: Robustness Analysis Framework.

Representations of uncertainty vary primarily in terms of the amount of structure they contain. This reflects the knowledge of the physical mechanisms that cause differences between the model and the plant and the ability to represent these mechanisms in a way that facilitates convenient manipulation [ZHO96]. Knowledge of these physical mechanisms allows the uncertainty to be modelled with structure. Parametric uncertainty is often called *structured uncertainty* as it models the uncertainty in a structured manner. Unstructured perturbations are often used to get a simple uncertainty model when limited knowledge exists of how the uncertainty dynamics enter the model. Neglected or unmodelled dynamics uncertainties are often called *unstructured uncertainty*. However, one should be careful in the use of these terms because there are several levels of structure, especially with MIMO systems [POST96].

3.4.1 $\mathcal{H}_\infty$ -Norm

It is necessary to introduce the notion of a system norm as a measure of the size of a system. Two important norms are used in this thesis for matrices $G(s)$ whose entries are *analytic* functions of the complex variable s in the right half s -plane. The matrix entries of which are real rational functions of s ; thus the only points where the entries can fail to

be *analytic* are at their poles. The $\mathcal{H}_2$ -norm of $G(s)$ is denoted as $\|G(s)\|_2$ and will be defined in a later chapter. The $\mathcal{H}_\infty$ -norm of $G(s)$ is denoted $\|G(s)\|_\infty$ and is defined

$$\|G\|_\infty := \sup_{\omega} \bar{\sigma}(G(j\omega)) \quad (3.11)$$

where $\bar{\sigma}(M)$ is the largest singular value of M . In particular, a matrix $G(s)$ is said to belong to the matrix *Hardy space* $\mathcal{H}_\infty$ if it is bounded with *analytic* continuation into the right half s -plane. The real rational subspace of $\mathcal{H}_\infty$ denoted by $\mathcal{RH}_\infty$ consists of all proper and real rational stable transfer matrices.

3.4.2 Small Gain Theorem

This section considers the stability test of a nominally stable system under unstructured perturbations. *Unstructured uncertainty* is defined as the use of a “full” complex perturbation matrix Δ , where at each frequency any $\Delta(j\omega)$ satisfying $\bar{\sigma}(\Delta(j\omega)) \leq 1$ is allowed. The basis for the robust stability criteria described here is the *small gain theorem* [ZHO96]. Consider the interconnected system shown in Fig. 3.2 with $M(s)$ a stable $p \times q$ transfer matrix.

Theorem 3.1 (Small Gain Theorem): Suppose $M \in \mathcal{RH}_\infty$ and let $\gamma > 0$. Then the interconnected system shown in Fig. 3.2 is well-posed and internally stable for all $\Delta(s) \in \mathcal{RH}_\infty$ with either

- a) $\|\Delta\|_\infty \leq 1/\gamma$ if and only if $\|M(s)\|_\infty < \gamma$;
- b) $\|\Delta\|_\infty < 1/\gamma$ if and only if $\|M(s)\|_\infty \leq \gamma$.

Proof: [ZAM63]

Thus an uncertain system described by Fig. 3.2 with an unstructured uncertainty $\Delta(s) \in \mathcal{RH}_\infty$ with $\|\Delta\|_\infty \leq 1$ is robustly stable if and only if the transfer function from w to z has $\mathcal{H}_\infty$ -norm no greater than 1. $\|\Delta\|_\infty \leq 1$ implies stability, but not conversely, because this test ignores the known block diagonal structure of the uncertainty and is equivalent to regarding Δ as unstructured. This can be arbitrarily conservative in that stable systems can have arbitrarily large $\|\Delta\|_\infty$ [DOY82].

3.4.3 μ -Analysis

The structured singular value (SSV) also denoted by μ , is a function which provides a generalization of the singular value, $\bar{\sigma}$. Consider again the robust analysis framework of Fig. 3.2 with stable $M(s)$ and $\Delta(s)$. The SSV or μ is used to give necessary and sufficient conditions for robust stability. A statement of the definition of μ is defined here [SKO96].

Find the smallest structured Δ (measured in terms of $\bar{\sigma}(\Delta)$) which makes $\det(I - M\Delta) = 0$ in Fig. 3.2; then $\mu(M) = 1 / \bar{\sigma}(\Delta)$.

where $\det(M)$ denotes the *determinant* of the matrix M . Mathematically,

$$\mu(M)^{-1} := \min_{\Delta} \{ \bar{\sigma}(\Delta) \mid \det(I - M\Delta) = 0 \text{ for structured } \Delta \}$$

such that $\mu(M)$ depends not only on M but also on the allowed structure of Δ .

The most well-known use of μ as a robustness analysis tool is in the frequency domain. Suppose $M(s)$ in Fig 3.2 is a stable real rational, MIMO transfer function of a linear system. Then the following theorem is useful for robust stability analysis.

Theorem 3.2 (Robust stability for block diagonal perturbations: real or complex):

Assume that the nominal system M and the perturbation Δ are stable. Then the $M\Delta$ - system in Fig 3.2 is stable for all allowable perturbations with $\bar{\sigma}(\Delta) \leq 1, \forall \omega$, if and only if

$$\sup_{\omega} \mu(M(j\omega)) < 1$$

Proof: [SKO96]

The structured singular value μ is useful to assess the robust stability of time-invariant LFT interconnections. The reciprocal

$$K_m = \frac{1}{\sup_{\omega} \mu(M(j\omega))}$$

represents the robust stability margin, that is, the largest amount of uncertainty Δ that can be tolerated without losing stability [BAL94]. Robust analysis using μ is performed using μ -TOOLS™ where the variable gains may be real, representing true physical parameters, or complex, representing uncertain dynamics in the system components. The real perturbations can also be given some complex value in the analysis providing a dynamical uncertainty flavour to the analysis [BAL94].

3.5 The Bell 205 Helicopter

The aircraft considered in this thesis is a Bell 205 helicopter, fully instrumented so that all relevant aircraft states are measurable. The aircraft can be configured to allow direct manual pilot control via hydraulic actuators or full authority fly-by-wire actuation [SAT84]. The Bell 205 is a single-turbine powered aircraft with a two-bladed teetering main rotor and an anti-torque tail rotor depicted in Fig. 3.3 [PAD96].

Figure 3.3: The Canadian NRC fly-by-wire Bell 205.

3.5.1 Low-Order Helicopter Model

Six-degree-of-freedom linear models of the Bell 205 are available at a variety of trim points [HEF79] and these models are to be used as nominal flight conditions. The steady hover condition is used as the trimmed operating condition as this trim condition is the starting point for a variety of important helicopter tasks such as precision hovering. The low-order model given in (2.1) for the Bell 205 by [HEF79] represents the aircraft trimmed at a nominal $+5^\circ$ pitch attitude, with a mid-range weight, a mid-position centre of gravity and operating in-ground effect at near sea level. The model is described by

$$\dot{x}_h = A_h x_h + B_h u \quad (3.12)$$

where the pair (A_h, B_h) and the state and control vectors are defined as follows:

$$A_h = \begin{bmatrix} 0 & 0.03 & 0.18 & -0.01 & -0.42 & -0.08 & -9.81 & 0 \\ -0.10 & -0.39 & 0.09 & -0.10 & -0.12 & 0.68 & 0 & 0 \\ 0.01 & -0.01 & -0.19 & 0 & 0.23 & 0.04 & 0 & 0 \\ 0.02 & 0 & -0.41 & -0.05 & -0.27 & 0.27 & 0 & 9.81 \\ 0.03 & -0.02 & -0.88 & -0.04 & -0.57 & 0.14 & 0 & 0 \\ -0.01 & -0.02 & -0.06 & 0.07 & -0.32 & -0.71 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \quad (3.13)$$

$$B_h = \begin{bmatrix} 0.08 & 0.13 & 0 & 0 \\ -1.17 & 0.04 & 0 & 0.01 \\ 0 & -0.07 & 0 & 0.01 \\ -0.04 & 0 & 0.11 & 0.20 \\ -0.04 & 0 & 0.22 & 0.17 \\ 0.17 & 0 & 0.03 & -0.47 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (3.14)$$

$$x_h = \begin{bmatrix} U \\ W \\ Q \\ V \\ P \\ R \\ \theta \\ \phi \end{bmatrix} = \begin{bmatrix} \text{forward velocity} \\ \text{vertical velocity} \\ \text{pitch rate} \\ \text{lateral velocity} \\ \text{roll rate} \\ \text{yaw rate} \\ \text{pitch attitude} \\ \text{roll attitude} \end{bmatrix}, u = \begin{bmatrix} \delta_c \\ \delta_B \\ \delta_A \\ \delta_P \end{bmatrix} = \begin{bmatrix} \text{collective} \\ \text{longitudinal cyclic} \\ \text{lateral cyclic} \\ \text{tail rotor collective} \end{bmatrix}. \quad (3.15-16)$$

Angular rates are in units of radians per second; translational rates in meters per second; and attitudes in radians. The units of control input are in terms of pilot stick movement measured in centimeters.

3.5.2 Experimental Discrepancy Description

The significance of the teetering rotor in the helicopter is that body rotational and translational motion are a result of a high order process involving cockpit stick dynamics, actuator dynamics, rotor tip path plane dynamics and finally body motion. Low-order aircraft models do not fully describe these characteristics and so it is necessary to study the associated uncertainty dynamics. In order that the closed loop system achieves desired performance the linear design model should be sufficiently accurate to reduce the trade-off between performance and robustness. It was decided to use the Heffley model in (3.12) as nominal and to quantify the uncertain error dynamics between it and the actual aircraft. By adding this information to the Heffley model it is proposed that the uncertainty in the linear design model will be smaller and the trade-off between performance and robustness reduced resulting in higher achievable performance. The uncertain error dynamics in the Heffley model result from a variety of sources including:

1. unmodelled rotor modes
2. unmodelled high frequency body and rotor mast flexing modes
3. unmodelled sensor and actuator dynamics
4. other discrepancies between the actual aircraft body motion and the model due to high order effects.

In order to assess the model fidelity and to quantify the error dynamics associated with the true aircraft, frequency domain comparisons between the Heffley model and the aircraft were conducted. The frequency domain response of the Heffley model is relatively simple to compute and graph using standard commercial software routines in MATLAB[®], a product of the The Mathworks, Inc. The experimental frequency domain aircraft response was found by examining Fourier transform input-to-output transfer functions for the aircraft evaluated across the frequency spectrum of interest from flight time histories. It is important that the time histories contain sufficient frequency information to allow confidence in the resulting Fourier transform transfer functions. Experience with this type of testing, conducted at the National Research Council, has shown that a manually governed pilot-induced frequency sweep of each input in turn gives the desired result. The pilot attempted to minimize off-axis inputs during the frequency sweeps so that cross-coupling stability and control derivatives could be more accurately analyzed. The Fourier transforms are taken in the discrete time domain through numerical techniques associated with Fast Fourier Transforms (FFT).

Examples of experimental results of frequency sweep testing are illustrated in Fig. 3.4 through 3.13 with the experimental points indicating the transfer function magnitude and phase from input to output. The coherence function is a measure of the reliability of the experimental data. This measure is found by taking several overlapping windows (10 each of length 64 seconds for these results) on the time histories and computing the FFT for each window. The variance of these results evaluated at each frequency point and normalized gives a measure of the likelihood that the data is in some sense corrupt. Subtracting the value from unity gives the likely coherence measure of the data.

Figure 3.4: Experimental Frequency in Hover, Collective to Vertical Velocity.

Figure 3.5: Experimental Frequency in Hover, Collective to Yaw Rate.

Figure 3.6: Experimental Frequency in Hover, Longitudinal Cyclic to Forward Velocity.

Figure 3.7: Experimental Frequency in Hover, Longitudinal Cyclic to Pitch Rate.

Figure 3.8: Experimental Frequency in Hover, Longitudinal Cyclic to Roll Rate.

Figure 3.9: Experimental Frequency in Hover, Lateral Cyclic to Pitch Rate.

Figure 3.10: Experimental Frequency in Hover, Lateral Cyclic to Lateral Velocity.

Figure 3.11: Experimental Frequency in Hover, Lateral Cyclic to Roll Rate.

Figure 3.12: Experimental Frequency in Hover, Tail Rotor Collective to Lateral Velocity.

Figure 3.13: Experimental Frequency in Hover, Tail Rotor Collective to Yaw Rate.

Comparison of the Heffley model response to the actual aircraft response reveals modelling discrepancies as a result of uncertain error dynamics from the various cited sources. Initially it is noted that the Heffley model does not include sensor and actuator dynamics. These dynamics are typically first order and fairly fast. They affect all channels or axes equally with the exception of sensor dynamics in pitch and roll attitudes. Measurement of pitch and roll attitudes comes from computer integration of the respective rates with negligible dynamics. Noise effects in the sensors will also be ignored in the nominal model. The sensor and actuator dynamics will be modelled as

$$E_{sen}(s) = \frac{1}{\tau_s s + 1} \quad (3.17)$$

$$E_{act}(s) = \frac{1}{\tau_a s + 1} \quad (3.18)$$

respectively, where $\tau_s = 0.06$ s and $\tau_a = 0.03$ s. Thus these dynamics are typically well beyond the region of pilot closed loop control. State space realizations for the MIMO systems representing sensor and actuator dynamics are:

$$\begin{aligned} \dot{x}_s &= -1/\tau_s \cdot I_6 x_s + I_6 u_s \\ &= A_s x_s + B_s u_s \\ y_s &= 1/\tau_s \cdot I_6 x_s + 0_{6 \times 6} u_s \\ &= C_s x_s + D_s u_s \end{aligned} \quad (3.19)$$

$$\begin{aligned} \dot{x}_a &= -1/\tau_a \cdot I_4 x_a + I_4 u_a \\ &= A_a x_a + B_a u_a \\ y_a &= 1/\tau_a \cdot I_4 x_a + 0_{4 \times 4} u_a \\ &= C_a x_a + D_a u_a \end{aligned} \quad (3.20)$$

where I_i denotes the identity matrix size $i \times i$; and $0_{i \times j}$ the zero matrix size $i \times j$.

Of primary importance in achieving high-bandwidth control of the Bell 205 is consideration of the rotor mast flexing mode. This primary response involves movement of the transmission and rotor mast constrained by the engine mounts and other linkages. The mode can be seen on the experimental frequency response plots as a resonant peak at about 14 rad/s. From pilot experience and comparison with standard second order system response curves, the damping ratio of this mode was estimated at 0.15 for the Bell 205. Thus the mode is described by the error dynamics

$$E_{flex}(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \quad (3.21)$$

where $\omega_n = 14.0$ rad/s and $\zeta = 0.15$. The rotor mast flexing mode occurs when collective, longitudinal or lateral cyclic controls are applied. The mode is relatively fast and therefore appears in the rate modes of the aircraft associated with their most influential input. It is assumed that this mode will affect the following strong input-output channels equally:

1. collective and vertical velocity
2. longitudinal cyclic and pitch rate
3. lateral cyclic and roll rate

This mode is also included more conservatively in the following strongly coupled channels:

4. longitudinal cyclic and roll rate
5. lateral cyclic and pitch rate

The state space realization representing rotor mast flexing mode dynamics is:

$$\begin{aligned}
\dot{x}_f &= \begin{bmatrix} -2\zeta\omega_n & -\omega_n^2 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2\zeta\omega_n & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2\zeta\omega_n & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} x_f + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} u_f \\
&= A_f x_f + B_f u_f \\
y_f &= \begin{bmatrix} 0 & \omega_n^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega_n^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \omega_n^2 \end{bmatrix} x_f + 0_{3 \times 3} u_f \\
&= C_f x_f + D_f u_f
\end{aligned} \tag{3.22}$$

An important consideration in modelling the uncertainty of this type of system is the inclusion of high frequency unmodelled dynamics. These dynamics include rotor modes, structural modes and mechanical component responses. They are manifested in the frequency response curves as additional high frequency phase lag. The dynamics will be modelled as pure delay with a conservative estimate of 100 ms delay. This delay is modelled as a first order Padé approximation:

$$E_{del}(s) = \frac{-(\tau_d/2)s + 1}{(\tau_d/2)s + 1} \tag{3.23}$$

with $\tau_d = 0.1$ s. This mode is the fastest and manifests itself mainly in the rotational rate modes of the aircraft, which are the first aircraft modes to respond to helicopter inputs. It is assumed that this mode will affect the following strong input-output channels equally:

1. longitudinal cyclic and pitch rate
2. lateral cyclic and roll rate
3. tail rotor collective and yaw rate

and again more conservatively in the following strongly coupled channels:

4. longitudinal cyclic and roll rate
5. lateral cyclic and pitch rate

Its state space realization is:

$$\begin{aligned}
 \dot{x}_d &= -2/\tau_d \cdot I_3 x_d + I_3 u_d \\
 &= A_d x_d + B_d u_d \\
 y_d &= 4/\tau_d \cdot I_3 x_d - I_3 u_d \\
 &= C_d x_d + D_d u_d
 \end{aligned} \tag{3.24}$$

Finally, low frequency system/model mismatch is corrected by the inclusion of a lead-lag filter. Typically, it is observed that the model break frequency is lower than that of the actual plant. Usually, the model break frequency is about 1 rad/s while the actual system breaks at 2 rad/s. Although this perturbation appears very significant, the exact values of these uncertainties are not critical, as this factor is not as dominant as the shaft flexing mode. Using this observed data, the low frequency error uncertainty can be described by

$$E_{low}(s) = \frac{\tau_m s + 1}{\tau_l s + 1} \tag{3.25}$$

with $\tau_m = 1.0$ and $\tau_l = 2.0$ to effectively speed up the model to match the true system. Upon investigation of the data it is necessary to include the low frequency dynamics in the following input-output channels:

1. collective and vertical velocity
2. longitudinal cyclic and pitch rate
3. lateral cyclic and roll rate

and more conservatively in:

4. longitudinal cyclic and roll rate
5. lateral cyclic and pitch rate

with the resulting state space realization

$$\begin{aligned}
 \dot{x}_l &= -1/\tau_l \cdot I_4 x_l + I_4 u_l \\
 &= A_l x_l + B_l u_l \\
 y_l &= (\tau_l - \tau_m)/\tau_l^2 \cdot I_4 x_l - I_4 u_l \\
 &= C_l x_l + D_l u_l
 \end{aligned} \tag{3.26}$$

3.5.3 Higher-Order Nominal Model

The equations governing the motion of the helicopter are complex and difficult to formulate with high levels of precision. The low-order Heffley model is undoubtedly in error because of neglected and unmodelled dynamics, or simply error dynamics. A detailed study of the associated uncertainty error dynamics was performed. Transfer function descriptions of the error dynamics between the low-order Heffley model and the actual aircraft were created using information gathered from both experimental data of the actual aircraft and pilot experience. The transfer function descriptions of the error dynamics provide an accurate and reliable representation of the neglected or unmodelled dynamics in the Heffley model.

Piecing together the error dynamics of the Heffley model with the model itself results in a higher-order model whose frequency domain response now more closely matches the experimental frequency domain aircraft response. Although the interconnection structure is quite complicated for this complex system, many software packages, such as μ - TOOLS™ used here, are available which can be used to generate the interconnection structure from system components. A diagram of the interconnection structure of the higher-order model is depicted in Figure 3.14. The blocks in the diagram represent dynamical system components with block diagonal structures described by (3.17-3.26).

Figure 3.14: Interconnection Structure of the Higher-Order Model.

They are interconnected with 4 copies of the Heffley model (3.12) to account for all reasonable dynamics of the helicopter. The result is a higher-order model of the Bell 205 with reduced dynamical uncertainty. It better represents the Bell 205 helicopter trimmed at a nominal $+5^\circ$ pitch attitude, with a mid-range weight, a mid-position centre of gravity and operating in-ground effect at near sea level. The higher-order model, the augmentation of the Heffley model with its error dynamics model, is described by

$$\dot{x}_{h/e} = A_{h/e} x_{h/e} + B_{h/e} u \quad (3.27)$$

where the pair $(A_{h/e}, B_{h/e})$ is defined with consistent units.

3.5.4 Uncertainty Description of the Linear Design Model

It is unreasonable to assume that dynamical uncertainty can be completely removed from the nominal model of the helicopter. However effort has been extended to explicitly include unmodelled or error dynamics using reliable sources to produce a nominal linear model with smaller dynamical uncertainty that more accurately describes the Bell 205 helicopter.

Parametric uncertainty exists in the modelling of sensor and actuator dynamics from the imperfect knowledge of the physical parameter values τ_s and τ_a used to model these first order systems. The uncertainty used in these parameters is $\pm 1/10$ of their nominal values. This uncertainty may also account for variations of these parameters during operation such that each of the uncertain physical parameter values τ_s and τ_a are bounded respectively in the region

$$[\tau_s - 1/10 \tau_s, \tau_s + 1/10 \tau_s], \quad [\tau_a - 1/10 \tau_a, \tau_a + 1/10 \tau_a].$$

Uncertainty also exists in the parameters ω_n and ζ used to model the neglected or unmodelled rotor mast flexing mode. The mode can be seen on the experimental frequency response plots as a resonant peak and from pilot experience and comparison with standard second order systems the damping ratio of this mode can be identified. The description of this mode is considered quite reliable but a larger uncertainty in the parameters ω_n and ζ is approximated conservatively at $\pm 1/5$ of their nominal values. Therefore the parameters ω_n and ξ are believed to lie in the bounded region

$$[\omega_n - \frac{1}{5}\omega_n, \omega_n + \frac{1}{5}\omega_n], \quad [\zeta - \frac{1}{5}\zeta, \zeta + \frac{1}{5}\zeta].$$

It is important to note that dynamical uncertainty is not completely removed from the higher-order model. Instead it is embedded in the higher-order model as the inclusion of low frequency system/model mismatch and high frequency unmodelled dynamics. To allow these dynamics to cover a larger range of frequencies, parametric uncertainty is included in the coefficients used to model these dynamics. The coefficients, τ_m and τ_l , which model the low frequency system/model mismatch are extended to cover the range of values

$$[\tau_m - \frac{1}{3}\tau_m, \tau_m + \frac{1}{3}\tau_m], \quad [\tau_l - \frac{1}{3}\tau_l, \tau_l + \frac{1}{3}\tau_l].$$

This is also true of the coefficient τ_d used to model the high frequency unmodelled dynamics. Uncertainty increases at high frequencies thus the coefficient τ_d is assumed to lie in the larger interval

$$[\tau_d - \frac{1}{2}\tau_d, \tau_d + \frac{1}{2}\tau_d].$$

Thus the model represented by the pair $(A_{h/e}, B_{h/e})$ is ready for a robust control design to handle its uncertainty description.

3.6 Summary

This chapter demonstrates how to use the LFT to map realistic plant uncertainties into the appropriate mathematical framework for robust control. The structured singular value and μ -analysis as a tool for determining the robust stability characteristics of closed loop feedback systems are also presented. The combination of LFT modelling and μ -analysis provides a powerful approach for worst-case stability analysis of control systems. Model uncertainty will always exist in any real system. Thus no mathematical system can precisely model a real physical system. Uncertainty arises from two classes; dynamical and parametric uncertainty. Dynamical uncertainty results from missing or neglected dynamics and parametric uncertainty stems from the imperfect knowledge of physical parameter values or variation of these parameters during operation.

The **HELICOPTER PROBLEM** is posed as a problem of optimal nominal performance with the additional constraint of robust stability. Experimental frequency domain responses of the actual helicopter provide a clear and reliable picture of the uncertainty between the nominal model and actual physical plant. The result is a higher-order model that more accurately describes the helicopter. It is unreasonable to assume that uncertainty can be completely removed from the higher-order nominal model and so the uncertainty of the higher-order linear design model is also discussed.

CHAPTER 4

$\mathcal{H}_2$ and $\mathcal{H}_\infty$ -OPTIMIZATION AND MIXED NORM CONTROL

4.1 Introduction

This chapter reviews the state space solutions to standard $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control problems. The purpose of this chapter is to motivate the use of mixed norm control and present the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -optimal synthesis theory and to explore its effectiveness in application to a realistic control problem.

One way to describe the control system performance is in terms of the size of certain signals of interest. For example, the performance of a tracking system could be measured by the size of the error signal. In §4.2 norms for signals are introduced as a measure for defining the size of a signal. The $\mathcal{H}_2$ and $\mathcal{H}_\infty$ are defined as system norms for evaluating the performance of systems and in §4.3 show their usefulness given their signal interpretation. Controllers to achieve optimality with respect to these norms will be developed. The Riccati operator is introduced in §4.4 as algebraic Riccati equations are used to compute state space realizations for both $\mathcal{H}_2$ and $\mathcal{H}_\infty$ optimal control laws.

Optimization is introduced in §4.5 with the general $\mathcal{H}_2$ problem statement and review of optimization theory in optimal control of linear time invariant systems with a quadratic performance criterion. Review of the general $\mathcal{H}_\infty$ -optimal synthesis theory developed by is treated in §4.6. Mixed norm control is introduced in §4.7 with motivation for its use followed by the presentation of mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -optimal synthesis theory.

The effectiveness of the mixed $\mathcal{H}_2 / \mathcal{H}_\infty$ -optimal control design is investigated in application to a benchmark problem in §4.8 and used to determine the potential for general application. The results of controller synthesis are presented for a closed loop system that achieves $\mathcal{H}_\infty$ robust stability with optimal nominal $\mathcal{H}_2$ performance. Detailed analysis, including μ -analysis of the robust stability characteristics for the closed loop system is also presented and used to evaluate the effectiveness of the controller synthesis.

It is concluded that the mixed norm control synthesis can be expected to be conservative yet effective in general application. The mixed $\mathcal{H}_2 / \mathcal{H}_\infty$ control design allows the designer to simultaneously address $\mathcal{H}_2$ and $\mathcal{H}_\infty$ design goals and synthesize a controller with an acceptable trade-off between their objectives.

4.2 Norms for Signals and Systems

It is useful to have a single number which gives an overall measure of the size of a vector, a matrix, a signal, or system. For this purpose functions called norms are used.

Definition 4.1: A norm of e (which may be a vector, matrix, signal, or system) is a real number, denoted $\|e\|$, that satisfies the following properties:

1. *Non-negative:* $\|e\| \geq 0$.
2. *Positive:* $\|e\| = 0 \Leftrightarrow e = 0$
3. *Homogenous:* $\|\alpha \cdot e\| = |\alpha| \cdot \|e\|$ for all complex scalars α .
4. *Triangle inequality:* $\|e_1 + e_2\| \leq \|e_1\| + \|e_2\|$

More precisely, e is an element in a vector space V over the field $\mathbb{C}$ of complex numbers, and the properties above must be satisfied $\forall e, e_1, e_2 \in V$ and $\forall \alpha \in \mathbb{C}$.

Consider a vector a with m elements, that is, the vector space is $V \in \mathbb{C}^m$. The vector p -norm is defined as:

$$\|a\|_p = \left(\sum_i |a_i|^p \right)^{1/p}$$

where $p \geq 1$ to satisfy the triangle inequality (property 4 of a norm). Two norms are considered here which are special cases of the vector p -norm. The first is the vector 2-norm (Euclidean norm):

$$\|a\|_2 := \sqrt{\sum_i |a_i|^2}$$

which also satisfies the property

$$a^* a = \|a\|_2^2$$

where a^* denotes the complex conjugate transpose of the vector a . The vector ∞ -norm is the largest element magnitude in the vector so that

$$\|a\|_\infty := \max_i |a_i|.$$

The temporal p -norm, $\|e(t)\|_p$, of a time-varying vector is defined as

$$l_p \text{ -norm: } \|e(t)\|_p = \left(\int_{-\infty}^{\infty} \sum_i |e_i(\tau)|^p d\tau \right)^{1/p}$$

The most commonly used temporal norms of signals are the 1-norm in time (integral absolute error)

$$\|e(t)\|_1 = \int_{-\infty}^{\infty} \sum_i |e_i(\tau)| d\tau$$

the 2-norm in time (quadratic norm, integral square error and “energy” of signal)

$$\|e(t)\|_2 = \sqrt{\int_{-\infty}^{\infty} \sum_i |e_i(\tau)|^2 d\tau}$$

and the ∞ -norm in time

$$\|e(t)\|_{\infty} = \max_i \left(\max_{\tau} |e_i(\tau)| \right)$$

Norms for systems may be defined analogously for which a treatment of this topic may be found in [SKO96].

Two important norms are defined for matrices $G(s)$ whose entries are *analytic* functions of the complex variable s in the right half s -plane. The matrix entries of which are real rational functions of s ; thus the only points where the entries can fail to be *analytic* are at their poles. The $\mathcal{H}_2$ -norm of $G(s)$ is denoted as $\|G\|_2$ and is defined to be

$$\|G\|_2 := \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \text{trace}[G(j\omega) * G(j\omega)] d\omega \right)^{1/2}$$

where *trace* M denotes the trace of the matrix M . The $\mathcal{H}_{\infty}$ -norm of $G(s)$ is denoted $\|G(s)\|_{\infty}$ and is defined to be

$$\|G\|_{\infty} := \sup_{\omega} \bar{\sigma}(G(j\omega))$$

In particular, a matrix $G(s)$ is said to belong to the matrix *Hardy space* $\mathcal{H}_2$ if it is square-integrable on the imaginary axis with *analytic* continuation into the right half s -plane. Similarly, a matrix $G(s)$ is said to belong to the matrix *Hardy space* $\mathcal{H}_\infty$ if it is bounded with *analytic* continuation into the right half s -plane.

4.3 Signal Interpretation of the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ System Norms

Consider a system with stable transfer function matrix $G(s)$ and impulse response matrix $g(t)$. The input and output of the system are d and e , respectively, such that

$$e = Gd$$

For performance it may be that the output signal e be kept “small” for any allowed input signals d . It is therefore necessary to specify:

1. What d s are allowed. (Which set does d belong to?)
2. What is meant by “small”. (Which norm should be used for e ?)

Some possible input signal sets are:

1. $d(t)$ is a white noise process with zero mean.
2. $d(t)$ consists of impulses $\delta(t)$, which generate step changes in the states.
3. $d(t)$ is any signal bounded in energy, $\|d(t)\|_2 \leq 1$.

The first two sets of inputs are specific signals, whereas the latter is a class of inputs with bounded norm. The physical problem at hand determines which of these input classes is the most reasonable.

To measure the output signal one may consider the following norms:

1. 2-norm (energy), $\|e(t)\|_2$
2. ∞ -norm (peak magnitude), $\|e(t)\|_\infty$

Other norms are possible but it is engineering issues that determine which norm is most appropriate.

4.3.1 System Norms for the Set of Specific and Norm Bounded Input Signals

With the signal norms defined for the output signals and the various classes of input signals it is possible to define the resulting system norms. With the input $d(t)$ as a white noise process with zero mean the result is $\|e(t)\|_2 = \|G(s)\|_2$. This is the stochastic interpretation of the $\mathcal{H}_2$ -norm in terms of the quadratic criterion in LQG control. Here the expected root mean square (rms) value of the output is measured in response to white noise excitation. This type of control problem is not addressed in this thesis. Instead the following deterministic performance interpretation of the $\mathcal{H}_2$ -norm is used:

$$\|G(s)\|_2 = \max_{d(t)=\text{unit impulses}} \|e(t)\|_2$$

The $\mathcal{H}_2$ -norm does have a number of good mathematical and numerical properties, and its minimization has important engineering implications. However, the $\mathcal{H}_2$ -norm is *not* an induced norm and does *not* satisfy the multiplicative property. This is the source of its shortcoming in expressing plant uncertainty. The appropriate system gain for the set of norm bounded input signals defined in §4.2 and the usual 2-norm for the output is the $\mathcal{H}_\infty$ -norm:

$$\|G(s)\|_\infty = \max_{d(t) \neq 0} \frac{\|e(t)\|_2}{\|d(t)\|_2} = \max_{\|d(t)\|_2=1} \|e(t)\|_2$$

Note in particular that the $\mathcal{H}_\infty$ -norm is equal to the induced (worst-case) 2-norm in the time domain [DES75]. The reason for the popularity of the $\mathcal{H}_\infty$ -norm and its use in robust control is mainly attributed to the convenience for representing unstructured model uncertainty, and because it satisfies the multiplicative property:

$$\|A(s)B(s)\|_\infty \leq \|A(s)\|_\infty \|B(s)\|_\infty.$$

4.4 The Riccati Operator

The algebraic Riccati matrix equation is fundamental to linear optimal control and robust control theory. It is used with the definition of the Riccati operator to compute state space realizations for optimal control laws in both $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control.

Let A, Q, R be real $n \times n$ matrices with Q and R symmetric. Define the $2n \times 2n$ Hamiltonian matrix:

$$H := \begin{bmatrix} A & R \\ -Q & -A^T \end{bmatrix}.$$

Assume H has no eigenvalues on the imaginary axis. Then it must have n eigenvalues in $\text{Re } s < 0$ and n in $\text{Re } s > 0$. Consider the two n -dimensional spectral subspaces $\mathfrak{N}_-(H)$ and $\mathfrak{N}_+(H)$: the former is the invariant subspace corresponding to eigenvalues in $\text{Re } s < 0$; the latter, to eigenvalues in $\text{Re } s > 0$. Finding a basis for $\mathfrak{N}_-(H)$, stacking the basis vectors up to form a matrix, and partitioning the matrix, results in

$$\mathfrak{N}_-(H) = \text{Im} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

where $X_1, X_2 \in \mathfrak{R}^{n \times n}$. If X_1 is nonsingular, or equivalently, if the two subspaces:

$$\mathfrak{N}_-(H), \quad \text{Im} \begin{bmatrix} 0 \\ I \end{bmatrix} \quad (4.1)$$

are complimentary, then it is possible to set $X := X_2 X_1^{-1}$. Then X is uniquely determined by H and is thus a function denoted by Ric ; thus $X = Ric(H)$. Take the domain of Ric , denoted $dom(Ric)$, to consist of Hamiltonian matrices H with two properties, namely, H

has no eigenvalues on the imaginary axis and the two subspaces in (4.1) are complimentary. The following well-known results give some properties of X .

Lemma 4.1: Suppose $H \in \text{dom}(\text{Ric})$ and $X = \text{Ric}(H)$. Then:

- a) X is symmetric
- b) X satisfies the algebraic Riccati equation

$$A^T X + XA + XRX - Q = 0;$$

- c) $A + RX$ is stable.

Proof: [DOY89]

4.5 Control Systems Designs As $\mathcal{H}_2$ -Optimizations

With the $\mathcal{H}_2$ -norm defined, many practical feedback design problems can be formulated as the minimization of the $\mathcal{H}_2$ -norm of an appropriately weighted closed loop transfer function matrix. The $\mathcal{H}_2$ -norm is probably the most popular performance measure in optimal control theory due to its mathematical tractability and ease of solution based on state space methods [DOY89].

4.5.1 Review of $\mathcal{H}_2$ -Optimization Theory

The system considered in this section is described by GCC block diagram of Fig. 3.1. Both G and K are real rational and proper. K is constrained to provide internal stability, with state models of G and K defined. Then *internal stability* will mean that the states of G and K go to zero from all initial values when $w = 0$. Since controllers are restricted to being proper and real rational which are also stabilizable and detectable, these properties

will be assumed throughout. Thus, the term controller will be taken to mean a controller that satisfies these properties. Controllers that have the additional property of being internally stabilizable will said to be admissible. The realization of the transfer matrix G is taken to be of the form

$$G = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{bmatrix}.$$

Notice the special off-diagonal structure of D : D_{22} is assumed to be zero so that G_{22} is strictly proper; also, D_{11} is assumed to be zero in order to guarantee that the $\mathcal{H}_2$ problem is properly posed. The following additional assumptions are made for the output feedback $\mathcal{H}_2$ problem:

- i) (A, B_1) is stabilizable and (C_1, A) is detectable;
- ii) (A, B_2) is stabilizable and (C_2, A) is detectable;
- iii) $D_{12}^T [C_1 \ D_{12}] = [0 \ I]$.
- iv) $\begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} D_{21}^T = \begin{bmatrix} 0 \\ I \end{bmatrix}$.

The first assumption is made for a technical reason: together with ii) it guarantees that the two Hamiltonian matrices associated with the $\mathcal{H}_2$ problem belong to $\text{dom}(\text{Ric})$. Assumption ii) is necessary and sufficient for G to be internally stabilizable. Assumption iii) implies that $C_1 x$ and $D_{12} u$ are orthogonal so that the penalty on $z = C_1 x + D_{12} u$ includes a nonsingular, normalized component on the control u . Assumption iv) is dual to iii) and concerns how the exogenous signal w enters G . The $\mathcal{H}_2$ problem statement is then summarized as:

$\mathcal{H}_2$ Problem: The $\mathcal{H}_2$ control problem is to find a proper, real rational controller K which stabilizes G internally and minimizes the $\mathcal{H}_2$ -norm of the transfer matrix $F_r(G, K)$ in $z = F_r(G, K)w$.

By Lemma 4.1, the Hamiltonian matrices

$$H_2 := \begin{bmatrix} A & -B_2 B_2^T \\ -C_1^T C_1 & -A^T \end{bmatrix}, \quad J_2 := \begin{bmatrix} A^T & -C_2^T C_2 \\ -B_1 B_1^T & -A \end{bmatrix}$$

belong to $\text{dom}(\text{Ric})$, and, moreover, $X_2 := \text{Ric}(H_2)$ and $Y_2 := \text{Ric}(J_2)$ are positive semi-definite. Define

$$F_2 := -B_2^T X_2, \quad L_2 := -Y_2 C_2^T$$

and

$$\begin{aligned} A_{F_2} &:= A + B_2 F_2, & C_{1F_2} &:= C_1 + D_{12} F_2 \\ A_{L_2} &:= A + L_2 C_2, & B_{1L_2} &:= B_1 + L_2 D_{21} \\ \hat{A}_2 &:= A + B_2 F_2 + L_2 C_2 \end{aligned}$$

$$G_c(s) := \begin{bmatrix} A_{F_2} & I \\ C_{1F_2} & 0 \end{bmatrix}, \quad G_f(s) := \begin{bmatrix} A_{L_2} & B_{1L_2} \\ I & 0 \end{bmatrix}.$$

Theorem 4.1: The unique optimal controller is

$$K_{opt}(s) := \begin{bmatrix} \hat{A}_2 & -L_2 \\ F_2 & 0 \end{bmatrix}$$

Proof: [DOY89]

The controller K_{opt} has the well-known separation structure, which is useful for the special case of state feedback. If the state of the plant, x , is known with $C_2 = I$ and $D_{21} = 0$ such that $y = x$ then K_{opt} reduces to

$$K_{opt}(s) := \begin{bmatrix} 0 & 0 \\ F_2 & 0 \end{bmatrix}$$

4.5.2 Linear Quadratic Regulator Problem

Consider the following dynamical system:

$$\dot{x} = Ax + B_2 u, \quad x(t_0) = x_0$$

where x_0 is given but arbitrary. Our objective is to find a control function $u(t)$ defined on $[t_0, t_f]$ which can be a function of the state $x(t)$ such that the state $x(t)$ is driven to a small neighbourhood of the origin at time t_f . This is the so-called *Regulator Problem*. One might suggest that this regulator problem can be trivially solved for any $t_f > t_0$ if the system is controllable. This is indeed the case in theory if the controller can provide arbitrarily large amounts of energy since, by definition of controllability, one can immediately construct a cost function that will drive the state to zero in an arbitrarily short time. This however, is not practical since any physical system has the energy limitation, whereby the actuator will eventually saturate. Furthermore, large control action can easily drive the system out of the region where the given linear model is valid. Hence certain limitations have to be imposed on the control in practical engineering implementation [ZHO96].

The constraints on $u(t)$ may be measured in many different ways; for example,

$$\int_{t_0}^{t_f} u^T(t) u(t) dt, \quad \int_{t_0}^{t_f} [u^T(t) u(t)]^{1/2} dt, \quad \sup_{t \in [t_0, t_f]} \|u(t)\|_2$$

or

$$\int_{t_0}^{t_f} u^T(t) R u(t) dt$$

where R is a positive definite matrix, which, without loss of generality, can always be taken as symmetric. Similarly one might also want to impose some constraint on the transient response $x(t)$ in a similar fashion

$$\int_{t_0}^{t_f} x^T(t) Q x(t) dt$$

where Q is symmetric positive definite. Hence the regulator problem can be posed as an optimal control problem with certain combined performance index on $u(t)$ and $x(t)$ as given above. In defining the constraints on $u(t)$ and $x(t)$ it is desired to seek constraints that have an engineering sense in that the appropriate quantities are penalized. If the input vector $u(t)$ consists of capacitor voltages $v(t)$ and inductor currents $i(t)$, then $u^T(t) u(t)$ will contain terms like $v^2(t)$ and $i^2(t)$. Similarly, if velocity $s(t)$ is a state component then $x^T(t) x(t)$ will contain terms like $s^2(t)$.

Moreover focus on the infinite time regulator problem (i.e. $t_f \rightarrow \infty$) and without loss of generality assume $t_0 = 0$. In this case, the problem is as follows: Find a control $u(t)$ defined on $[0, \infty)$ such that the state $x(t)$ is driven to the origin at $t \rightarrow \infty$ and the following performance index is minimized:

$$\min_u \int_0^\infty x^T Q x + u^T R u \quad dt$$

for some $Q = Q^T \geq 0$ and $R = R^T > 0$. This problem is traditionally called a linear quadratic regulator (LQR) problem and represents a generalized minimum energy problem where the concern is minimizing the energy in the states without using too much control energy.

4.5.3 GCC for LQR State Feedback

Consider the *traditional problem statement* for LQR state feedback, which is to minimize the time domain cost function

$$J = \int_0^\infty z_{perf}^T Q z_{perf} + u^T R u \, dt \quad (4.2)$$

for the feedback system described by:

$$\begin{aligned} \dot{x} &= Ax + B_2 u \\ z_{perf} &= Cx \\ u &= F_2 x \end{aligned}$$

for any arbitrary initial condition $x(0) = x_0$, where $x(t)$ is the state vector, $u(t)$ is the input vector commanded by the unique optimal state feedback controller $K = F_2$, and $z_{perf}(t)$ is a criterion or performance output vector.

The GCC becomes LQR full state feedback when the objective to find a stabilizing K such that the performance measure $\|F_r(G, K)\|_2$ is minimized when y is assumed to be the full state vector and z has a form equivalent to the Q and R weighting matrices on outputs and controls. The *equivalent frequency domain problem* in terms of the GCC in Fig. 3.1 is to solve for K to minimize $\|F_r(G, K)\|_2$ where G has the state space realization:

$$G_{SF,2}(s) = \begin{bmatrix} A & I & B_2 \\ Q^{1/2}C & 0 & 0 \\ 0 & 0 & R^{1/2} \\ I & 0 & 0 \end{bmatrix}$$

where the matrix square roots are square matrices satisfying

$$(Q^{1/2})^T Q^{1/2} = Q, \quad (R^{1/2})^T R^{1/2} = R$$

and where $z = Q^{1/2}Cx + R^{1/2}u$ forms the external output vector, with $y = x$ forming the measurement vector available for feedback. Minimizing the 2-norm of the closed loop system transfer function from z to w is equivalent to minimizing the traditional cost function in (4.2) because the squared 2-norm of the output vector z at any point in time is

$$z^T z = x^T C^T Q C x + u^T R u = z_{\text{perf}}^T Q z_{\text{perf}} + u^T R u$$

4.6 Control Systems Designs As $\mathcal{H}_\infty$ -Optimizations

Many practical feedback design problems can be formulated as the minimization of the $\mathcal{H}_\infty$ -norm of an appropriately weighted transfer function matrix. The minimization of such a norm is now well understood and all the computations can be carried out using state space representations of the transfer function matrices [DOY89]. With reference to the GCC of Fig. 3.1, the standard $\mathcal{H}_\infty$ -optimal control problem is to find all stabilizing controllers K which minimize $\|F_r(G, K)\|_\infty$.

As discussed in §4.3.1 the $\mathcal{H}_\infty$ -norm has several interpretations in terms of performance. One is that it minimizes the peak of the maximum singular value of $F_r(G, K)(j\omega)$. It also has the time domain interpretation as the induced (worst-case) 2-norm. The important

property of the $\mathcal{H}_\infty$ -norm comes from the application of the small gain theorem as a robust stability test under various model uncertainty assumptions.

4.6.1 Review of $\mathcal{H}_\infty$ -Optimization Theory

In practice, it is usually not necessary to obtain an optimal controller for the $\mathcal{H}_\infty$ problem, and it is often computationally (and theoretically) simpler to design a suboptimal one. Let $\gamma_{\min}$ be the minimal value of $\|F_r(G, K)\|_\infty$ over all stabilizing controllers K . The problem considered in this subsection is the suboptimal $\mathcal{H}_\infty$ control problem:

Suboptimal $\mathcal{H}_\infty$ Problem: Given a $\gamma > \gamma_{\min}$, find all admissible K such that

$$\|F_r(G, K)\|_\infty < \gamma.$$

For the GCC of Fig. 3.1 with assumptions i-iv) of §4.5.1, the $\mathcal{H}_\infty$ solution involves two Hamiltonian matrices

$$H_\infty := \begin{bmatrix} A & \gamma^{-2} B_1 B_1^T - B_2 B_2^T \\ -C_1^T C_1 & -A^T \end{bmatrix}, \quad J_\infty := \begin{bmatrix} A^T & \gamma^{-2} C_1^T C_1 - C_2^T C_2 \\ -B_1 B_1^T & -A \end{bmatrix}.$$

There exists a stabilizing controller K such that $\|F_r(G, K)\|_\infty < \gamma$ if and only if the following three conditions hold:

- i) $H_\infty \in \text{dom}(\text{Ric})$ and $X_\infty := \text{Ric}(H_\infty) \geq 0$.
- ii) $J_\infty \in \text{dom}(\text{Ric})$ and $Y_\infty := \text{Ric}(J_\infty) \geq 0$.
- iii) $\rho(X_\infty Y_\infty) < \gamma^2$

where $\rho(M)$ is the spectral radius or maximum absolute valued eigenvalue of M . Moreover, when these conditions hold, one such controller is

$$K_{sub}(s) := \begin{bmatrix} \hat{A}_\infty & -Z_\infty L_\infty \\ F_\infty & 0 \end{bmatrix}$$

where

$$\begin{aligned} \hat{A}_\infty &:= A + \gamma^{-2} B_1 B_1^T X_\infty + B_2 F_\infty + Z_\infty L_\infty C_2 \\ F_\infty &:= -B_2^T X_\infty, \quad L_\infty := -Y_\infty C_2^T, \quad Z_\infty := (I - \gamma^{-2} Y_\infty X_\infty)^{-1}. \end{aligned}$$

4.6.2 $\mathcal{H}_\infty$ State Feedback

If full state information is available the design is simplified. Consider an open loop system transfer matrix

$$G_{SF/\infty}(s) = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & 0 & D_{12} \\ I & 0 & 0 \end{bmatrix}$$

with the same assumptions inherited from the output feedback problem. There exists a stabilizing controller K such that $\|F_\ell(P, K)\|_\infty < \gamma$ if and only if the following condition holds.

- i) $H_\infty \in \text{dom}(\text{Ric})$ and $X_\infty := \text{Ric}(H_\infty) \geq 0$. The resulting suboptimal controller in this case is then:

$$K_{sub}(s) := \begin{bmatrix} 0 & 0 \\ F_\infty & 0 \end{bmatrix}.$$

4.7 $\mathcal{H}_2/\mathcal{H}_\infty$ Control

The purpose of this section is to motivate the use of mixed norm control and present a mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -optimal control problem and its solution. The “secret” of optimal control is to construct formal cost functions that tend to cause these genuine system requirements to be satisfied. Although the $\mathcal{H}_2$ -norm or the $\mathcal{H}_\infty$ -norm may often be genuine design objectives, neither the $\mathcal{H}_2$ -norm nor the $\mathcal{H}_\infty$ -norm alone usually will not exactly correspond to the true requirements. To combine all the genuine objectives using a simplified norm as a cost function, as with LQ and $\mathcal{H}_\infty$ synthesis, requires a compromise: the norm does not directly include all of the design objectives.

One of the most attractive features of the recently developed mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -optimal control design by [ROT91] is the ability to address both $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance objectives simultaneously. This eliminates the need to compromise the design objectives and establishes a link between $\mathcal{H}_2$ and $\mathcal{H}_\infty$.

4.7.1 Motivation for the Use of Mixed Norm Control

Actual engineering control problems invariably include a set of conflicting requirements on time response, frequency response, actuator activity, disturbance rejection, and robustness to changing conditions and plant uncertainty. The key to successful application of optimal control is to construct formal cost functions that tend to cause these genuine system requirements to be satisfied.

Although the $\mathcal{H}_2$ -norm may be a meaningful performance measure and although LQ theory can give efficient design compromises under certain disturbance and plant assumptions, the major limitation of the theory is the lack of formal treatment of uncertainty in the plant itself. The result is a controller adept at attenuating impulse disturbances but with potentially weak robustness characteristics. Much of modern control theory relies upon a mathematical model of the plant to be controlled. In the real

world, there is always uncertainty in any such model: the actual response of the plant may be quite different from the assumed model. Furthermore the plant may change as it ages or as operating conditions vary. $\mathcal{H}_\infty$ optimization focuses on the minimization of system output energy to unknown, but bounded, energy inputs. $\mathcal{H}_\infty$ control can be formulated to give robust stability to model uncertainty. The $\mathcal{H}_\infty$ -optimal controller results in a highly robust system; however, one which can be notably deficient in providing optimal nominal performance.

The mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control design methodology reviewed in this section, allows $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance objectives to be addressed simultaneously. It establishes a link between $\mathcal{H}_2$ and $\mathcal{H}_\infty$ -optimizations, allowing a trade-off between their objectives. In this manner it is the desired to synthesize an admissible robust state feedback controller that minimizes an $\mathcal{H}_2$ performance objective subject to a constraint on the $\mathcal{H}_\infty$ -norm of a different transfer function matrix.

4.7.2 $\mathcal{H}_2$ -Optimal Control with an $\mathcal{H}_\infty$ Constraint: The State Feedback Case

The control problem addressed in this paper concerns the FDLTI feedback system depicted in Fig. 4.1.

Figure 4.1. The Synthesis Framework.

The states of the generalized plant G are assumed available for feedback. A state space description of G is given by:

$$\begin{aligned}
\dot{x} &= Ax + B_1 w_1 + B_2 w_2 + B_3 u \\
z_1 &= C_1 x + D_1 u \\
z_2 &= C_2 x + D_2 u \\
y &= x.
\end{aligned} \tag{4.3}$$

Let $T_1(K)$ and $T_2(K)$ denote the closed loop transfer matrices from w_1 to z_1 and w_2 to z_2 , respectively. A given controller K is called admissible (for the plant G) if K is real rational proper, and the minimal realization of K internally stabilizes the state space realization of G . The two problems considered in this chapter are defined as follows:

Problem A: Minimal $\mathcal{H}_2$ -norm subject to an $\mathcal{H}_\infty$ -norm constraint. For the plant G defined in (4.3), find an admissible controller K that achieves

$$\inf \{ \|T_1(K)\|_2 : K \text{ admissible and } \|T_2(K)\|_\infty < 1 \}.$$

Problem B: Simultaneous $\mathcal{H}_2/\mathcal{H}_\infty$ -optimal control. For the plant G defined in (4.3) find an admissible controller that achieves

$$\inf \{ \|T_1(K)\|_2 : K \text{ admissible} \} \text{ and such that } \|T_2(K)\|_\infty < 1.$$

Problem A represents a constrained optimization problem. *Problem B* is to find (if it exists), a solution to the unconstrained problem of minimizing an $\mathcal{H}_2$ performance measure that also satisfies an $\mathcal{H}_\infty$ -norm bound. The key point is that a solution to *Problem B* is also a solution to *Problem A*, however the converse need not be true.

Recently *Problem A* has received a great deal of attention, mainly because it represents a problem of optimal nominal performance with robust stability. Indeed, consider a stable (possibly nonlinear) perturbation Δ connected from z_2 to w_2 , then the small gain theorem ensures stability of the perturbed system if the nominal system ($\Delta = 0$) is internally stable

and $\|T_2\|_\infty < 1$, provided that the induced operator norm of Δ is less than or equal to one. Among all the admissible controllers K that provide robust stability, *Problem A* is to find a controller that minimizes the variance of the output z_1 (with $\Delta = 0$) when w_1 is zero mean unit variance white noise. The deterministic interpretation of *Problem A* is to find a controller that minimizes the 2-norm of the output z_1 (with $\Delta = 0$) when subject to impulse inputs, w_1 .

The following standard assumptions are made on the plant G

$$\text{i)} \quad \{A, B_3\} \text{ is stabilizable.} \quad (4.4a)$$

$$\text{ii)} \quad D_1 \text{ and } D_2 \text{ both have full column rank.} \quad (4.4b)$$

$$\text{iii)} \quad \text{For each } \omega \in \Re \text{ and for } k = 1, 2$$

$$\begin{bmatrix} A - j\omega I & B_3 \\ C_k & D_k \end{bmatrix} \quad (4.4c)$$

has full column rank.

Assumption (4.4a) ensures that the set of admissible controllers is nonempty while assumptions (4.4b, c) are standard and guarantee the LQR problem corresponding to the quadratic cost $\|z_k\|_2$ has an admissible solution. Without loss of further generality further assume that

$$\text{iv)} \quad D_2^T [C_2 \ D_2] = [0 \ I]. \quad (4.4d)$$

4.7.2.1 Parameterization of All $\mathcal{H}_2$ -Optimal Controllers

It is a classical fact that under the assumptions (4.4a-c) for the plant G there exists an admissible controller that minimizes $\|T_1(K)\|_2$. The unique optimal controller, $K_o = F$ is computed according to

$$F := -(D_1^T D_1)(D_1^T C_1 + B_3^T X_1) \quad (4.5)$$

where the Hamiltonian matrix is

$$H_{SF/2} := \begin{bmatrix} A - B_3(D_1^T D_1)^{-1}(D_1^T C_1) & -B_3(D_1^T D_1)^{-1} B_3^T \\ -C_1^T C_1 + (C_1^T D_1)(D_1^T D_1)^{-1}(D_1^T C_1) & -(A - B_3(D_1^T D_1)^{-1}(D_1^T C_1))^T \end{bmatrix}$$

belongs to $\text{dom}(\text{Ric})$, and, moreover, $X_1 := \text{Ric}(H_{SF/2})$.

Perhaps it is less well known that (4.5) is not the only admissible controller that minimizes $\|T_1(K)\|_2$. Theorem 4.2 gives a complete parameterization of all solutions such that $\|T_1(K)\|_2 < \gamma$. The suboptimal controllers are parameterized by a fixed (independent of γ) linear fractional transformation with a free parameter Q . With reference to the realization of the plant G given by (4.3) and with F given by (4.5), define

$$\begin{aligned} \Pi_1 &:= I - B_1 B_1^+ \\ A_F &:= A + B_3 F \\ C_{kF} &:= C_k + D_k F, k = 1, 2. \end{aligned} \quad (4.6-4.8)$$

where M^+ denotes the Moore-Penrose pseudoinverse of the matrix M . The matrix Π_1 is the orthogonal projection onto $(\text{im} B_1)^\perp$, where $\text{im} M$ denotes the image (or range space) of M and $M^\perp$ denotes the orthogonal complement of M . Note that A_F is a stability matrix

since F solves a well-posed LQR problem with pair (A, B_3) . Define the set of transfer matrices:

$$S := \{ Q \in \mathcal{RH}_\infty : Q = W\Pi_1(sI - A_F), W \in \mathcal{RH}_2^+ \} \quad (4.9)$$

Theorem 4.2: Consider the feedback system of Fig. 4.1, with the plant G given by (4.3). Let K denote an admissible controller and T_1 the corresponding closed loop transfer matrix from w_1 to z_1 . Then, K minimizes $\|T_1(K)\|_2$ if and only if K equals the transfer matrix from y to u in

where

$$J := \begin{bmatrix} A_F & 0 & B_3 \\ 0 & F & I \\ -I & I & 0 \end{bmatrix} \quad (4.10)$$

for some $Q \in S$.

Proof: [ROT91]

Note that if $\text{im}B_1 = \mathbb{R}^n$ ($n := \text{state dimension}$) then $\Pi_1 = 0$ and (4.10) reduces to the unique optimal controller $K_0 = F$. On the other hand, if $\text{im}B_1$ is a proper subspace of $\mathbb{R}^n$ then (4.10) generates a family of controllers parameterized by W . This extra freedom can be used to satisfy some additional constraints. A state space representation for the

controller K of Theorem 4.2 is given that will be useful for establishing one of the main result of this section (Theorem 4.3). Let $W \in \mathcal{RH}_2$ be given by

$$W = \begin{bmatrix} A_w & B_w \\ C_w & 0 \end{bmatrix}. \quad (4.11)$$

Then it can be shown that $Q = W\Pi_1(sI - A_F)$ is given by

$$Q = \begin{bmatrix} A_w & A_w B_w \Pi_1 - B_w \Pi_1 A_F \\ C_w & C_w B_w \Pi_1 \end{bmatrix}. \quad (4.12)$$

Substituting this realization of Q in (4.10), the controller K of Theorem 4.2 is given by

$$\begin{aligned} A_K &= A_w - B_w \Pi_1 B_3 C_w, \\ K &= \begin{bmatrix} A_K & A_K B_w \Pi_1 - B_w \Pi_1 A_F \\ C_w & F + C_w B_w \Pi_1 \end{bmatrix}. \end{aligned} \quad (4.13a-b)$$

4.7.2.2 The Simultaneous $\mathcal{H}_2/\mathcal{H}_\infty$ Problem Solution

In this section the solution of the simultaneous $\mathcal{H}_2/\mathcal{H}_\infty$ optimization problem (*Problem B*) is presented. The development is carried out using Theorem 4.2 along with the recent solution to the $\mathcal{H}_\infty$ -optimization problem [DOY89]. Define

$$V_2 := \Pi_1 B_2. \quad (4.14)$$

Note that $V_2 = 0$ if and only if $\text{im} B_2 \subset \text{im} B_1$. Also, define the Hamiltonian matrices

$$H_{2,\infty} := \begin{bmatrix} A & B_2 B_2^T - B_3 B_3^T \\ -C_2^T C_2 & -A^T \end{bmatrix} \quad (4.15)$$

$$J_{2/\infty} := \begin{bmatrix} A_F^T & C_{2F}^T C_{2F} \\ -B_2(I - V_2^* V_2)B_2^T & -A \end{bmatrix} \quad (4.16)$$

Theorem 4.3 to follow provides the necessary and sufficient conditions for the existence of a solution to *Problem B*.

Theorem 4.3: Consider the feedback system of Fig. 4.1, with the plant G given by (4.3). Then there exists an admissible controller K that solves *Problem B* if and only if the following conditions hold:

1. $H_{2/\infty} \in \text{dom}(\text{Ric})$ and $X_{2/\infty} := \text{Ric}(H_{2/\infty}) \geq 0$. (4.17a)
2. $J_{2/\infty} \in \text{dom}(\text{Ric})$ and $Y_{2/\infty} := \text{Ric}(J_{2/\infty})$. (4.17b)
3. $\rho(X_{2/\infty} Y_{2/\infty}) < 1$ (4.17c)

Proof: [ROT91]

When these conditions are met, one solution to *Problem B* is given by

$$K := \begin{bmatrix} A_0 & A_0 \Sigma - \Sigma A_F \\ H - F & F(I - \Sigma) + H \Sigma \end{bmatrix} \quad (4.18a)$$

where

$$\begin{aligned} A_0 &:= A + (I - \Sigma)B_3 H + \Sigma B_3 F + (I - \Sigma)B_2 B_2^T X_2 \\ \Sigma &:= Z_2 B_2 V_2^* \Pi_1 \\ H &:= -B_3^T X_2 \\ Z_2 &:= (I - Y_2 X_2)^{-1}. \end{aligned} \quad (4.18b-e)$$

Proof: [ROT91]

Under assumption (4.4d), condition (4.17a) is equivalent to the existence of an admissible controller K such that $\|T_2(K)\|_\infty < 1$. The other two conditions (4.17b,c) reflect the fact that one of these admissible controllers must also minimize $\|T_1(K)\|_2$. When $\text{im}B_2 \subset \text{im}B_1$, *Problem B* has a solution if and only if condition (4.17b) holds. In this case, F defined in (4.5) is a solution to *Problem B*. Since a solution to *Problem B* is also a solution to *Problem A*, Theorem 4.3 may be used to produce a sufficient condition under which *Problem A* can be solved. If (4.17a) is not satisfied then there is no state feedback controller that internally stabilizes the plant G and yields $\|T_2\|_\infty < 1$. Thus (4.17a) is necessary for a solution to *Problem A* to exist.

An immediate consequence of Theorem 4.3 is the following:

Corollary 4.1: Consider the feedback system of Fig. 4.1, with the plant G given by (4.3). Suppose that conditions (4.17) hold. Then the controller given by (4.18) is an admissible controller that solves *Problem A*. Furthermore,

$$\inf \{ \|T_1(K)\|_2 : K \text{ admissible and } \|T_2(K)\|_\infty < 1 \} = \inf \{ \|T_1(K)\|_2 : K \text{ admissible} \}.$$

4.7.2.3 Special Controller Case

Consider the following two cases for Theorem 4.3.

Case 1 Suppose that $\text{im}B_2 \subset \text{im}B_1$. In this case, $V_2 \neq 0$. Among the admissible controllers that minimize $\|T_1(K)\|_2$ there exists one such that $\|T_2(K)\|_\infty < 1$ if and only if conditions (4.17) hold. It follows that a solution to *Problem B* is given by (4.18).

Case 2 Suppose that $\text{im}B_2 \subset \text{im}B_1$. In this case V_2 and V_2^+ are both zero. When $\text{im}B_2 \subset \text{im}B_1$, *Problem B* has a solution if and only if condition (4.17b) holds. In this case, the controller K in (4.18) reduces to the constant gain F defined in (4.5) and is a solution to *Problem B*.

The focus in this section is on *Problem A*. Corollary 4.1 states that if conditions (4.17) hold, then there is a solution to the unconstrained problem $\inf \{ \|T_1(K)\|_2 : K \text{ admissible} \}$ that also solves *Problem A*. If $\text{im}B_1$ and $\text{im}B_2$ are linearly independent then conditions (4.17b,c) hold. In fact, under this geometric condition, a much stronger result given by Lemma 4.1 is true.

Lemma 4.1: Consider the feedback system of Fig. 4.1, with the plant G given by (4.3). Suppose that $\text{im}B_2 \cap \text{im}B_1 = 0$. Then *Problem A* is solvable if and only if there exists an admissible controller K such that $\|T_2(K)\|_\infty < 1$. In this case, a solution to *Problem A* is given by

$$K := \begin{bmatrix} A_1 & A_1\Omega - \Omega A_{F_1} \\ H - F & F(I - \Omega) + H\Omega \end{bmatrix} \quad (4.19a)$$

where

$$\begin{aligned} A_1 &:= A + (I - \Omega)B_3H + \Omega B_3F \\ A_{F_1} &:= A + B_3F \\ \Omega &:= B_2V_2^+\Pi_1 \end{aligned} \quad (4.19b-d)$$

Proof: [ROT91]

Problem A is a constrained optimization problem. As stated in §4.7.2 it represents a problem of optimal nominal performance with robust stability. As such it is the problem of interest although no analytic solution to *Problem A* currently exists. However a solution to *Problem B*, the unconstrained problem of minimizing an $\mathcal{H}_2$ performance

measure that also satisfies an $\mathcal{H}_\infty$ -norm bound does exist. Although it does not exactly represent the problem it does possess some useful properties:

1. A solution to *Problem B* exists.
2. A solution to *Problem B* is also a solution to *Problem A*.
3. Necessary and sufficient conditions are expressed for the existence of a solution to *Problem B*.
4. Since a solution to *Problem B* is a solution to *Problem A*, these conditions suffice for the solvability of *Problem A*.

Sufficient conditions for the existence of solutions to *Problem A* involve two algebraic Riccati equations (AREs) and a coupling condition, as stated in §4.7.2.2. When these conditions are satisfied, a dynamic state feedback controller that solves *Problem A* exists. The controller is highly problem specific, for example:

1. if $\text{im}B_1 \subset \text{im}B_2$, the solution to *Problem B* reduces to the static LQR state feedback controller.
2. if $\text{im}B_2 \cap \text{im}B_1 = 0$, *Problem B* and *Problem A* are equivalent. In other words, under this mild condition of linear independence of $\text{im}B_1$ and $\text{im}B_2$, one gets a complete solution to *Problem A*. Then *Problem A* is solvable if and only (4.17a) holds.

4.8 Case Study of Multi-Objective Control: Synthesis and Analysis

In this section research efforts carried out on a benchmark problem to determine the effectiveness of the mixed norm controller design for application to real world control problems are presented. The benchmark problem is formulated into the controller setup via LFTs to include $\mathcal{H}_\infty$ robust stability and $\mathcal{H}_2$ performance objectives for a system with a reliable uncertainty description. Through detailed robust stability analysis, involving μ -analysis, it is possible to conclude whether the mixed norm control synthesis is effective. It is hoped that the benchmark problem will provide insight and provide answers to the

potential of the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ controller synthesis in application to the **HELICOPTER PROBLEM**.

4.8.1 Benchmark Problem for Robust Control

The benchmark problem is depicted in Fig. 4.2 as proposed in [WIE90].

Figure 4.2: Benchmark Problem.

It is primarily a disturbance rejection problem in the presence of parametric uncertainty. The sensor and actuator are noncollocated in this uncertain dynamical system. A representative state space model is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k/m_1 & k/m_1 & 0 & 0 \\ k/m_2 & -k/m_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1/m_1 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/m_2 \end{bmatrix} w \quad (4.20)$$

$$y = x \quad (4.21)$$

$$z = Cx = x_2 \quad (4.22)$$

where x_1 and x_2 are the positions of body 1 and body 2, respectively. The output y is modified to provide full state feedback; z is the performance variable, and w a plant

disturbance acting on body 2 only. It is assumed that for the nominal system $m_1 = m_2 = k = 1$ with appropriate units and time measured in seconds.

The benchmark problem, modified to include state feedback, represents many of the salient robust control issues of more complex aerospace applications. State feedback is reasonable for the modified design problem, as ultimately these conflicting issues are addressed for the helicopter problem. The modified benchmark problem is posed for the design of a constant gain, linear, state feedback controller K of the form:

$$K(s) = \begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} \quad (4.23)$$

4.8.2 Benchmark Controller Specifications

The benchmark problem is posed as a robust stability problem, with state space parameter uncertainty and disturbance rejection as the performance criterion. Unit impulse disturbances act on the mechanical system; the uncertainty is seen as the variation of a spring constant. The parametric uncertainty is given in terms of a Δ -representation using linear fractional transformations [GRE95].

According to the 1990 American Control Conference (ACC) Benchmark Problem 1, outlined in [WIE90], stability of the closed loop system should be achieved for $m_1 = m_2 = 1$, and $0.5 \leq k \leq 2.0$. For a unit impulse disturbance $w(t)$ applied to the nominal system, the performance variable z should have a settling time of less than 15 s. Define the settling time as the time it takes to reach and stay within 10% of the maximum displacement.

4.8.3 Benchmark Controller Design: LFT Uncertainty and Performance Description

The linear fractional transformation (LFT) is a natural way to model many types of uncertainty in linear systems, including state space parameter uncertainty [PAC93]. The LFT approach is to model plant variations as variable linear gains in a feedback structure [DAI95]. The variable gains may be real, representing true physical parameters, or complex, representing uncertain dynamics in the system components. The physical parameter k is not exactly known, but is believed to lie in the interval $0.5 \leq k \leq 2.0$. Thus the actual spring constant k is within ± 0.75 of a nominal value, $k_n = 1.25$. The design model is thus chosen with the parameters $m_1 = m_2 = 1$, and $k_n = 1.25$. The uncertainty in k is represented additively as

$$k = k_n \pm 0.75\delta_k \quad (4.24)$$

such that δ_k is unknown, but real and $\|\delta_k\|_\infty < 1$. Since the LFT differential equation description of the system shows that k_n affects both body 1 and body 2, it is necessary to model the uncertainty in δ_k as a set of values, or by multiple perturbations. It is possible to “pull out” the uncertainties from the nominal system, and collect them in a diagonal matrix Δ with repeated real δ_k ’s. The matrix may be written as

$$\Delta = \begin{bmatrix} \delta_k & 0 \\ 0 & \delta_k \end{bmatrix} \quad (4.25)$$

whose inputs and outputs are z_2 and w_2 , respectively so as to fit in with the controller setup of §4.7.2.3. As such, the signal w_2 is the output of the model error A , not an exogenous input [GRE95]. The complete system model is:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k_n/m_1 & k_n/m_1 & 0 & 0 \\ k_n/m_2 & -k_n/m_{21} & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/m_2 \end{bmatrix} w_1$$

$$+ \frac{1}{\gamma} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1/m_1 & 1/m_1 & 0 \\ 1/m_2 & -1/m_2 & 0 \end{bmatrix} w_2 + \begin{bmatrix} 0 \\ 0 \\ 1/m_1 \\ 0 \end{bmatrix} u$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + B_1 w_1 + \frac{1}{\gamma} B_2 w_2 + B_3 u \quad (4.26)$$

$$z_1 = \begin{bmatrix} Q^{1/2} C \\ 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ R^{1/2} \end{bmatrix} u \quad (4.27)$$

$$z_2 = \gamma \begin{bmatrix} 0.75 & 0 & 0 & 0 \\ 0 & 0.75 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \quad (4.28)$$

$$\Delta' = \begin{bmatrix} \delta_k & 0 & 0 \\ 0 & \delta_k & 0 \\ 0 & 0 & \delta_u \end{bmatrix} \quad (4.29)$$

Some other important notes concerning the LFT description (4.26-4.29) include:

1. Q and R in (4.27) are scalars with C defined in (4.22). Q and R are selected similar to the standard LQR design.
2. A fictitious uncertainty in the control effort u is introduced in order to satisfy assumption (4.4d). The result is an added real scalar perturbation δ_u added to (4.29). The matrices C_2 and B_2 are slightly modified to satisfy (4.4d) and

strategically scaled by γ to preserve the physical significance of the posed problem.

3. The $\mathcal{H}_2/\mathcal{H}_\infty$ controller synthesis ignores the known block diagonal structure of the uncertainties. It is thus reasonable to expect a certain degree of conservativeness in the chosen synthesis method to robustness.
4. The traditional problem statement for LQR synthesis is to minimize the time domain cost function:

$$J = \int_0^{\infty} \dot{z}^T Q \dot{z} + u^T R u \, dt$$

where z is the performance output (4.22). The equivalent frequency domain problem in terms of the LFT controller setup is accomplished via (4.27).

4.8.4 Case Study Results

A mixed norm controller is synthesized using the LFT model description (4.26-4.29). With the design model defined above, imB_1 and imB_2 are linearly independent resulting in a complete solution to *Problem A*. As such, *Problem A* has a solution if and only if condition (4.17a) is satisfied. When this condition is satisfied, an admissible controller that solves *Problem A* is given by (4.19).

The only parameters to define are the $\mathcal{H}_2$ performance index weights. The state and control weightings are adjusted until a compromise is reached between performance and control effort. The $\mathcal{H}_2$ performance index weights are chosen as $Q = I$ and $R = I.7$. As such, the ARE (4.17a) admits a stabilizing solution guaranteeing closed loop stability for $0.5 \leq k \leq 2.0$, and completely defines the controller (4.19) as:

$$\dot{x}_x(t) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1.8 & 1.2 & -7.9 \times 10^{-1} & -5.8 \times 10^{-1} \\ -7.5 \times 10^{-7} & -3.5 \times 10^{-7} & -1.2 \times 10^{-4} & -1.2 \times 10^{-8} \end{bmatrix} x_c(t) \\ + \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 1.7 & -1.2 & 5.8 \times 10^{-1} & 5.8 \times 10^{-1} \\ -1.7 & 1.2 & 1.2 \times 10^8 & -5.8 \times 10^{-1} \end{bmatrix} x(t)$$

$$\dot{x}_c(t) = A_c x_c(t) + B_c x(t)$$

$$u(t) = \begin{bmatrix} -7.5 \times 10^{-7} & -3.5 \times 10^{-7} & -1.2 \times 10^{-4} & -1.2 \times 10^{-8} \end{bmatrix} x_c(t) \\ + \begin{bmatrix} -5.4 \times 10^{-1} & -5.3 \times 10^{-2} & -1.2 \times 10^8 & -5.8 \times 10^{-1} \end{bmatrix} x(t)$$

$$u(t) = C_c x_c(t) + D_c x(t)$$

The response of z to unit impulse disturbances is depicted in Fig. 4.3 for various spring constants. A settling time of 15 s. as defined in §4.8.2, is achieved for the nominal system with reasonable control effort. The responses are accompanied by the corresponding control histories.

Figure 4.3: Unit Impulse Disturbance for $k = 0.5, 1.0$ and 2.0 .

An LQR design was implemented to minimize the $\mathcal{H}_2$ -norm and draw some comparisons to the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ design. The LQR design displayed well known robustness margins with $\|T_2\|_\infty$ significantly larger than 1. By effectively decreasing the uncertainty seen by the controller in the design it is possible to approach the minimal $\mathcal{H}_2$ -norm value achieved by the LQR design. This is done at the obvious expense of the $\mathcal{H}_\infty$ -norm but not the stability margins. Thus by increasing and decreasing the effective uncertainty seen by the control design, the norms move in the appropriate direction. A decrease in the $\mathcal{H}_2$ -norm comes at the expense of an increase in the $\mathcal{H}_\infty$ -norm.

The trade-off involving parameter robustness is not evident without the use of robust stability analysis tests. This is due to the inherent conservativeness of the controller formulation. Recall that $\|T_2\|_\infty < 1$, implies stability but not conversely, because it ignores the known block diagonal structure of the uncertainties. This can be arbitrarily conservative in that stable systems can have arbitrarily large $\|T_2\|_\infty$ [ZHO96]. Thus it is necessary to turn to an analysis tool that makes use of the structure of the uncertainty.

4.8.5 Robust Stability Analysis

Using μ developed in [DOY82] as reviewed in §3.4.3, the robust stability characteristics of the closed loop system are evaluated. For analysis, the fictitious uncertainty in the control u is removed. It is present in the controller synthesis to satisfy an assumption on the design model; it is removed in the analysis of the “true” system characteristics. The uncertainty is represented as

$$\bar{z}_2 = \begin{bmatrix} 0.75 & 0 & 0 & 0 \\ 0 & 0.75 & 0 & 0 \end{bmatrix} x, \quad w_2 = \begin{bmatrix} \delta_k & 0 \\ 0 & \delta_k \end{bmatrix} \bar{z}_2$$

The matrix B_2 is modified

$$B_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -1/m_1 & 1/m_1 \\ 1/m_2 & -1/m_2 \end{bmatrix}$$

to allow compatibility of the input signal w_2 . The most well known use of μ as a robustness analysis tool is in the frequency domain. The peak value on the μ -plot of the frequency response determines the size of perturbations that the loop is robustly stable against [ZHO96].

With repeated real perturbations using the μ -TOOLS™ software package [BAL94] a peak value for μ approximately equal to 0.6 is calculated. The μ -plot of the frequency response is illustrated in Fig. 4.4 including the plot with the real parameter δ_k replaced by $\delta_k = \delta_k (1 + 0.05j)$. For complex perturbations, μ may be computed relatively easily [BAL94]. The result is a conservative value for systems with real parameter variations. The lower bound for μ will generally have better convergence behaviour, however the upper bound will invariably increase as the percentage of allowed complex perturbation is increased [PAC93].

The upper and lower bounds for the pure real perturbations are tight for the peak value of μ , near a frequency of 5×10^{-4} rad/s. This implies that real μ -analysis provides a tight estimate on the size of the smallest destabilizing real perturbation. The peak of the lower bound in Fig. 4.4 implies that there is a diagonal, real perturbation of size $1/0.6$ that causes instability.

Figure 4.4: μ -Plot for Robust Stability of Benchmark Problem.

In fact for $\delta_k = 1$, $k = 1.25 \pm (0.75/0.6) \delta_k = [0, 2.5]$. The result is a zero spring constant yielding internal physical instability. However, the use of μ implies a worst-case analysis, so it is possible that further analysis may allow for larger sized perturbations in the closed loop system. μ -analysis shows that the resulting controller is conservative. The developed $\mathcal{H}_2/\mathcal{H}_\infty$ controller only guarantees stability for $0.5 \leq k \leq 2.0$. μ -analysis provides the magnitude of the smallest destabilizing perturbation. In this case it is in the negative direction. The analysis gives no insight into stability margins for positive variations of k .

For further analysis graphical eigenvalue mapping is used. In general it is not possible to use this form of analysis for higher order systems with multiple uncertain parameters. However, it is appropriate for this low order single parameter variant system. The root locus with respect to the spring constant k is shown in Fig. 4.5.

Figure 4.5: Root locus with respect to k for $0 < k < 7482$.

Closer inspection of Fig. 4.5 reveals that the closed loop system is not only stable for the required range of spring constants, but for the extended range, $0 < k < 7482$. The controller is conservative in its ability to guarantee stability margins, which may be attributed to the fact that the synthesis method ignores the known block diagonal structure of the uncertainty. μ -analysis provides further insight into the robust stability characteristics of the system. However, this analysis method is also conservative compared to graphical eigenvalue mapping. Eigenvalue mapping demonstrates that the closed loop system has significantly larger robustness margins than design conditions suggest.

4.8.6 Case Study Conclusions

The case study is concerned with the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control system design in application to a real world control problem formulated into the controller setup via LFTs. The LFTs are genuine design objectives including $\mathcal{H}_\infty$ robust stability and $\mathcal{H}_2$ performance objectives for a system with real parameter variations. Through robust stability analysis, including μ -analysis, it is shown that the mixed norm control synthesis is conservative but effective. The mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control design allows the designer to simultaneously address $\mathcal{H}_2$ and $\mathcal{H}_\infty$ design goals and synthesize a controller with an acceptable trade-off

between their objectives. Some of the advantages and disadvantages are apparent in the combined synthesis approach. The controller synthesis is highly dependent on the design task. For example, it is possible that the mixed norm controller reduces to the LQR state feedback controller for a given plant setup. The primary advantage however, is the freedom to tailor the method to meet the requirements of nominal performance and the ability to achieve stability robustness in the face of parameter uncertainty. The disadvantages lie in the need for design iteration and the possible requirement to solve for two Riccati equations and a coupling condition. Like many multivariable designs, the controller is of the same order as the plant.

4.9 Summary

This chapter reviews the general $\mathcal{H}_2$ and $\mathcal{H}_\infty$ -optimal synthesis theory and motivates the use of mixed norm control. It presents the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -optimal synthesis theory and explores its effectiveness in application to a realistic control problem to determine its potential for general application. Norms for signals are introduced as a measure for defining a signals size and the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ system norms as useful measures of evaluating system performance. The Riccati operator is introduced and used in the computation of state space realizations for both $\mathcal{H}_2$ and $\mathcal{H}_\infty$ -optimal control laws. $\mathcal{H}_2$ and $\mathcal{H}_\infty$ -optimal synthesis theory are presented as background for the presentation of the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -optimal synthesis theory. The effectiveness of the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -optimal control design is investigated in application to a benchmark problem. The mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control design allows the designer to simultaneously address $\mathcal{H}_2$ and $\mathcal{H}_\infty$ design goals and synthesize a controller with an acceptable trade-off between their objectives. It is expected that the mixed norm control synthesis would be conservative but effective in general application.

CHAPTER 5

MODEL FOLLOWING CONTROL FOR THE HELICOPTER

5.1 Introduction

This chapter presents model following control system design for the Bell 205 helicopter in hover. Two fundamentally different types of model following control designs, explicit and implicit, using linear quadratic optimal control theory are discussed. Results of a study comparing the relative merits of controllers using these techniques where the objective is to provide desired performance in simulation for the nominal plant model are presented. Based on these results a controller most suited to the **HELICOPTER PROBLEM** is selected.

The concern of the **HELICOPTER PROBLEM** is not to regulate the state near zero as with a SAS but in following a nonzero reference signal. Its purpose is to provide satisfactory dynamic response to pilot inputs and so it is of primary interest to design a control system for optimal step response shaping. Unfortunately, while the regulator has a direct solution, the tracker problem is not so easy.

Tracking with model following behaviour for a Bell 205 helicopter in hover is carried out in §5.2 using linear quadratic optimal control. The use of model following design is natural in situations where the closed loop performance objectives are expressed in terms of target transfer functions. The military handling qualities specifications found in §2.3 provide closed loop performance objectives for the Bell 205 helicopter in hover from

which an ideal target model is constructed in §2.4. The target model is used in the design where it is the goal of the designer to reproduce this performance in the actual aircraft and design a control system that ensures this behaviour. Two fundamentally different types of model following control designs, explicit and implicit, are discussed. The design may be based on explicit model following, in which the target model explicitly appears in the controller, or by way of implicit model following, for which a description of the model is used only in the synthesis process. The two designs result in controllers of different structures.

Results of a study comparing the relative merits of controllers using both explicit and implicit model following techniques where the objective is to provide desired performance in simulation for the nominal plant model are presented in §5.3. A selection is made between controllers using both explicit and implicit model following techniques of the one most suited to the **HELICOPTER PROBLEM**.

5.2 Model Following Control Design for the Helicopter in Hover

The use of model following design is natural in situations where the closed loop performance objectives are expressed in terms of target transfer functions. Military handling qualities specifications for the various aircraft modes found in §2.3 provide the closed loop performance objectives for the Bell 205 helicopter in hover. From these desirable handling qualities a target model demonstrating ideal performance is constructed in §2.4. It is the goal of the designer to reproduce this performance in the actual aircraft and design a control system that ensures this behaviour.

This section focuses on two objectives, the first of which is to present the results of a study where model following design techniques are used to make the nominal aircraft model behave like the target model. The principle behind this technique is that the task of designing a model following controller can be formulated in terms of a linear quadratic regulator problem by defining the error dynamics of a system as the difference between

the actual system dynamics and those of a target model. The design may be based on explicit model following in which the target model explicitly appears in the controller, or by way of implicit model following, for which only a description of the model is used in the design process. The second objective of this section is to compare the relative merits of each design technique by evaluating the designs on the basis of their nominal performance, control effort, controller complexity, and the expertise and tuning required of design parameters. Based on these comparisons a design technique that most aptly satisfies each of the four criteria is selected for use in the **HELICOPTER PROBLEM**.

In the following techniques, linear quadratic regulator theory is used to minimize the error transients between the responses of the actual plant and those of the desired target model. The controller is chosen so that the closed loop system dynamics attempt to match the ideal target loop dynamics. An $\mathcal{H}_2$ minimization criterion is selected for this design problem which will devise a controller that minimizes the area in the frequency domain of the system and target model responses over all frequencies for the steady state optimal control solution.

5.2.1 Explicit Model Following Design: Command Generator Technique

Recall the state space realization of the low-order plant assumed to be modelled exactly by (3.12) with the helicopter fully instrumented to provide measurements of all relevant aircraft states. The measured outputs $y(t)$ available for full state feedback are

$$y = Cx = I_3 x . \quad (5.1)$$

Control law design objectives are directly related to the aircraft mission and the corresponding handling qualities specifications. Recollect that to perform the precision hover task in a UCE of 2, the report specifies an attitude command attitude hold (ACAH) response in pitch and roll axes with a vertical rate command heading hold (RCHH) in

vertical and yaw axes. The choice of performance outputs $z(t)$ is apparent from the above specification,

$$z = Hx = \begin{bmatrix} \text{vertical velocity} \\ \text{pitch attitude} \\ \text{roll attitude} \\ \text{yaw rate} \end{bmatrix}. \quad (5.2)$$

For the helicopter system of §3.5.1 the performance output matrix required is

$$H = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}. \quad (5.3)$$

It is the objective of the design to make the plant behave like the target model (2.8). Notice that the target model has a reference input $r(t)$, which infers a tracker problem. In other words, the objective of the design is to produce the same forced response in the plant as in the target model. To produce this result, model states for feedback and model performance outputs are chosen to be identical to those selected for the plant

$$\underline{y} = \underline{C}\underline{x} \quad (5.4)$$

$$\underline{z} = \underline{H}\underline{x} \quad (5.5)$$

where

$$\underline{C} = C \quad (5.6)$$

$$\underline{H} = H. \quad (5.7)$$

This is done so that proper selection of the plant control $u(t)$ will minimize the difference between the model performance output $\underline{z}(t)$ and the plant performance output $z(t)$. By minimizing the model mismatch error

$$e_e = \underline{z} - z = \underline{H}\underline{x} - Hx \quad (5.8)$$

the helicopter will demonstrate the desired time responses of the target model in all four channels.

Collecting all of the above dynamics into one system, the augmented state may be written as:

$$\dot{x}' = A'x' + B'u + G'r \quad (5.9)$$

with

$$A' = \begin{bmatrix} A_h & 0 \\ 0 & \underline{A} \end{bmatrix}, B' = \begin{bmatrix} B_h \\ 0 \end{bmatrix}, G' = \begin{bmatrix} 0 \\ \underline{B} \end{bmatrix}. \quad (5.10)$$

The presence of the reference input $r(t)$ in the augmented system encumbers the LQR state feedback design. The tracking problem is modified using the command generator tracker (CGT) technique so that standard regulator theory may be used. A command generator system is supplied for the reference input that defines a differential equation for the command signal to follow.

Many reference command signals $r(t)$ of practical interest satisfy the differential equation

$$r^{(d)} + a_1 r^{(d-1)} + \dots + a_d r = 0 \quad (5.11)$$

where $r^{(d)} = dr^d/dt^d$ for any number of degrees d and set of coefficients a_i . The equation in (5.11) is called the *command generator* system. As an example, the unit step satisfies the differential equation

$$\dot{r} = 0 \quad (5.12)$$

when d is 1 and a_1 is set to zero. Thus to yield step commands in each of the four channels of the helicopter it is necessary to define,

$$\dot{r} = \begin{bmatrix} \dot{r}_H \\ \dot{r}_\theta \\ \dot{r}_\phi \\ \dot{r}_R \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \quad (5.13)$$

The command generator characteristic polynomial is defined as

$$f(s) = s^d + a_1 s^{d-1} + \dots + a_d s. \quad (5.14)$$

This function takes the first derivative of the argument with the degree d and coefficient a_1 defined to equal 1 and 0, respectively. Performing this operation on the augmented system dynamics

$$f(\dot{x}') = f(A'x' + B'u + G'r) \quad (5.15)$$

results in the modified system

$$\dot{\xi} = A'\xi + B'\mu \quad (5.16)$$

where the modified state and control input are

$$\xi = f(x') = \dot{x}' \quad (5.17)$$

$$\mu = f(u) = \dot{u}. \quad (5.18)$$

Most importantly the reference input $r(t)$ is known to satisfy:

$$f(r) = \dot{r} = 0 \quad (5.19)$$

as a result of this operation and so it does not appear in the modified system dynamics. To make it clear the state ξ is partitioned as

$$\xi = \begin{bmatrix} \xi_p \\ \xi_m \end{bmatrix} \quad (5.20)$$

where the subscripts p and m denote plant and model, respectively. Applying the above operation to the model mismatch error of (5.8) results in

$$\begin{aligned} f(e_e) &= \dot{e}_e \\ &= f(\underline{H}\underline{x} - Hx) \\ &= [-\underline{H} \quad \underline{H}]\xi \\ &= H'\xi. \end{aligned} \quad (5.21)$$

There are four model mismatch errors based on the selection of the performance outputs for the plant and model. In state variable form the modified errors are represented as

$$\dot{e}_e = \begin{bmatrix} \dot{e}_{e1} \\ \dot{e}_{e2} \\ \dot{e}_{e3} \\ \dot{e}_{e4} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} e_{e1} \\ e_{e2} \\ e_{e3} \\ e_{e4} \end{bmatrix} + [-H \quad \underline{H}] \xi \quad (5.22)$$

$$\dot{e}_e = Fe_e + H' \xi$$

Finally collecting the dynamics of

$$\dot{\xi} = A' \xi + B' \mu \quad (5.23)$$

$$\dot{e}_e = Fe_e + H' \xi \quad (5.24)$$

into one system produces

$$\begin{bmatrix} \dot{e}_e \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} F & H' \\ 0 & A' \end{bmatrix} \begin{bmatrix} e_e \\ \xi \end{bmatrix} + \begin{bmatrix} 0 \\ B' \end{bmatrix} \mu \quad (5.25)$$

$$\begin{bmatrix} \dot{e}_e \\ \dot{\xi} \end{bmatrix} = \tilde{A} \begin{bmatrix} e_e \\ \xi \end{bmatrix} + \tilde{B} \mu.$$

Through use of the command generator characteristic polynomial the augmented state, control input, reference input and error have all been prefiltered to produce the dynamic system described in (5.25). This prefiltering is required to obtain a system that is not driven by the reference input $r(t)$ so that LQR theory may be used. It is well known that the LQ regulator will drive the states and thus the model mismatch error to zero. Through modification of the original problem LQR theory may now be used to design a tracking system with model following behaviour.

Using full state feedback the objective is to minimize the error $e_e(t)$ without using excessive control energy $u(t)$. The performance index required to achieve this control objective would be

$$J = \frac{1}{2} \int_0^{\infty} e_e^T Q e_e + u^T R u \, dt. \quad (5.26)$$

It is necessary to compromise the performance index to incorporate the complicated tracking problem into a more familiar LQR framework. By simply replacing $u(t)$ with $\mu(t)$ in the performance index it is possible to perform an LQ regulator design. The new performance index is

$$J = \frac{1}{2} \int_0^{\infty} e_e^T Q e_e + \mu^T R \mu \, dt \quad (5.27)$$

which indirectly achieves the original objective. Instead it attempts to minimize the error $e(t)$ without using excessive change in control input with respect to time,

$$\mu(t) = \dot{u}(t). \quad (5.28)$$

The performance index is still not in the proper form to implement the LQR design. The state, not the error, must appear in the performance index. The error in terms of the state is written as

$$\begin{aligned} e_e &= [I_4 \quad 0_{4 \times 16}] \begin{bmatrix} e_e \\ \xi_p \\ \xi_m \end{bmatrix} \\ &= C_e \begin{bmatrix} e_e \\ \xi_p \\ \xi_m \end{bmatrix} \end{aligned} \quad (5.29)$$

such that substitution in the previous performance index yields

$$J = \frac{1}{2} \int_0^{\infty} \begin{bmatrix} e_e \\ \xi_p \\ \xi_m \end{bmatrix}^T C_e^T Q C_e \begin{bmatrix} e_e \\ \xi_p \\ \xi_m \end{bmatrix} + \mu^T R \mu \, dt \quad (5.30)$$

Now it is possible to apply the LQR solution to the following standard setup

System Model:	$\begin{bmatrix} \dot{e}_e \\ \dot{\xi} \end{bmatrix} = \tilde{A} \begin{bmatrix} e_e \\ \xi \end{bmatrix} + \tilde{B} \mu$
Full State Feedback Control:	$\mu = -[K_e \quad K_p \quad K_m] \begin{bmatrix} e_e \\ \xi \end{bmatrix}$
Performance Index:	$J = \frac{1}{2} \int_0^{\infty} \begin{bmatrix} e_e \\ \xi \end{bmatrix}^T \tilde{Q} \begin{bmatrix} e_e \\ \xi \end{bmatrix} + \mu^T R \mu \, dt$

where

$$\tilde{Q} = C_e^T Q C_e. \quad (5.31)$$

The resulting pseudo-control is

$$\mu = -K_e e_e - K_p \xi_p - K_m \xi_m. \quad (5.32)$$

Integrating the above equation yields the true control input

$$u = \int \mu \, dt = -K_e \int e_e \, dt - K_p x - K_m \underline{x}. \quad (5.33)$$

Thus explicit model following design results in a dynamic controller since the model appears in the control structure as a feedforward compensator. The model mismatch error $e_e(t)$ is integrated and also used as part of the feedforward compensation.

5.2.2 Explicit Model Following Design: Reference Command Feedforward

An attempt is made in this section to approach the previous problem from a slightly different angle. The objective remains to modify the tracker problem into one that can be handled by LQ regulator theory. Once again the dynamics of the plant and target model are collected under one system. This time the augmented state is defined as

$$x' = \begin{bmatrix} x \\ \underline{x} \\ r \end{bmatrix} \quad (5.34)$$

and the augmented system as

$$\dot{x}' = A'x' + B'u \quad (5.35)$$

where

$$A' = \begin{bmatrix} A_h & 0 & 0 \\ 0 & \underline{A} & \underline{B} \\ 0 & 0 & L \end{bmatrix}, B' = \begin{bmatrix} B_h \\ 0 \\ 0 \end{bmatrix}. \quad (5.36)$$

It is important to note that the differential equations

$$\dot{r} = \begin{bmatrix} \dot{r}_W \\ r_\theta \\ r_\phi \\ r_R \end{bmatrix} = Lr = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (5.37)$$

where

$$L = 0_{4 \times 4} \quad (5.38)$$

defined for the command signals to follow are included as states of the augmented system. This methodology renders an alternative system to the latter for which LQR theory is suitable.

The system, however, is uncontrollable and ineffective in terms of accomplishing the required objective. Modification of (5.37) to include the model mismatch error $e_e(t)$ yields

$$\dot{r} = Lr + e_e = -Hx + \underline{H}\underline{x} + Lr . \quad (5.39)$$

Including these new dynamics in the augmented system necessitates revision of the state transition matrix A' such that

$$A' = \begin{bmatrix} A_h & 0 & 0 \\ 0 & \underline{A} & \underline{B} \\ -H & \underline{H} & L \end{bmatrix}. \quad (5.40)$$

The pair (A', B') is now controllable. Most importantly this modification fits in well with the desired objective. Once again the LQ regulator will drive the states and thus the model mismatch error to zero.

The objective remains firm, which is to use full state feedback to minimize the error $e_e(t)$ without using excessive, control energy $u(t)$. The performance index required to achieve this control objective is

$$J = \frac{1}{2} \int_0^{\infty} e_e^T Q e_e + u^T R u \, dt . \quad (5.41)$$

Replacing the plant state for the model mismatch error $e_e(t)$ in the performance index yields

$$J = \frac{1}{2} \int_0^{\infty} \begin{bmatrix} \dot{x} \\ \underline{x} \\ r \end{bmatrix}^T \underline{Q}' \begin{bmatrix} \dot{x} \\ \underline{x} \\ r \end{bmatrix} + u^T R u \, dt. \quad (5.42)$$

Applying the LQR solution to the following standard setup

System Model:	$\begin{bmatrix} \dot{x} \\ \underline{x} \\ r \end{bmatrix} = A' \begin{bmatrix} x \\ \underline{x} \\ r \end{bmatrix} + B' u$
Full State Feedback Control:	$u = -[K_p \quad K_m \quad K_r] \begin{bmatrix} x \\ \underline{x} \\ r \end{bmatrix}$
Performance Index:	$J = \frac{1}{2} \int_0^{\infty} \begin{bmatrix} \dot{x} \\ \underline{x} \\ r \end{bmatrix}^T \underline{Q}' \begin{bmatrix} \dot{x} \\ \underline{x} \\ r \end{bmatrix} + u^T R u \, dt$

where

$$\underline{Q}' = \begin{bmatrix} H^T Q H & -H^T Q \underline{H} & 0 \\ -\underline{H}^T Q H & \underline{H}^T Q \underline{H} & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (5.43)$$

The resulting control is

$$u = -K_p x - K_m \underline{x} - K_r r. \quad (5.44)$$

This model following design is also explicit. The model appears in the control structure as a feedforward compensator however, the model mismatch error $e_e(t)$ is not used.

Instead the reference command signal $r(t)$ is used as the additional feedforward compensator.

5.2.3 Implicit Model Following Design

The control system structure is assumed made up of state feedback with an added command feedforward compensator. The control system design problem is broken into two parts. The controller development is adapted from [KREIN76] and [STE92]. The first part of the problem is to select the matrix K for the feedback portion of the control.

The system to be controlled is again assumed to be exactly modelled by (3.12) for use in an implicit model following design. The performance index associated with the plant

$$J = \frac{1}{2} \int_0^{\infty} x^T Q x + u^T R u \, dt \quad (5.45)$$

is to be minimized resulting in the familiar linear control law

$$u = Kx. \quad (5.46)$$

The target model is once more described by (2.8) with the pair $(\underline{A}, \underline{B})$ and the reference input $r(t)$ as defined in §2.4. If the closed loop system were to follow the target model perfectly with zero input then

$$\dot{x} = \underline{A}x. \quad (5.47)$$

This condition will not hold in general and so a model following error $e_i(t)$ is defined

$$\begin{aligned}
e_i &= \dot{x} - \underline{A}x \\
&= A_h x + B_h u - \underline{A}x \\
&= (A_h - \underline{A})x + B_h u.
\end{aligned} \tag{5.48}$$

Note that this error is different than the one used in the explicit model following design. To make the model following error $e_i(t)$ small without using excessive control energy it is necessary to minimize

$$J = \frac{1}{2} \int_0^\infty e_i^T Q_i e_i + u^T R u \, dt. \tag{5.49}$$

Substituting (5.48) into the performance index gives

$$J = \frac{1}{2} \int_0^\infty x^T \hat{Q} x + 2x^T \hat{S} u + u^T \hat{R} u \, dt \tag{5.50}$$

where

$$\hat{Q} = (A_h - \underline{A})^T Q_i (A_h - \underline{A}) \tag{5.51}$$

$$\hat{S} = (A_h - \underline{A})^T Q_i B \tag{5.52}$$

$$\hat{R} = R + B^T Q_i B. \tag{5.53}$$

The state feedback control, which minimizes J from (5.50) is derived from

$$K = -R^{-1} B^T P \tag{5.54}$$

$$P \hat{A} + \hat{A}^T P - P B \hat{R}^{-1} B^T P + Q^* = 0 \tag{5.55}$$

$$Q^* = \hat{Q} - \hat{S} \hat{R}^{-1} \hat{S}^T \tag{5.56}$$

$$\hat{A} = A_h - B R^{-1} \hat{S}^T \tag{5.57}$$

The feedback portion of the applied control is given by

$$u_{fb} = -(\hat{R}^{-1}B^T P + \hat{S}\hat{R}^{-1})x. \quad (5.58)$$

As mentioned a feedforward compensator is added to the state feedback control structure derived above. The control input $u(t)$ is thus assumed made up of state feedback with a command feedforward compensator

$$u = Kx + Fr. \quad (5.59)$$

The second part of the control system design is not complete. With the matrix K secured, it remains to determine the matrix F to achieve the desired target model behaviour in the aircraft. The command feedforward processing is accomplished via the matrix

$$F = \hat{R}^{-1}B^T(A_h + BK)^{-T}\underline{A}^T\underline{Q}, \underline{B} \quad (5.60)$$

as specified in [KREIC72].

It remains to find the matrices $(\underline{Q}, R)$ for the performance index given by (5.49). The weighting matrices are chosen based on an inverse square rule [AST90]. These matrices are selected as diagonal with elements chosen to normalize contributions based on maximum deviations in the corresponding state or control.

All control movements are in units of centimeters and a reasonable maximum expected deviation is 15 cm. Thus the control weighting matrix is selected as

$$R = \frac{1}{15^2} \times I_4. \quad (5.61)$$

Recall that the forward and lateral velocities are not directly controlled. Therefore, the elements of Q corresponding to these terms, the first and fourth rows, are set to zero. The pitch and roll rates are similarly not directly controlled. However, to preserve controllability in the LQR problem the third and fifth diagonal elements are selected as the small positives 0.01. Vertical velocity is measured in meters per second, with a model tracking error of 1m/s a reasonable maximum. Therefore, the second diagonal element is selected as unity. A maximum yaw rate deviation of 0.1 radians per second is considered a reasonable goal. Thus the sixth diagonal element is selected as 100. Lastly the pitch and roll attitudes are to be kept as close as possible to the target model states. They are both weighted 400 corresponding to a deviation of only 0.05 radians. This results in the following weighting matrices

$$Q_c = \text{diag}(0,1,0.01,0,0.01,100,400,400) . \quad (5.62)$$

5.3 Results and Comparison of Model Following Control System Designs

The results of controller simulation are used to compare the relative merits of the various designs. It is first necessary, however, to define the criteria for the purpose of controller assessment. Four criteria are used to determine the effectiveness and practicality of the model following control system designs. The first criterion is the closed loop performance of the helicopter; the second criterion is the control effort required of the controller; the third a measure of controller complexity; and lastly a measure of the expertise and tuning required in the selection of system design parameters. These criteria are developed further below.

5.3.1 Nominal Performance

The answer to the obvious question of how well the helicopter performs with the given controller is addressed. The dynamics of the actual aircraft are compared to those of the

ideal target model. A closed loop system whose time responses more closely resemble the responses of the ideal target model exhibits preferred performance.

The EMF design techniques provide the desired time responses in each of the four channels. Fig. 5.1 depicts the desired pitch attitude response to a 0.1 radian step input supplied by the CGT controller.

Figure 5.1: EMF Pitch Attitude Response for the CGT Controller.

Virtually no coupling is exhibited in the time responses. The implicit model following technique does equally well with the vertical and yaw rate command responses. No coupling is present here. However, the pitch and roll attitude command responses are not up to par with the explicit technique. Its pitch attitude command response is poor, evident in Fig. 5.2.

Figure 5.2: IMF Pitch Attitude Response.

As a matter of fact the pitch attitude only reaches 0.02 radians for the 0.1 radian step input. This difference tends to increase with time. Pitch and roll attitudes are slightly coupled in each of these responses. The roll attitude command response is better, reaching a value of 0.07 radians for the same step input. The response still moves away from the input with time at a slightly lesser rate. Pitch and roll attitudes remain slightly coupled in each of these responses.

Both the explicit model following controllers provide the best nominal performance. Moreover, they both provide the exact responses desired in each of the four input channels.

5.3.2 Control Effort

Assessment of the control effort is necessary to guarantee that the controllers do not exceed deflection limits of control surface actuators. If one control design yields a controller that drives the actuators with large deflections of control surface actuators, then it should be discarded in light of a second design that requires smaller deflections [MAN90].

Comparing the pitch attitude response of the CGT controller in Fig. 5.1 to that of the reference feedforward controller depicted in Fig. 5.3

Figure 5.3: EMF Pitch Attitude Response for the Reference Feedforward Controller.

it is evident that deflections of dissimilar magnitudes are required for similar nominal performance. The maximum size of deflections required of the CGT controller is smaller than that required of the reference feedforward controller where control actions are large for this simulation study.

The implicit control design yields a controller that drives the actuators with smaller control movements in the pitch attitude response. In fact it drives the actuators with a third of the size of the CGT controller which has the smallest control movements of the explicit design techniques. Recall that the pitch attitude command response for the implicit design however, is very poor. The CGT controller has slightly smaller sized control movements in each of the other three responses.

5.3.3 Controller Complexity

Control design methodologies yield controllers of different structures of which some are more complex than others. As an example, one design methodology may yield a

controller that requires dynamic compensation opposed to static feedback. In these terms the controller that proves easiest to implement should be chosen.

The two fundamentally different types of model following control, explicit and implicit, produce controllers of different structures. The IMF design merely uses the target model in defining the performance index. The ideal target model states do not appear in the control input $u(t)$ of (5.59). The IMF controller uses static feedback and reference command feedforward only, thus there is no need to simulate the target model during implementation. It is evident from (5.33) and (5.34) that simulation of the model is required as part of the feedforward compensators in the two EMF controllers. The EMF controllers developed here are much like standard PI controllers and so are not very complex. Controller complexity is therefore not a significant factor in choosing between the two designs.

5.3.4 Expertise in Selecting and Tuning of Design Parameters

In LQ regulator theory it is necessary to define the performance index weighting matrices Q and R . Experience and expertise are often required of some design methodologies to provide the desired performance objectives. Selection of the weighting matrices Q and R can be considerably complicated and often becomes more of an art than a science. Their selection is imprecise and so the originally favoured weighting matrices may require modification. Designers thus need be prepared to combine selection of Q and R with iteration.

Design methodologies that simplify the selection of the performance index weighting matrices are preferable. They may reduce the number of design parameters, even for those systems with many states and numerous undetermined control gains. This reduces the design time required to achieve performance specifications. A design methodology that simplifies this task is preferable over one that requires advanced expertise and excessive tuning to achieve like goals.

In the explicit model following design the only parameters left to the designer are the performance index weighting matrices Q and R in (5.27). The performance indices for the two versions of controllers are quite similar. The ultimate control laws differ due only to slight variations in design methodology used to obtain the modified system dynamics.

The objective in each design is similar. That is to use full state feedback to minimize the model mismatch error without using excessive control energy. Diagonal Q and R are chosen which readily reflect the objective. The weighting matrices are both chosen to be I_4 . This ensures that each of the four model mismatch errors are weighted equally. This is also true for the inputs. The only further requirement is to determine the relative weighting of the two identity matrices to achieve the design objective. By minimizing the model mismatch error the design ensures the helicopter will exhibit the desired time response of the target model. Few iterations are required to obtain the desired result. Nothing more is asked of the designer.

This is a much simpler task required of the designer compared to the implicit design. Again diagonal Q_i and R are chosen as the weighting matrices for the performance index in (5.49). The size of Q_i is larger to accommodate the eight model following errors. The performance index is structured such that selection of the diagonal elements requires expertise and ingenuity. The selection of the matrix Q_i is not as automatic as the last design. Furthermore command feedforward processing is accomplished by the matrix F defined by (5.60).

Further tuning does not afford any improvement in the pitch and roll attitude responses of the system. This design requires a higher level of expertise and tuning from the designer compared to the explicit design procedure. The explicit design is much more clear and concise in the selection of its design parameters.

The selection of the weighting matrices in the EMF design is simple and iteration is needed only to determine their relative weighting. The performance index in the IMF design is structured such that selection of the diagonal elements requires experience and ingenuity. The EMF design methodology simplifies the desired task and is preferable to the IMF design where advanced expertise and significant tuning are needed to achieve like goals.

5.3.5 Conclusion of Model Following Design Study

The appropriateness of the model following technique is recognized and used to achieve $\mathcal{H}_2$ nominal performance for the nominal helicopter model. Considering the criteria the CGT controller is best suited for the application.

5.4 Summary

This chapter presents model following control system design for the Bell 205 helicopter in hover. The use of model following design is natural in situations where the closed loop performance objectives are expressed in terms of target transfer functions. Two fundamentally different types of model following control designs, explicit and implicit, are discussed. The design may be based on explicit model following, in which the target model explicitly appears in the controller, or by way of implicit model following, for which a description of the model is used only in the synthesis process. Results of a study comparing the relative merits of controllers using both explicit and implicit model following techniques are used to select a controller most suited to the **HELICOPTER PROBLEM**.

CHAPTER 6

MIXED NORM CONTROL FOR THE HELICOPTER

6.1 Introduction

This chapter presents the application of mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -optimization for control system design of a Bell 205 helicopter. It is shown that a flight controller designed using mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -optimization improves the handling qualities of the aircraft according to US Army handling qualities specifications. The controller provides this performance against the constraint of robust stability for the aircraft not included in the specifications.

In §6.2 formulation of the **HELICOPTER PROBLEM** incorporating $\mathcal{H}_\infty$ robust stability and optimal nominal $\mathcal{H}_2$ model following performance is presented to synthesize a controller designed to improve the handling qualities of the helicopter. The problem is formulated using mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -optimization where optimal nominal performance is represented as the model following design optimization of an ideal target model based on military handling qualities specifications. $\mathcal{H}_\infty$ -optimization is addressed via the incorporation of uncertain aircraft dynamics using experimental data to include an uncertainty description with a prescribed degree of structure.

A control law is synthesized to improve the handling qualities of the helicopter in simulation. The control law provides the desired decoupling with fast and tailored responses to pilot command inputs like those exhibited by the target model with robust stability to model variation. The control laws are analyzed with frequency and time

domain responses for the nominal and perturbed plants in §6.3.1 and §6.3.2, respectively. It is shown that a controller designed using mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -optimization improves the handling qualities of the aircraft according to US Army handling qualities specifications. Stability robustness of the closed loop system is assessed using structured singular value techniques in §6.3.3.

6.2 Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Control Design for the Helicopter

The **HELICOPTER PROBLEM** is cast into the mixed norm control problem reviewed in §4.7.2. More specifically the problem is cast into the mathematical framework of *Problem A* to represent a problem of optimal nominal performance with robust stability represented by Fig. 6.1.

Figure 6.1: Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Synthesis Framework.

The state space description of G given by (4.3) is repeated below.

$$\begin{aligned}\dot{x} &= Ax + B_1 w_1 + B_2 w_2 + B_3 u \\ z_1 &= C_1 x + D_1 u \\ z_2 &= C_2 x + D_2 u \\ y &= x.\end{aligned}$$

It is necessary to piece together the uncertainty description of the linear design model of §3.5.4 and the selected model following control design of §5.2.1 into the appropriate framework for solution of the **HELICOPTER PROBLEM**. This is done by casting the **HELICOPTER PROBLEM** into the mathematical framework of *Problem A* for which a solution is given by Lemma 5.1 in §4.7.2.3. The uncertainty description of the helicopter is discussed first.

6.2.1 Model Uncertainty Description

The model represented by the pair $(A_{h,e}, B_{h,e})$ in (3.27) better represents the Bell 205 helicopter than the low-order Heffley model of §3.5.1. This is accomplished by piecing together the error dynamics of the Heffley model using a reliable description of the uncertainty with the Heffley model itself as described in §3.5.3. As mentioned earlier, it is unreasonable to assume that the higher-order model given by (3.27) exactly describes the true aircraft. An uncertainty description of the higher-order model is detailed in §3.5.4, which together with LFTs discussed in §3.2.2 allows a linear fractional model of the uncertainty to be formulated.

Specifically, state space parameter uncertainty of the higher-order model is modelled as variable linear gains in a feedback structure via LFTs represented by the matrix Δ in Fig. 6.1. Parametric uncertainty exists in the modelling of sensor and actuator dynamics; the rotor mast flexing mode; low frequency system/model mismatch; and high frequency unmodelled dynamics. These parameters are not known but are believed to lie in the intervals defined in §3.5.4. It is possible to “pull out” the uncertainties from the system and collect them in a diagonal matrix Δ as in §3.2.2 and the benchmark example in §4.8.3. Inputs and outputs to Δ are z_2 and w_2 , respectively so as to fit in with the controller setup of §4.7.2. As such, the signal w_2 is the output of the model error A , not an exogenous input [GRE95]. The linear design model is then assumed described by:

$$\dot{x}_{h/e} = A_{h/e/nom} x_{h/e} + B_{dev:A} w_2 + B_{h/e} u \quad (6.1)$$

To arrive at the state space description in (6.1) the following algorithm is used for each parametric uncertainty component in (3.27):

1. Starting with $(A_{h/e}, B_{h/e})$ identify the component in the state transition matrix $A_{h/e}$ associated with the uncertainty specified in §3.5.4
2. Find a range on the uncertain parameter about the nominal midpoint to form a mean state transition matrix $A_{h/e/mean}$.
3. Construct a B_{dev} in the range space of the uncertain parameter so that $A_{h/e/mean} + \delta B_{dev}$ covers the range of all possible uncertainty for the identified component of uncertainty with $|\delta| \leq 1$.
4. Replace the uncertain component in $A_{h/e}$ with its corresponding uncertainty description, $A_{h/e/mean} + \delta B_{dev}$.
5. Pull out the δB_{dev} elements to form a matrix $B_{dev:A}$.
6. $A_{h/e/mean}$ forms a new nominal component in the state transition matrix.
7. Repeat this for each uncertain component in $A_{h/e}$.

When the algorithm is completed the state transition matrix, $A_{h/e/nom}$ is created for the system described in (6.1) which represents the helicopter dynamics when the uncertain parameters are believed to equal their mean values. The matrix $B_{dev:A}$ represents the deviation in the matrix $A_{h/e/nom}$ to fit in with the controller setup of §4.7.2.

6.2.2 Model Following Performance Objective

The model following objective is defined here for the **HELICOPTER PROBLEM**. This is accomplished by replacing the Heffley model used for controller synthesis with the system described by $(A_{h/e/nom}, B_{h/e})$ to represent the higher-order model of the Bell

205 helicopter with mean parameter values. The model following design objective is described by:

$$\text{System Model:} \quad \begin{bmatrix} \dot{e}_e \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} F & H' \\ 0 & A' \end{bmatrix} \begin{bmatrix} e_e \\ \xi \end{bmatrix} + \begin{bmatrix} 0 \\ B' \end{bmatrix} \mu \quad (6.2)$$

$$\text{Performance Index:} \quad J = \frac{1}{2} \int_0^\infty \begin{bmatrix} e_e \\ \xi \end{bmatrix}^T \bar{Q} \begin{bmatrix} e_e \\ \xi \end{bmatrix} + \mu^T R \mu \, dt \quad (6.3)$$

as in §5.2.1 where

$$A' = \begin{bmatrix} A_{h.e./nom} & 0 \\ 0 & \underline{A} \end{bmatrix}, \quad B' = \begin{bmatrix} B_{h.e.} \\ 0 \end{bmatrix}$$

corresponding to the states of ξ partitioned as

$$\xi = \begin{bmatrix} \xi_p \\ \xi_m \end{bmatrix}$$

where the subscripts p and m denote helicopter plant and target model, respectively. All other matrices remain the same and are appropriately augmented in size to retain their dimensional compatibility.

6.2.3 Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Problem Description

The complete system model for the **HELICOPTER MODEL** is thus:

$$\begin{bmatrix} \dot{e}_e \\ \xi_p \\ \xi_m \end{bmatrix} = \begin{bmatrix} F & -H & \underline{H} \\ 0 & A_{h/e/nom} & 0 \\ 0 & 0 & \underline{A} \end{bmatrix} \begin{bmatrix} e_e \\ \xi_p \\ \xi_m \end{bmatrix} + B_1 w_1 + \frac{1}{\gamma} \begin{bmatrix} 0 \\ B_{dev/A} \\ 0 \end{bmatrix} w_2 + \begin{bmatrix} 0 \\ B_{h/e} \\ 0 \end{bmatrix} \mu \quad (6.4)$$

$$z_1 = \begin{bmatrix} Q^{1/2} C \\ 0 \end{bmatrix} \begin{bmatrix} e \\ \xi_p \\ \xi_m \end{bmatrix} + \begin{bmatrix} 0 \\ R^{1/2} \end{bmatrix} \mu \quad (6.5)$$

$$z_2 = \gamma \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} e_e \\ \xi_p \\ \xi_m \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \mu \quad (6.6)$$

Some important notes concerning the LFT description (6.4 – 6.6) learned from the benchmark problem include:

1. Q and R in (6.5) are selected through design iteration to achieve desired nominal performance for the aircraft and are selected similar to the standard LQR design.
2. A fictitious uncertainty in the control effort μ is introduced with the same scaling strategy applied here as in the benchmark problem to preserve the physical significance of the posed problem.
3. The $\mathcal{H}_2/\mathcal{H}_\infty$ controller synthesis ignores the known block diagonal structure of the uncertainties. It is thus reasonable to expect a certain degree of conservativeness in the chosen synthesis method to robustness.
4. The traditional problem statement for LQR synthesis is to minimize the time domain cost function is accomplished. The equivalent frequency domain problem in terms of the LFT controller setup is accomplished via (6.5).
5. The matrix B_1 is selected so that the range space of B_1 and the range space of

$$\begin{bmatrix} 0 \\ B_{dev/A} \\ 0 \end{bmatrix} \text{ are linearly independent.}$$

6.3 Results of the Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Controller Synthesis

The results of controller synthesis are presented below through the use of frequency response and time domain simulations. The robust stability characteristics of the closed loop system are also investigated using μ -analysis. It is noted that a controller is synthesized here which satisfies the three conditions in (4.17a-c) if the size of the uncertainty described in §3.5.4 is reduced by a factor of a thousand. An optimistic statement of this result is that the controller is merely conservative.

6.3.1 Frequency Response Results

Results of the frequency response for the closed loop system using the mixed norm controller are presented. Frequency response plots of the nominal and perturbed plant systems are presented in Figs. 6.2 to 6.13 for each of the reference command inputs; vertical velocity, pitch attitude, roll attitude, and yaw rate.

Quality of response and speed of response information are available in these types of plots for evaluating closed loop system performance. The former is inferred from the magnitude of the closed loop system transfer functions and the latter deduced from the bandwidth frequency of the closed loop system. The (*closed loop*) *bandwidth* defined here for the purpose of system performance evaluation is the frequency where the magnitude starts to fall off from its low frequency value. Specifically it is the frequency at which the magnitude first crosses $1/\sqrt{2} = 0.707$ ($\approx -3\text{db}$) from below.

The quality in the response refers to effectiveness of the control in achieving the desired closed loop bandwidth and decoupling specifications. For desired response to reference command inputs the on-axis frequency response should be 0 dB across the desired bandwidth frequency range with off-axis magnitudes no greater than -20 dB. On-axis response implies the response that corresponds to the particular command input. For example, the on-axis response to vertical velocity command inputs is vertical velocity

leaving the pitch attitude, roll attitude and yaw rate response as the off-axis response for this input. The first requirement on the magnitude response ensures that the controller is effective, allowing the reference command to significantly change the desired output response. The second requirement is the decoupling of off-axis responses. The on-axis and off-axis responses are sufficiently decoupled (less than 10%) if off-axis responses have a magnitude of at most -20 dB over the bandwidth. Higher magnitudes in off-axis responses will result in stronger coupling seen in the time domain.

In general, a large bandwidth corresponds to a faster rise time in the time domain, since high frequency signals are more easily passed on to the outputs. A closed loop system bandwidth that approaches the desired bandwidth of the ideal target model is desirable. The bandwidth of the ideal target model is equal to 3 rad/s in vertical, 4 rad/s in pitch and roll, and 5 rad/s in yaw.

6.3.1.1 Frequency Response of the Nominal Plant

The nominal plant frequency response is depicted following this discussion in Figs. 6.2 to 6.5. The vertical velocity command input frequency response is shown first in Fig. 6.2. The response is highly decoupled with desired closed loop system bandwidth of 3 rad/s. The expected result is a fast speed of response in the time domain with significantly small coupling apparent. The pitch attitude command response of Fig. 6.3 exhibits desired decoupling of the response with bandwidth frequency near 1 rad/s. A highly decoupled response in the time domain is to be expected with a slightly larger rise time resulting. Conversely, the roll attitude command response in Fig. 6.4 is lightly coupled with a larger closed loop system bandwidth frequency of 2 rad/s. A quicker speed of response with a smaller degree of coupling should appear in the time domain response. The yaw rate command frequency response of Fig. 6.5 illustrates low coupling with a bandwidth frequency near the desired 5 rad/s. The on-axis time domain response is expected to exhibit low coupling and quick response to yaw rate command inputs.

Figure 6.2: Frequency response of closed loop system to vertical velocity command input for the nominal plant, $\Delta = 0_{67 \times 67}$.

Figure 6.3: Frequency response of closed loop system to pitch attitude command input for the nominal plant, $\Delta = 0_{67 \times 67}$.

Figure 6.4: Frequency response of closed loop system to roll attitude command input for the nominal plant, $\Delta = 0_{67 \times 67}$.

Figure 6.5: Frequency response of closed loop system to yaw rate command input for the nominal plant, $\Delta = 0_{67 \times 67}$.

6.3.1.2 Frequency Response of the Perturbed Plant

Frequency response plots for the two extreme cases of uncertainty are presented in Figs. 6.6 to 6.13 following the discussion of their results presented below. The first set of perturbed plant, Figs. 6.6 - 6.9, represent the frequency response plots for the case where $\Delta = I_{67}$ and the last set, Figs. 6.10 – 13, where $\Delta = -0.6 \times I_{67}$. They are treated separately in turn beginning with the case $\Delta = I_{67}$.

The vertical velocity command input frequency response is shown in Fig. 6.6. The response is highly decoupled with desired closed loop system bandwidth of 3 rad/s. A resonant peak exists at roughly 14 rad/s in the magnitude plot. This represents the strong influence of the rotor mast flexing mode and is a good indication that the uncertainty model description is correct. Pitch attitude command response depicted in Fig. 6.7 exhibits the desired decoupling and bandwidth frequency of 0.5 rad/s. The roll attitude command response illustrated in Fig. 6.8 is also decoupled with a bandwidth frequency approximately at 1 rad/s. The yaw rate command input response of Fig. 6.9 is slightly coupled with close to desired bandwidth frequency of 4 rad/s.

When the uncertainty is changed to $\Delta = -0.6 \times I_{67}$ the vertical velocity command frequency response, Fig. 6.10, exhibits coupling and reduced bandwidth frequency of 1.5 rad/s. Similar coupling occurrences result in the frequency response plots for pitch and roll attitude command inputs of Figs. 6.11 and 6.12. Their respective bandwidths are 1 and 1.5 rad/s, respectively. Lastly Fig. 6.13 demonstrates coupling in the yaw rate command input with close to desired bandwidth frequency 4.5 rad/s.

Figure 6.6: Frequency response of closed loop system to vertical velocity command input for the perturbed plant, $\Delta = I_{67}$.

Figure 6.7: Frequency response of closed loop system to pitch attitude command input for the perturbed plant, $\Delta = I_{67}$.

Figure 6.8: Frequency response of closed loop system to roll attitude command input for the perturbed plant, $\Delta = I_{67}$.

Figure 6.9: Frequency response of closed loop system to yaw rate command input for the perturbed plant, $\Delta = I_{67}$.

Figure 6.10: Frequency response of closed loop system to vertical velocity command input for the perturbed plant, $\Delta = -0.6 \times I_{67}$.

Figure 6.11: Frequency response of closed loop system to pitch attitude command input for the perturbed plant, $\Delta = -0.6 \times I_{67}$.

Figure 6.12: Frequency response of closed loop system to roll attitude command input for the perturbed plant, $\Delta = -0.6 \times I_{67}$.

Figure 6.13: Frequency response of closed loop system to yaw rate command input for the perturbed plant, $\Delta = -0.6 \times I_{67}$.

6.3.2 Time Domain Response Results

Time domain responses for the closed loop system using the mixed norm controller are presented for nominal and perturbed plant systems along with their corresponding control histories. They are used to confer the observations made from the frequency response plots for the nominal and perturbed plants. The time domain simulation of performance outputs are presented in Figs. 6.2 to 6.21 for each of the reference command inputs; vertical velocity, pitch attitude, roll attitude, and yaw rate.

For the nominal plant response the desired on-axis time domain response to vertical velocity command inputs should be first order. Pitch and roll attitude command inputs should result in desired second order on-axis responses. Lastly, the vertical velocity input should command a first order on-axis response. It is desired that on-axis and off-axis responses exhibit the desired decoupling (less than 10%). The first order responses in vertical and yaw should possess a time constant of $1/3$ s and $1/5$ s, respectively, to match the ideal target model response in vertical and yaw. The *time constant* for a first order system is defined as the time required for the response to reach 63% of its final value. Similarly for the second order responses in pitch and roll the time responses should each possess a rise time of a 0.45 s and a settling time of 1.5 s, to match the ideal target model response in pitch and roll. Assessment of the control effort is necessary to guarantee that the controllers do not exceed deflection limits of control surface actuators. Control activity within a ± 20 cm range is considered within the accepted deflection limits of the control surface actuators.

The on-axis performance of the perturbed plant responses should possess a relative degree of insensitivity to plant variations if it is desired to retain performance relative to that of the target model. In addition, the decoupling of on-axis and off-axis responses should exhibit a relative degree of insensitivity to plant perturbations so that the desired decoupling qualities are not lost. As well control effort should be insensitive to plant variation so that deflection limits do not exceed ± 20 cm.

6.3.2.1 Time Domain Response of the Nominal Plant

The time domain responses for the nominal plant are found in Figs. 6.14-17. The vertical velocity command input of Fig. 6.14 results in the desired on-axis response which is nearly first order and possesses the desired time constant of 1/3 seconds. Pitch and roll attitudes are illustrated in Figs. 6.15 and 6.16, respectively. The rise times are larger than desired confirming the conclusions drawn from the nominal frequency response plots. Both responses possess settling times that match the target model. The yaw rate command input results in the desired on-axis first order response evident in Fig. 6.17 with a time constant close to 1/5 s. The coupling is below 10% in all cases especially in the vertical channel with acceptable deflection of control surface actuators in all channels.

6.3.2.2 Time Domain Response of the Perturbed Plant

The ‘envelope’ of closed loop system time responses to step command inputs for perturbed plants, $\Delta = -0.6 \times I_{67}, -0.45 \times I_{67}, -0.3 \times I_{67}, -0.15 \times I_{67}, 0.25 \times I_{67}, 0.5 \times I_{67}, 0.75 \times I_{67}$, I_{67} are presented in Figs. 6.18-6.21 with their corresponding control histories. From these plots several conclusions can be made concerning the relative degree of sensitivity of on-axis response, decoupling of on-axis and off-axis responses, and control effort to model variation.

It is evident from the time domain responses that on-axis performance in vertical is sensitive to plant variations and that the decoupling qualities are rather insensitive to plant perturbations. On-axis performance in pitch and roll are sensitive to uncertainties in the plant with good decoupling qualities over the uncertainty range. On-axis performance in yaw is insensitive however the decoupling is sensitive to plant perturbations. Control is rather insensitive to plant variations except in yaw where control is sensitive to plant variations. Despite this sensitivity deflections of the actuators are all within the accepted limits.

Figure 6.14: Time response of closed loop system to vertical velocity step input command for the nominal plant, $\Delta = 0_{67 \times 67}$.

Figure 6.15: Time response of closed loop system to pitch attitude step input command for the nominal plant, $\Delta = 0_{67 \times 67}$.

Figure 6.16: Time response of closed loop system to roll attitude step input command for the nominal plant, $\Delta = 0_{67 \times 67}$.

Figure 6.17: Time response of closed loop system to yaw rate step input command for the nominal plant, $\Delta = 0_{67 \times 67}$.

Figure 6.18: Envelope of closed loop system time responses to vertical velocity step input command for perturbed plants, $\Delta = -0.6 \times I_{67}, -0.45 \times I_{67}, -0.3 \times I_{67}, -0.15 \times I_{67}, 0.25 \times I_{67}, 0.5 \times I_{67}, 0.75 \times I_{67}, I_{67}$.

Figure 6.19: Envelope of closed loop system time responses to pitch attitude step input command for perturbed plants, $\Delta = -0.6 \times I_{67}, -0.45 \times I_{67}, -0.3 \times I_{67}, -0.15 \times I_{67}, 0.25 \times I_{67}, 0.5 \times I_{67}, 0.75 \times I_{67}, I_{67}$.

Figure 6.20: Envelope of closed loop system time responses to roll attitude step input command for perturbed plants, $\Delta = -0.6 \times I_{67}, -0.45 \times I_{67}, -0.3 \times I_{67}, -0.15 \times I_{67}, 0.25 \times I_{67}, 0.5 \times I_{67}, 0.75 \times I_{67}, I_{67}$.

Figure 6.21: Envelope of closed loop system time responses to yaw rate step input command for perturbed plants, $\Delta = -0.6 \times I_{67}, -0.45 \times I_{67}, -0.3 \times I_{67}, -0.15 \times I_{67}, 0.25 \times I_{67}, 0.5 \times I_{67}, 0.75 \times I_{67}, I_{67}$.

6.3.3 *Robust Stability Analysis*

Using μ -analysis methods presented in §3.4.3 the robust stability characteristics of the closed loop system for the **HELICOPTER PROBLEM** are evaluated. For analysis, the fictitious uncertainty in the control μ is removed. It is present in the controller synthesis to satisfy an assumption on the design model and removed in the analysis of the “true” system characteristics. The use of μ as a robustness analysis tool in the frequency domain is used here for the **HELICOPTER PROBLEM**. Again it is the peak value from the μ -plot that determines the size of perturbations for which the closed loop system is guaranteed stable.

The μ -plot of the frequency response with real perturbations is illustrated in figure 6.22. Using μ -TOOLS™ the peak value of the upper bound on μ is approximately equal to 1.3 with a lower bound at zero over all frequencies. When the percentage of allowed complex perturbation is increased to 5% the lower bound has better convergence especially at the peak value where the lower and upper bounds of the software agree well with one another. The peak value increases by 15% demonstrating that the analysis is sensitive to the percentage of allowable complex perturbation. Increase the percentage of allowable complex perturbations to 50% as in Fig. 6.24 and μ demonstrates the improved convergence of the upper and lower bounds. The tight peak of the upper bound and lower bounds increases by 62% resulting in a conservative estimate for a system with real parameter variations.

The peak value of the upper bound on μ in Fig. 6.22 with pure real uncertainty is similar to the peak value of the upper and lower bounds on μ in Fig. 6.23 where the percentage of allowed complex perturbations is small. This implies that the upper bound in Fig. 6.22 with pure real uncertainty provides a good estimate on the size of the smallest destabilizing real perturbation. The peak value of 1.3 for the upper bound of μ implies that there is a diagonal, real perturbation of size $1/1.3$ or approximately 0.77 that causes

instability. In other words the use of μ implies that the range of values covered by the uncertain parameters in §3.5.4 must be decreased by a factor of roughly 0.77 to ensure stability.

The use of μ -analysis implies a worst-case analysis. Dealing with the worst-case scenario is very pessimistic and it is likely that the worst-case perturbation would not occur. The use of μ -analysis must be weighed with these qualitative arguments to determine the significance of the results. The value confirms well with simulation results where the worst-case time simulation occurs in the negative direction of the uncertainty. When μ -analysis is given a good framework in terms of the uncertainty structure it is able to provide precise results.

The $\mathcal{H}_2/\mathcal{H}_\infty$ controller provides a very conservative estimate of the allowable uncertainty. with the largest destabilizing perturbation equal to 1/1000, essentially zero uncertainty in the model. The controller is conservative in its ability to guarantee stability margins, which may be attributed to the fact that the synthesis method ignores the known block diagonal structure of the uncertainty. Further insight into the robust stability characteristics of the closed loop system is afforded through the use of μ -analysis.

Figure 6.22: μ -plot for robust stability analysis with real uncertainty.

Figure 6.23: μ -plot for robust stability analysis with mixed real/complex uncertainty for allowable complex perturbations of 5%.

Figure 6.24: μ -plot for robust stability analysis with mixed real/complex uncertainty for allowable complex perturbations of 50%.

6.4 Comparison with $\mathcal{H}_2$ Solution

For comparison, an LQR solution using an $\mathcal{H}_2$ performance criterion is investigated for the **HELICOPTER PROBLEM**. This is accomplished by allowing γ in (6.4) to be an arbitrarily large value approaching infinity. The resulting $\mathcal{H}_2/\mathcal{H}_\infty$ problem is solved with Q and R unchanged from §6.2.3 to represent an equivalent LQR solution for the **HELICOPTER PROBLEM**. The LQR solution minimizes the $\mathcal{H}_2$ -norm providing desired optimal nominal performance only without any explicit optimization of robustness.

A μ -analysis is performed on the closed loop system to determine the robust stability characteristics of the LQR solution. Analysis results in a peak value of the upper bound on μ equal to 1.4. This is comparable to the value of μ equal to 1.3 obtained for the closed loop system with mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -optimal control. The peak values of the upper bounds suggest that the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ and LQR design possess similar robust stability characteristics. However, as seen in the vertical velocity command response depicted in Figure 6.25, decoupling is somewhat degraded and a potentially problematic oscillation in the output occurs at approximately 2.5 Hz or 14 rad/s, the natural frequency of the rotor mast flexing mode. This vibration in the response while guaranteed to be stable from μ -analysis is not acceptable in terms of the desired handling qualities. Moreover, these types of repeated vibrations introduce undesirable fatigue concerns from material science of helicopter components. An LQR design would require the weighting on Q to be relaxed in order to rid the helicopter of this vibration in the perturbed response. This occurs at the cost of degraded optimal nominal performance for the aircraft and is not desired where the objective is to meet handling qualities specifications.

An increase in the weighting of Q provides desired optimal nominal performance but there are consequences in terms of robust stability. As Q is increased from 1×10^5 to 1×10^9 , Figure 6.26 demonstrates the significant reduction in stability margins for the

Figure 6.25: Envelope of closed loop system time responses to vertical velocity step input command for perturbed plants, $\Delta = -0.6 \times I_{67}, -0.45 \times I_{67}, -0.3 \times I_{67}, -0.15 \times I_{67}, 0.25 \times I_{67}, 0.5 \times I_{67}, 0.75 \times I_{67}, I_{67}$, using $\mathcal{H}_2$ -optimal control.

Figure 6.26: μ -plot for robust stability analysis with real uncertainty for $\mathcal{H}_2$ -optimal control with increased weighting on Q .

LQR design. This is evident from the peak value of μ approximately equal to 3.0 at roughly 14 rad/s, corresponding to the rotor mast flexing mode. This is significantly larger than the peak value of 1.3 found for both the $\mathcal{H}_2/\mathcal{H}_\infty$ and LQR solutions with $Q = 1 \times 10^5$. This demonstrates the sensitivity of μ to changes in Q for the LQR solution. More importantly it identifies the rotor mast flexing mode uncertainty as the source of the performance versus robustness trade-off for the **HELICOPTER PROBLEM**.

The LQR solution provides optimal nominal performance and stability robustness comparable to that of the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ design for certain weightings in the cost function. However, in the presence of model variation the quality of performance provided by the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ controller is more aptly maintained than that afforded by the LQR controller. In other words, the LQR solution while stable results in an unacceptable level of vibration of the rotor flexing mode for perturbed plants. This is not seen in the $\mathcal{H}_2/\mathcal{H}_\infty$ system under the given simulation studies. A more rigorous simulation covering the entire envelope of the uncertainty is necessary to increase the strength of conclusions that can be drawn from simulation.

6.5 Summary of Results

This chapter presents the application of mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -optimization for control system design of a Bell 205 helicopter. The problem of robust flight controller synthesis is formulated as a mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -optimization where optimal nominal performance is represented as the model following design optimization of an ideal target model based on military handling qualities specifications. $\mathcal{H}_\infty$ -optimization is addressed via the incorporation of uncertain aircraft dynamics in to the model with a prescribed degree of structure to the uncertainty. It is shown that a controller is synthesized to provide improved handling qualities of the aircraft that closely match US Army handling qualities specifications against the constraint of robust stability for the aircraft not included in the specifications.

CHAPTER 7

CONCLUSIONS

7.1 *Summary*

This thesis has presented the design and analysis of a robust full authority flight control system for a fly-by-wire actuated Bell 205 helicopter using mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -optimization. The flight control system was found to improve the handling qualities of the aircraft by providing dynamic responses in simulation for the nominal aircraft that closely match stringent US Army handling qualities specifications. The improvement of handling qualities has been achieved against the constraint of robust stability for the aircraft not included in ADS-33C.

The addition of a robust stability constraint creates a multi-objective control problem. The real world control problem has been formulated into a mathematical framework as $\mathcal{H}_\infty$ robust stability with optimal nominal $\mathcal{H}_2$ model following performance for the helicopter, which realistically represents the genuine design objectives. Robustness to plant uncertainty is based on experimental and analytical uncertainty models to produce a realistic and reliable description of the uncertainty in the mathematical model. A linear fractional transformational description of the uncertain system dynamics was used as a natural way of modelling structured uncertainty in the linear design model. The $\mathcal{H}_\infty$ -norm is shown as a very natural norm to use in describing stability margins in view of the small gain theorem.

The use of model following design is appropriate in situations where the closed loop performance objectives are expressed in terms of target transfer functions. Military handling qualities specifications for the various aircraft modes found in ADS-33C provide the closed loop performance objectives from which a target model demonstrating ideal performance was constructed. It is a designer goal to reproduce this performance in the actual aircraft and design a control system that ensures this behaviour. $\mathcal{H}_2$ - optimization is appropriate in the minimization of error transients between the responses of the actual plant and those of the desired target model. The popular $\mathcal{H}_\infty$ -optimization would attempt to minimize the worst case or peak value of the difference between actual and target transfer functions in the frequency domain. This criterion is not suitable for the type of response required in the aircraft where it is desirable to have the system follow the target model closely over a frequency interval. Accordingly, an $\mathcal{H}_2$ minimization criterion is selected for the **HELICOPTER PROBLEM** which devises a controller that minimizes the area in the frequency domain between the system and target model responses over all frequencies for the steady state optimal control solution.

An outstanding problem in helicopter flight control has been related to the unrealistic mapping of multiple design objectives into a single norm. Although the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ - norm may often be genuine design objectives, to combine all these true objectives using a single norm as a cost function requires compromise. As with LQ and $\mathcal{H}_\infty$ synthesis, the norms do not directly include all of the design objectives. For example, LQ synthesis results in a controller that is adept at providing optimal nominal performance with potentially weak robustness characteristics. Similarly, an $\mathcal{H}_\infty$ controller results in a highly robust system with potentially poor performance. This problem has been addressed in this thesis with the use of a mixed $\mathcal{H}_2 / \mathcal{H}_\infty$ -norm problem with two exogenous input vectors and two controlled output vectors developed for full state feedback. The most attractive feature of the mixed $\mathcal{H}_2 / \mathcal{H}_\infty$ -optimization for the **HELICOPTER PROBLEM** has been the ability to address $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance objectives simultaneously. Mixed $\mathcal{H}_2 / \mathcal{H}_\infty$ -optimization has eliminated the need to compromise genuine design objectives in the **HELICOPTER PROBLEM** making it

possible to synthesize, in an optimal fashion, a robust flight controller that directly addresses both the issue of $\mathcal{H}_\infty$ robust stability and optimal nominal $\mathcal{H}_2$ model following performance.

In order that the closed loop system achieves desired performance, the linear design model should be sufficiently accurate to reduce the trade-off between performance and robustness. The uncertainty of available low-order models is seen as high-order unmodelled dynamics. Experimental frequency domain responses of the actual helicopter provide a clear and reliable picture of the uncertainty between the nominal model and actual physical plant. It was decided to use a low-order model of the helicopter as nominal and to quantify the uncertain error dynamics between it and the actual aircraft. By adding this information to the low-order model, the uncertainty in the linear design model was significantly decreased and the trade-off between performance and robustness reduced resulting in higher achievable performance.

It is unreasonable to assume that uncertainty can be completely removed from the higher-order nominal model resulting in the formulation of an uncertainty description for the linear design model. The uncertainty in aircraft dynamics between the model and the actual helicopter has been meticulously addressed in this thesis with a well-defined structure that captures their differences. Finally, structured singular value analysis of system robustness was used to identify acceptable variation in aircraft dynamics within this well-defined uncertainty structure for which stability is assured.

7.2 Recommendations

The vast majority of control theory to date has focused on *stability*, and consequently *performance* guarantees have received much less attention in this regard. *Robust performance* means that not only do stability guarantees hold in the presence of specified errors in the plant model but that certain performance guarantees hold as well. This is a

current area of research for which the helicopter would show improved handling qualities that are less sensitive to model variation.

Modern controller design methods including $\mathcal{H}_\infty$, LQG and the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ controller synthesis in this thesis produce high-order controllers at least equal to the plant and usually higher. This occurs with the inclusion of weighting functions such as the inclusion of target model dynamics in the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ controller synthesis for the **HELICOPTER PROBLEM**. Methods exist to reduce the order of the controller and may be investigated here to save on the computational expenses required for practical implementation of control.

Simulation of the helicopter FCS that incorporates the nonlinear dynamics of the aircraft would be a more effective validation of the design offering more realism to the simulation results. After successful validation from ground-based simulations to include the nonlinear dynamics of the aircraft the only true measure to determine the effectiveness of the controller is viz. experimental flight tests. A full evaluation of ADS-33C manoeuvres may be performed to determine the effectiveness of the FCS in improving the handling qualities of the Bell 205 helicopter.

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