Risk Premiums and Their Applications in Ruin Probabilities

A Master's Thesis

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Risk Premiums and Their Applications in Ruin Probabilities

BY

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of

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Abstract:

Some useful properties of the nth stop-loss order and the exponential order will be given in this paper. These results will be applied to the study of losses L_i ($i = 1, 2, \cdots$), L and ruin probability $\psi(u)$. A relationship between the claim amount random variables and ruin probabilities will also be found. The concepts of the nth stop-loss distance and the ruin probability distance will be introduced. A formula for ruin probabilities for heterogeneous portfolios will be given.

Key Words:

nth stop-loss transform, nth stop-loss order, exponential order, ruin probability.

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Contents

0 Introduction

For an insurance company, each contract of insurance brings a risk with it. A claim may occur some time in the future and the amount of the claim is a nonnegative random variable which is called a risk. One of the main tasks of actuaries is to compare the attractiveness of different risks. This helps them to determine insurance premiums and to decide on the reinsurance needed. Another task of actuaries is to calculate the risk premiums. The basis of insurance is the hypothesis that claims can be compensated by fixed payments called premiums. Premiums are calculated by a premium calculation principle. The partial orders on a family of risks are called risk orders. The theorey of risk orders is a useful mathematical tool for comparing risks and risk premium principles.

From Bowers (1997), we know if the decision maker has decided on the fixed amount to be paid for insurance, also the expected claims is a fixed value, the stop-loss insurance will maximize the expected utility of the decision maker. Consequently, we concern more with the feature of the stop-loss insurance. The properties of nth stop-loss orders and exponential orders provide much more information for studying the stop-loss insurance, since the 1st stop-loss transforms are the stop-loss premiums.

This paper is based upon the works of Goovaerts et al. (1990) and Cheng and Pai (1999a, 1999b, 1999c). Many kinds of partial orders were discussed in Goovaerts et al. (1990). The nth stop-loss order and the exponential order are two of them. In Cheng and Pai (1999a, 1999b, 1999c). the concept of stop-loss transforms was generalized to the nth stop-loss transforms. The maintenance properties of the nth stop-loss order under the individual risk model and the collective risk model were developed. In this paper, we first discuss the properties of the nth stop-loss order and the exponential order, later apply them in risk premium principles and ruin probabilities.

This paper is organized as follows. In Section 1, we introduce some definitions and results of Goovaerts et al. (1990) and Cheng and Pai (1999a, 1999b, 1999c). In Section 2, we continue the study by Cheng and Pai (1999a, 1999b, 1999c) on nth stop-loss orders. We give a necessary condition and a sufficent condition for nth stop-loss orders. They are convenient tools to construct risk pairs that can have nth stop-loss orders. The maintenance properties of nth stop-loss orders under the operation of compound, in the situation where counting variables N_1 and N_2 are not identical, are to be proved. In Section 3, we study exponential orders which are weaker than nth stop-loss orders. A necessary condition and a sufficient condition for exponential orders will be given. The maintenance properties of exponential orders under the operations of compound and mixture are developed. In Section 4, the results from Section 2 and 3 will be used to study the losses L_i, the maximal aggregate losses L, ruin probabilities and risk premium principles. The necessary condition for nth stop-loss orders will be applied in the valuation of risk premium principles. We will prove that exponential premium principles can differentiate between losses more finely than the net premium principles under some conditions. Consequently, if some proper forms for premium rates c in the classical risk models being chosen, we can discuss the properties of ruin probabilities. This topic is worth further study. The relationship between the claim amount random variable and L, the relationship between the claim amount random variable and the ruin probability will be given. These result are worth further study for finding upper bounds of ruin probabilities.Some properties of $E[L^i]$ will be developed. These results may provide approximation methods to estimate the ruin probability functions. The concepts of the nth stop-loss distance and the ruin probability distance will also be introduced. In Section 5, a formula for ruin probabilities for heterogeneous portfolios will be given.

1 Groundwork

This article deals with risks to be insured, which are defined as non-negative random variables. Here we cite some definitions and results of Goovaerts et al. (1990) and Cheng and Pai (1999a, 1999b, 1999c).

Definition 1.1 (**nth Stop-Loss Transform**) Suppose loss random variable X is nonnegative with its distribution function being $F(x)$, its survival function being $\overline{F}(x) = 1 - F(x)$, and $E[X^n] < \infty$. Let

$$\Pi^{(n)}(u) = E[\{(X-u)_+\}^n], \quad u \geq 0,\ n = 1, 2, \cdots, \tag{1}$$

where

$$(x-u)_+ = \begin{cases} 0, & \text{for } x \leq u, \\ x-u, & \text{for } x > u, \end{cases}$$

$$\Pi^{(0)}(u) = \overline{F}(u) = 1 - F(u). \tag{2}$$

As a function of u, $\Pi^{(n)}(u)$, $n = 1, 2, \cdots$, will have domain $[0, \infty)$. We call function $\Pi^{(n)}(u)$ the nth stop-loss transform of X.

Definition 1.2. (**nth Stop-Loss Order**) We say that X is less than Y in the meaning of the nth stop-loss order, denoted by $X <_{sl(n)} Y$, if

$$E[X^k] \leq E[Y^k], \quad k = 1, 2, \cdots, n-1, \tag{3}$$

and

$$\Pi_X^{(n)}(u) \leq \Pi_Y^{(n)}(u), \quad \text{for all } u \geq 0. \tag{4}$$

When $n = 0$, the formula (3) disappears and formula (4) becomes

$$\overline{F}_X(u) \leq \overline{F}_Y(u), \quad \text{for all } u \geq 0.$$

When $n = 1$, the formula (3) is trivial and formula (4) becomes

$$\int_u^\infty \overline{F}_X(x)dx \leq \int_u^\infty \overline{F}_Y(x)dx, \quad \text{for all } u \geq 0.$$

Definition 1.3. (**Weak nth Stop-Loss Order**) Let

$$\Omega = \{H(x),\ x \geq 0 :\ H(x) \geq 0 \text{ monotonous decreasing and } \lim_{x\to\infty} H(x) = 0\ \}.$$

Suppose $H(x),\ G(x) \in \Omega$. We say that $H(x)$ is less than $G(x)$ in the meaning of weak nth stop-loss order, denoted by $H <_{wsl(n)} G$, if

$$\Pi_H^{(n)}(u) \leq \Pi_G^{(n)}(u), \quad \text{for all } u \geq 0. \tag{5}$$

Definition 1.4. (Exponential Order) Risk X precedes Y in the exponential order, written $X <_e Y$, if for each $\alpha > 0$, we have

$$M_X(\alpha) = E[e^{\alpha X}] \leq E[e^{\alpha Y}] = M_Y(\alpha) \text{ (can be } \infty). \tag{6}$$

Theorem 1.1.

$$\frac{d}{du}[\Pi_X^{(n)}(u)] = -n\Pi_X^{(n-1)}(u), \tag{7}$$

or

$$\Pi_X^{(n)}(u) = n\int_u^\infty \Pi_X^{(n-1)}(x)dx. \tag{8}$$

(see Cheng and Pai (1999a), Theorem 6)

Theorem 1.2. Let $n = 0, 1, 2, \cdots$ and $m > n$. Suppose risk $X <_{sl(n)} Y$. Then $X <_{sl(m)} Y$.

(see Goovaerts et al. (1990), Theorem 4.2.2)

Theorem 1.3. Suppose $u(x)$ is a utility function having $n-1$ continuous derivatives of alternating sign:

$$(-1)^{(k-1)}u^{(k)}(x) \quad \geq \quad 0, \; k = 1, 2, \cdots, n-1, \tag{9}$$

$$(-1)^{(n-1)}u^{(n)}(x) \quad \geq \quad 0, \text{ and non-decreasing in } x. \tag{10}$$

Let $U_n = \{u(x) : u(x) \text{ satisfies (9) and (10)}\}$, $w(x) = -u(-x)$, and $W_n = \{w(x) : w^{(k)}(x) = (-1)^{(k+1)}u^{(k)}(-x) \geq 0\}$. Then $X <_{sl(n)} Y$, if and only if

$$E[u(-X)] \geq E[u(-Y)], \text{ for all } u \in U_n,$$

if and only if

$$E[w(X)] \leq E[w(Y)], \text{ for all } w \in W_n.$$

(see Cheng and Pai (1999a), Theorem 10)

Theorem 1.4. The nth stop-loss order is maintained under the summation of independent random variables. That is, if

$$X_i <_{sl(n)} Y_i, \quad i = 1, 2, \cdots, k,$$

where k is a positive integer, then

$$\sum_{i=1}^{k} X_i <_{sl(n)} \sum_{i=1}^{k} Y_i, \quad n = 0, 1, 2, \cdots.$$

(see Cheng and Pai (1999a), Theorem 19)

2 Properties of nth Stop-Loss Orders

nth

From Theorem 1.3, we can see that the nth stop-loss order can be characterized as the common preferences of a group of decision makers with increasingly regular utility functions $u(x) \in U_n$. We will continue the work of Goovaerts et al. (1990) and Cheng and Pai (1999), to give more features of the nth stop-loss order.

Theorem 2.1 will be used to compare the differences of the net premium principle and the exponential premium principle in Section 4.

Theorem 2.1. (Necessary Condition) Suppose X , Y are not identically distributed risks. If $X <_{wsl(n)} Y$ and $E[X^{n+i}] < \infty$, then

$$E[X^{n+k}] < E[Y^{n+k}], \quad k = 1, 2, \cdots, i.$$

Proof

If $E[Y^{n+i}] = \infty$, the result is obvious. If $E[Y^{n+i}] < \infty$, we first show that for $k = 1$ we have

$$E[X^{n+1}] < E[Y^{n+1}].$$

Indeed, let

$$g(u) = \Pi_X^{(n+1)}(u) - \Pi_Y^{(n+1)}(u).$$

From Definition 1.3 and Theorem 1.1 , we have for all $u > 0$,

$$g(u) \leq 0,$$

and

$$g'(u) = \frac{d}{du}[\Pi_X^{(n+1)}(u) - \Pi_Y^{(n+1)}(u)] = -(n+1)[\Pi_X^{(n)}(u) - \Pi_Y^{(n)}(u)] \geq 0.$$

Further more , there exists $u_\circ \geq 0$. such that

$$g'(u_\circ) = -(n+1)[\Pi_X^{(n)}(u_\circ) - \Pi_Y^{(n)}(u_\circ)] > 0.$$

(otherwise differentiate $g'(u)$ n times, we will have $F_X(u) = G_Y(u)$)

So the following inequality must be true

$$g(0) = \Pi_X^{(n+1)}(0) - \Pi_Y^{(n+1)}(0) = E[X^{n+1}] - E[Y^{n+1}] < 0.$$

Applying the same method and the fact that $\Pi_X^{(n+j)}(u) \leq \Pi_Y^{(n+j)}(u)$ for $j = 1, 2, \cdots$ and for all $u > 0$, we obtain the relation

$$E[X^{n+k}] < E[Y^{n+k}], \quad k = 2, \cdots, i. \blacksquare$$

A sufficient condition for the nth stop-loss order is given by Theorem 4.2.3 of Goovaerts (1990): $n+1$ sign changes in density functions implies the nth stop-loss order. Here we give another sufficient condition: n sign changes in distribution functions implies the nth stop-loss order.

Theorem 2.2. (Sufficient Condition) Suppose that for two risks X and Y there is a partition of $[0, \infty)$ into $n+1$ consecutive non-empty intervals(closed intervals containing only one point are acceptable) $I_0, I_1, \cdots, I_n$ such that

$$(-1)^{n+1-j}\{F_X(t) - F_Y(t)\} \leq 0 \text{ on } I_j.$$

If moreover the first n moments satisfy

$$E[X^j] = E[Y^j], \; j = 1, 2, \cdots, n,$$

then

$$X <_{sl(n)} Y.$$

Proof

For convinence, we let n be an even number. When n is an odd number, we can apply the same method to arrive at the result. Let

$$h_i(t) = \Pi_Y^{(i)}(t) - \Pi_X^{(i)}(t), \quad i = 0, 1, \cdots, n,$$

then from Theorem 1.1, we have

$$h_i{}'(t) = -ih_{i-1}(t).$$

We only need to show that

$$h_n(t) \geq 0, \quad \text{for all } t > 0. \tag{11}$$

First we know that

$$(-1)^j(F_Y(t) - F_X(t)) \leq 0, \; j = 1, 2, \cdots, n,$$

and

$$h_1'(t) \leq 0, \quad h_1(t) \downarrow \text{ on } I_0,$$

$$h_1'(t) \geq 0, \quad h_1(t) \uparrow \text{ on } I_1,$$

$$\cdots, \qquad \cdots \qquad \cdots,$$

$$h_1'(t) \leq 0, \quad h_1(t) \downarrow \text{ on } I_n.$$

On the other hand, from

$$h_n(0) = h_n(\infty) = 0,$$

we know that there exists $a_1 \in (0, \infty)$ such that $h_n'(a_1) = 0$, using Rolle's theorem and repeating this process, we have: there exist $b_1 < b_2 < \cdots < b_{n-1}$ such that

$$h_1(0) = h_1(b_1) = \cdots = h_1(b_{n-1}) = h_1(\infty) = 0.$$

Combin the discussions above, the following conclusion must be true: there exist $c_1 \in I_1, \cdots, c_{n-1} \in I_{n-1}$ such that

$$h_1(t) \leq 0 \quad \text{on } [0, c_1) = {I_0}^{(1)},$$

$$h_1(t) \geq 0 \quad \text{on } [c_1, c_2) = {I_1}^{(1)},$$

$$\cdots, \qquad \cdots \qquad \cdots,$$

$$h_1(t) \geq 0 \quad \text{on } [c_{n-1}, \infty) = {I_{n-1}}^{(1)}.$$

Repeat the same process, we finally have (11). ∎

We can see that the condition of Theorem 2.2 implies: $F_X(t) = F_Y(t)$ at least at n different points in $(0, \infty)$.

Theorem 2.1 and 2.2 are two useful tools to help us find out or construct the risk pairs which have nth stop-loss orders.

Compound risk was discussed in Theorem 20 of Cheng and Pai (1999a) where the counting variables N_1 and N_2 have identical probability distributions. Now we give another result where $N_1 <_{sl(1)} N_2$ but X_i and Y_i are two sequences of independent and identically distributed risks.

Theorem 2.3. (Compound Risks) Let $X_1, X_2, \cdots$ and $Y_1, Y_2, \cdots$ be two sequences of independent and identically distributed risks, $N_j (j = 1, 2)$ be counting variables independent of X_i and Y_i. In the collective risk models, S_1 and S_2 are defined as

$$S_1 = \sum_{i=1}^{N_1} X_i, \quad S_2 = \sum_{i=1}^{N_2} Y_i.$$

If

$$X_i <_{sl(n)} Y_i, \quad N_1 <_{sl(1)} N_2,$$

then we have

$$S_1 <_{sl(n)} S_2.$$

Proof

According to Definition 1.2, we need to prove

$$E[S_1^i] \leq E[S_2^i] \quad i = 1, 2, \cdots, n-1, \tag{12}$$

and

$$\Pi_{S_1}^{(n)}(u) \leq \Pi_{S_2}^{(n)}(u), \quad \text{for all } u \geq 0. \tag{13}$$

First we prove (13). From Theorem 1.4, we have for all $u \geq 0$,

$$\begin{aligned}
\Pi_{S_1}^{(n)}(u) &= E[\{(S_1 - u)_+\}^n] \\
&= \sum_{k=0}^{\infty} E[\{(S_1 - u)_+\}^n \mid N_1 = k] \cdot \Pr(N_1 = k) \\
&\leq \sum_{k=0}^{\infty} E[\{(S_2 - u)_+\}^n \mid N_1 = k] \cdot \Pr(N_1 = k) \\
&= \sum_{k=0}^{\infty} E[\{(\sum_{i=1}^{k} Y_i - u)_+\}^n] \cdot \Pr(N_1 = k).
\end{aligned} \tag{14}$$

(define $E[\{(\sum_{i=1}^{k} Y_i - u)_+\}^n] = 0$ when $k = 0$)

Let

$$w_1(k) = E[\{(\sum_{i=1}^{k} Y_i - u)_+\}^n].$$

It is obvious that

$$w_1(k) \leq w_1(k+1), \quad k = 0, 1, \cdots.$$

If

$$2w_1(k+1) \leq w_1(k) + w_1(k+2), \quad k = 0, 1, \cdots, \tag{15}$$

we can construct a convex function $w_2(t)$, such that

$$w_2(k) = w_1(k),$$

and

$$w_2'(t) \geq 0, \text{ and non-decreasing in } t.$$

Then from Theorem 1.3, we have

$$E[w_2(N_1)] \leq E[w_2(N_2)],$$

and (14) becomes

$$\begin{aligned}
\Pi_{S_1}^{(n)}(u) &\leq \sum_{k=0}^{\infty} E[\{(\sum_{i=1}^{k} Y_i - u)_+\}^n] \cdot \Pr(N_1 = k) \\
&\leq \sum_{k=0}^{\infty} E[\{(\sum_{i=1}^{k} Y_i - u)_+\}^n] \cdot \Pr(N_2 = k) \\
&= \Pi_{S_2}^{(n)}(u).
\end{aligned}$$

Now we only need to show (15). Let

$$A_k = \sum_{i=1}^{k} Y_i.$$

(15) is equivalent to the following inequality

$$\begin{aligned}
&E[\{(A_k + Y_{k+1} - u)_+\}^n] + E[\{(A_k + Y_{k+2} - u)_+\}^n] \\
&\quad \leq E[\{(A_k - u)_+\}^n] + E[\{(A_k + Y_{k+1} + Y_{k+2} - u)_+\}^n],
\end{aligned}$$

and this follows directly if we look at the conditional distribution with $A_k = a$, $Y_{k+1} = y$, $Y_{k+2} = z$, and use the following inequality

$$(a + y - u)_+^n + (a + z - u)_+^n \leq (a - u)_+^n + (a + y + z - u)_+^n. \tag{16}$$

It is easy to check (16). When $u \geq a$, (16) is obvious; when $u < a$, we can get (16) by using Binomial Theorem.

Applying the same method, we can prove (12). ∎

In the following Corollary, we generalized the result of Theorem 3.2.5 in Goovaerts et al. (1990) from stop-loss orders to nth stop-loss orders.

Corollary 2.4. (Conditional Compound Poisson Distribution) Let Λ_j be a non-negative structure variable, and N_j be an integer valued non-negative random variable. Their conditional distribution given $\Lambda_j = \lambda$ of N_j is Poisson(λ) distributed, $j = 1, 2$. Let $X_1, X_2, \cdots$ and $Y_1, Y_2, \cdots$ be two sequences of independent and identically distributed risks, $N_j (j = 1, 2)$ be counting variables independent of X_i and Y_i. In the collective risk models, S_1 and S_2 are defined as

$$S_1 = \sum_{i=1}^{N_1} X_i, \quad S_2 = \sum_{i=1}^{N_2} Y_i.$$

If

$$X_i <_{sl(n)} Y_i, \quad i = 1, 2, \cdots,$$

and

$$\Lambda_1 <_{sl(1)} \Lambda_2,$$

then

$$S_1 <_{sl(n)} S_2.$$

Proof

In view of Theorem 2.3, we only need to know

$$N_1 <_{sl(1)} N_2.$$

From the proof of Theorem 3.2.5 of Goovaerts et al. (1990),

$$\Lambda_1 <_{sl(1)} \Lambda_2$$

implies

$$N_1 <_{sl(1)} N_2. \blacksquare$$

3 Properties of Exponential Orders

Now we discuss another partial order-exponential order. If exponential utility functions are being used, the risk averters' attitude to risk does not change with the acquired capital. From Definition 1.4, we know that the exponential order can be characterized as the common preferences of the group of these decision makers. The following proposition indicates the exponential order is a weaker order than any nth stop-loss order. It is the limiting case of the nth stop-loss order.

Proposition 3.1. **(Sufficient Condition)** Let $n = 0, 1, 2, \cdots$. Suppose risk $X <_{sl(n)} Y$, then

$$X <_e Y.$$

Proof

Applying Theorem 1.3, we have for all $\alpha > 0$, let $w(x) = e^{\alpha x}$, then $w(x) \in W_n$, and

$$E[e^{\alpha X}] = E[w(X)] \leq E[w(Y)] = E[e^{\alpha Y}]. \blacksquare$$

Example 1. For a compound Poisson risk process with premium c per unit time and two risks X, Y, if $X <_{sl(n)} Y$, then the adjustment coefficients satisfy $R_X \geq R_Y$.

Proof

From Theorem 2.3.2 of Goovaerts et al. (1990), $X <_e Y$ implies $R_X \geq R_Y$ and by Proposition 3.1 our conclusion can be arrived at immediately. ∎

For convenience to use later, we prove the following proposition.

Proposition 3.2. **(Necessary Condition)** Let $A_Z = \sup\{\alpha : E[e^{\alpha Z}] < \infty\}$, and X, Y be two non-negative random variables. If

$$X <_e Y,$$

then

$$A_X \geq A_Y.$$

Proof

If $A_X = 0$, from $X <_e Y$ and the definition of A_Z, we have $A_Y = 0$.

If $0 < A_X < \infty$, we use the method of reduction to absurdity to prove the result.

If $A_X < A_Y$, then there exists $\alpha_\circ > 0$, such that

$$A_X < \alpha_\circ < A_Y,$$

and

$$E[e^{\alpha_\circ X}] = \infty, \quad E[e^{\alpha_\circ Y}] < \infty.$$

This is contrary to $X <_e Y$. The proof is complete. ∎

Proposition 3.1 and 3.2 provide some information for finding risk pairs that have exponential orders.

The following theorem is useful when we discuss the properties of L_i and L later.

Theorem 3.3. Let $A_Y > 0$. If $X <_e Y$ and $E[X^j] = E[Y^j]$, $j = 1, \cdots, k-1$, then

$$E[X^k] \leq E[Y^k].$$

Proof

We use the method of reduction to absurdity to prove this proposition. We know that for $\alpha < A_Y$, $M_X^{(j)}(\alpha) < \infty$, $M_Y X^{(j)}(\alpha) < \infty$ for $j = 1, 2, \cdots$(we will show it later in Proposition 3.9). If

$$E[X^k] > E[Y^k],$$

then

$$M_X^{(k)}(0) - M_Y^{(k)}(0) > 0.$$

From $E[X^{k-1}] = E[Y^{k-1}] = M_X^{(k-1)}(0) = M_Y^{(k-1)}(0)$, we know that there exists $\alpha_{k-1} < A_Y$ such that for all $0 < \alpha < \alpha_{k-1}$

$$M_X^{(k-1)}(\alpha) - M_Y^{(k-1)}(\alpha) > 0.$$

Repeat this process, we finally have there exists $\alpha_0 < A_Y$ such that for all $0 < \alpha < \alpha_0$

$$M_X(\alpha) - M_Y(\alpha) > 0.$$

This is contrary to $X <_e Y$. ∎

Like nth stop-loss orders, exponential orders are maintained under a compound operation and a mixture operation, we will show these properties in the following theorems.

Theorem 3.4. (Compound Risks) Let $X_1, X_2, \cdots$ and $Y_1, Y_2, \cdots$ be two sequences of independent distributed risks. N_1 and N_2 are counting variables independent of X_i and Y_i. In addition, N_1 and N_2 have identical probability distributions. In the collective risk models, S_1 and S_2 are defined as

$$S_1 = \sum_{i=1}^{N_1} X_i, \quad S_2 = \sum_{i=1}^{N_2} Y_i.$$

If $X_i <_e Y_i$ for all i, we have

$$S_1 <_e S_2. \tag{17}$$

Proof

For $\alpha > 0$ and $E[e^{\alpha S_2}] < \infty$, we have

$$\begin{aligned} E[e^{\alpha S_1}] &= \sum_{n=0}^{\infty} \{E[e^{\alpha X_i}]\}^n \cdot \Pr(N_1 = n) \\ &\leq \sum_{n=0}^{\infty} \{E[e^{\alpha Y_i}]\}^n \cdot \Pr(N_2 = n) \\ &= E[e^{\alpha S_2}]. \end{aligned}$$

That is

$$S_1 <_e S_2. \blacksquare$$

The identical assumption of N_1 and N_2 in Theorem 3.4 can be released if X_i and Y_i are i.i.d. respectively, $i = 1, 2, \cdots$. The same property is held if $N_1 <_e N_2$. The result is stated in Theorem 3.5.

Theorem 3.5. (Compound Risks) Let $X_1, X_2, \cdots$ and $Y_1, Y_2, \cdots$ be two sequences of identically distributed risks, N_1 and N_2 be counting variables independent of X_i and Y_i. In the collective risk models, S_1 and S_2 are defined as

$$S_1 = \sum_{i=1}^{N_1} X_i, \quad S_2 = \sum_{i=1}^{N_2} Y_i.$$

Let X and Y be the common random variable of X_i and Y_i respectively. If $X <_e Y$ and $N_1 <_e N_2$, we have

$$S_1 <_e S_2. \tag{18}$$

Proof

If $E[e^{\alpha Y_i}] = 1$, (18) is obvious. Now we consider the case $E[e^{\alpha Y_i}] > 1$. For $\alpha > 0$ and $E[e^{\alpha S_2}] < \infty$, let $\alpha_1 > 0$ such that

$$e^{\alpha_1} = E[e^{\alpha Y_i}].$$

Applying Proposition 3.2, we have

$$
\begin{aligned}
E[e^{\alpha S_1}] &= \sum_{n=0}^{\infty}\{E[e^{\alpha X_i}]\}^n \cdot \Pr(N_1 = n) \\
&\leq \sum_{n=0}^{\infty}\{E[e^{\alpha Y_i}]\}^n \cdot \Pr(N_1 = n) \\
&= \sum_{n=0}^{\infty} e^{\alpha_1 n} \cdot \Pr(N_1 = n) \\
&\leq \sum_{n=0}^{\infty} e^{\alpha_1 n} \cdot \Pr(N_2 = n) \\
&= \sum_{n=0}^{\infty}\{E[e^{\alpha Y_i}]\}^n \cdot \Pr(N_2 = n) \\
&= E[e^{\alpha S_2}].
\end{aligned}
$$

That is

$$S_1 <_e S_2. \blacksquare$$

Corollary 3.6. **(Conditional Compound Poisson Distribution)** Let Λ_j be a non-negative structure variable, and N_j be an integer valued non-negative random variable. Their conditional distribution given $\Lambda_j = \lambda$ of N_j is Poisson(λ) distributed, $j = 1, 2$. Let $X_1, X_2, \cdots, Y_1, Y_2, \cdots$ and S_1, S_2 be the same as Theorem 3.5. If

$$X_i <_e Y_i \quad i = 1, 2, \cdots,$$

and

$$\Lambda_1 <_e \Lambda_2,$$

then

$$S_1 <_e S_2.$$

Proof

In the view of Theorem 3.5, we only need to prove

$$N_1 <_e N_2.$$

For $\alpha > 0$ and $E[\alpha N_2] < \infty$, using Proposition 3.2, we have

$$\begin{aligned} E[e^{\alpha N_1}] &= E[E[e^{\alpha N_1} \mid \Lambda_1]] = E[\exp{(e^{\alpha} - 1)\Lambda_1}] \\ &\overset{\alpha_1 = e^{\alpha} - 1 > 0}{=} E[e^{\alpha_1 \Lambda_1}] \leq E[e^{\alpha_1 \Lambda_2}] = E[e^{\alpha N_2}]. \ \blacksquare \end{aligned}$$

In the following theorem the situation is studied where a risk is produced by one of m sources. The index i for which $I_i = 1$ indicates which source actually produces the risk. The resulting distribution is a mixed distribution. We formulate the maintenance of exponential orders for the random variables.

Theorem 3.7. (Mixing of Random Variables) Let $X_1, \cdots, X_m$ and $Y_1, \cdots, Y_m$ be two sequences of independent risks with $X_i <_e Y_i$ for all $i = 1, \cdots, m$. If $I_1, \cdots, I_m$ have a joint distribution such that $I_1 + \cdots + I_m \equiv 1$ and marginally, $P(I_i) = p_i = 1 - P(I_i = 0)$. Then

$$\sum_{i=1}^{m} I_i X_i <_e \sum_{i=1}^{m} I_i Y_i.$$

Proof

For all $\alpha > 0$,

$$\begin{aligned} E[\exp\{\alpha \sum_{i=1}^{m} I_i X_i\}] &= \sum_{i=1}^{m} E[e^{\alpha X_i}] \cdot p_i \\ &\leq \sum_{i=1}^{m} E[e^{\alpha Y_i}] \cdot p_i = E[\exp\{\alpha \sum_{i=1}^{m} I_i Y_i\}]. \ \blacksquare \end{aligned}$$

In order to prove Theorem 3.12, we need the following proposition.

Proposition 3.8 Let X be non-negative random variable. If there exists $\alpha_o > 0$ such that $E[e^{\alpha_o X}] < \infty$, then

$$\lim_{x \to \infty} e^{\alpha_o x} \overline{F}(x) = 0,$$

where $\overline{F}(x)$ is the survival function of X.

Proof

From $E[e^{\alpha_o x}] < \infty$, we have

$$\lim_{x \to \infty} \int_x^\infty e^{\alpha_o y} \, dF(y) = 0,$$

therefore

$$\lim_{x \to \infty} e^{\alpha_o x} \overline{F}(x) \leq \lim_{x \to \infty} \int_x^\infty e^{\alpha_o y} \, dF(y) = 0.$$

The proof is complete. $\blacksquare$

Now we prove a more general result as follows:

Proposition 3.9. If $\alpha < A_X$, then

$$\lim_{x\to\infty} e^{\alpha x}\Pi_X^{(n)}(x) = 0, \quad n = 1, 2, \cdots.$$

Proof

We first prove that $M_X^{(n)}(\alpha) < \infty$, $n = 1, 2, \cdots$. Since

$$M_X(\alpha) = \int_0^\infty \lim_{k\to\infty} (\sum_{i=0}^{k} \frac{x^i\alpha^i}{i!})\, dF_X(x),$$

let

$$g_k(x) = \sum_{i=0}^{k} \frac{x^i\alpha^i}{i!},$$

then

$$|g_k(x)| \le e^{\alpha x},$$

and

$$\int_0^\infty e^{\alpha x}\, dF_X(x) < \infty.$$

By dominated convergence theorem, we have

$$\begin{aligned} M_X(\alpha) &= \int_0^\infty \lim_{k\to\infty} g_k(x)\, dF_X(x) \\ &= \sum_{i=0}^{\infty} \frac{E[X^i]\alpha^i}{i!} < \infty. \end{aligned}$$

So $M_X(\alpha)$ has derivatives of all orders at α and $M_X^{(n)}(\alpha)$ can be calculated by term-by-term differentiation of the series. That is

$$M_X^{(n)}(\alpha) = \lim_{k\to\infty} \int_0^\infty \sum_{i=0}^{k} \frac{x^{n+i}\alpha^i}{i!}\, dF_X(x).$$

By monotone convergence theorem, we have

$$\begin{aligned} M_X^{(n)}(\alpha) &= \int_0^\infty \lim_{k\to\infty} \sum_{i=0}^{k} \frac{x^i\alpha^i}{i!} x^n\, dF_X(x) \\ &= \int_0^\infty x^n e^{\alpha x}\, dF_X(x) < \infty. \end{aligned}$$

Therefore

$$\lim_{x\to\infty} \int_x^\infty y^n e^{\alpha y}\, dF_X(y) = 0,$$

and

$$\begin{aligned} \lim_{x\to\infty} e^{\alpha x}\Pi_X^{(n)}(x) &= \lim_{x\to\infty} e^{\alpha x} \int_x^\infty (y-x)^n\, dF_X(y) \\ &\leq \lim_{x\to\infty} \int_x^\infty e^{\alpha y} y^n\, dF_X(y) = 0. \end{aligned}$$

The prove is completed. ∎

Let us generalize the concept of the moment generating function and the concept of the exponential order to the class of general nonnegative monotonous decreasing functions on $[0, \infty)$.

Let Ω be the same as in Definition 1.3. We have the following definitions:

Definition 3.10 Let $H(x) \in \Omega$, and

$$M_H(\alpha) = -\int_0^\infty e^{\alpha x}\, dH(x) \text{ (can be } \infty).$$

We call $M_H(\alpha)$ the Laplace transform of $H(x)$.

Definition 3.11 Suppose $H(x), G(x) \in \Omega$. We say that $H(x)$ is less than $G(x)$ in the meaning of exponential order, denoted by $H(x) <_e G(x)$, if

$$M_H(\alpha) \le M_G(\alpha) \quad \text{for all } \alpha > 0.$$

According to these definitions, we can discuss the maintenance of exponential order for the 1st stop-loss transform.

Theorem 3.12 If $X <_e Y$,$E[X] < \infty$ and $E[Y] < \infty$, then

$$\Pi_X^{(1)}(x) <_e \Pi_Y^{(1)}(x).$$

Proof

If $\alpha < A_X$, from Theorem 1.2.1 of Goovaerts et al. (1990) and Proposition 3.8, we have

$$\begin{aligned}
-\int_0^\infty e^{\alpha x}\, d\Pi_X^{(1)}(x) &= \int_0^\infty e^{\alpha x}\Pi_X^{(0)}(x)\, dx \\
&= -\frac{1}{\alpha} + \int_0^\infty \frac{e^{\alpha x}}{\alpha}\, dF_X(x) \\
&\le -\frac{1}{\alpha} + \int_0^\infty \frac{e^{\alpha x}}{\alpha}\, dG_Y(x) \\
&= -\int_0^\infty e^{\alpha x}\, d\Pi_Y^{(1)}(x).
\end{aligned}$$

If $\alpha > A_X$, from $X <_e Y$ and Proposition 3.2, we have

$$\begin{aligned}
\lim_{y\to\infty}\int_0^y e^{\alpha x}\Pi_X^{(0)}(x)\,dx &= \lim_{y\to\infty}[\frac{e^{\alpha x}}{\alpha}(1-F_X(x))|_0^y + \int_0^y \frac{e^{\alpha x}}{\alpha}\,dF_X(x)] \\
&\geq \lim_{y\to\infty}[-\frac{1}{\alpha}+\frac{1}{\alpha}\int_0^y e^{\alpha x}\,dF_X(x)] \to \infty,
\end{aligned}$$

and

$$\begin{aligned}
\lim_{y\to\infty}\int_0^y e^{\alpha x}\Pi_Y^{(0)}(x)\,dx &= \lim_{y\to\infty}[\frac{e^{\alpha x}}{\alpha}(1-G_Y(x))|_0^y + \int_0^y \frac{e^{\alpha x}}{\alpha}\,dG_Y(x)] \\
&\geq \lim_{y\to\infty}[-\frac{1}{\alpha}+\frac{1}{\alpha}\int_0^y e^{\alpha x}\,dG_Y(x)] \to \infty.
\end{aligned}$$

Therefore

$$-\int_0^\infty e^{\alpha x}\,d\Pi_X^{(1)}(x) = -\int_0^\infty e^{\alpha x}\,d\Pi_Y^{(1)}(x) = \infty.$$

If $\alpha = A_X$, and $M_X(\alpha) = \infty$, then from $X <_e Y$, we have $M_Y(\alpha) = \infty$, the prove is the same as the case $\alpha > A_X$.

If $\alpha = A_X$, and $M_X(\alpha) < \infty$, then from Proposition 3.8, we have

$$\begin{aligned}
\lim_{y\to\infty}\int_0^y e^{\alpha x}\Pi_X^{(0)}(x)\,dx &= \lim_{y\to\infty}[\frac{e^{\alpha x}}{\alpha}(1-F_X(x))|_0^y + \int_0^y \frac{e^{\alpha x}}{\alpha}\,dF_X(x)] \\
&= -\frac{1}{\alpha}+\lim_{y\to\infty}\int_0^y \frac{e^{\alpha x}}{\alpha}\,dF_X(x) \\
&\leq -\frac{1}{\alpha}+\lim_{y\to\infty}\int_0^y \frac{e^{\alpha x}}{\alpha}\,dG_Y(x) \\
&\leq \lim_{y\to\infty}\int_0^y e^{\alpha x}\Pi_Y^{(0)}(x)\,dx.
\end{aligned}$$

So we have

$$\Pi_X^{(1)}(x) <_e \Pi_Y^{(1)}(x).$$

The proof is completed. ∎

4 Applications in Ruin Probabilities

We have now established two ways to study the risk models. One way is by using the martingale theorey, we can find formulas of $\psi(u)$ and estimate the upper bound and lower bound of $\psi(u)$. We will give a formula of $\psi(u)$ in Section 5 by this way. The other way is by using the Renewal equation from which we can find the distribution of L_1 and therefore we can study the maximal aggregate loss random variable L and L provides much more information about $\psi(u)$. We use this method to discuss the properties of L and $\psi(u)$ in this section .

4.1 Surplus Process

Now we introduce some concepts related to ruin probabilities by Bowers et al. (1997).

4.1.1 Surplus Process

Let $U(t)$ denote an insurer's surplus at time t, u denote the initial surplus at time 0, $c(t)$ denote premiums collected through time t, and $S(t)$ denote aggregate claims paid through time t. $U(t)$ is given by

$$U(t) = u + c(t) - S(t). \tag{19}$$

We call $U(t)$ the surplus process and $S(t)$ the aggregate claims process. $S(t)$ is determined by the number of claims $N(t)$ that occured in $[0,\ t)$ and the amount of each claim $X_1, \cdots, X_{N(t)}$. In this section, we assume that the claim number process $N(t)$ is a homogeneous Poisson process with constant parameter λ, X_i, $i = 1, 2, \cdots$, are independent and identically distributed with common d.f. $F_X(x)$, premium rate is a constant, c, $c > 0$. $c(t) = ct$, $c = (1+\theta)\lambda E[X]$ where θ is the security loading. Consequently, $S(t)$ is a compound Poisson process, and it is expressed as follows:

$$S(t) = \sum_{i=1}^{N(t)} X_i. \tag{20}$$

4.1.2 Ruin Probability

When the surplus becomes negative for the first time, we say that ruin has occurred. Let

$$T = \min\{t :\ t \geq 0,\ U(t) < 0\} \tag{21}$$

denote the time of ruin with $T = \infty$ if $U(t) \geq 0$ for all t. Let

$$\psi(u) = \Pr(T < \infty \mid U(0) = u) \tag{22}$$

denote the probability of ruin which is a function of the initial surplus u, and

$$\psi(u,\ t) = \Pr(T < t \mid U(0) = u) \tag{23}$$

denote the probability of ruin before time t. Of course, $\psi(u)$ is an upper bound for $\psi(u,t)$.

4.1.3 The First Surplus below the Initial Level

Let L_1 be a random variable denoting the amount by which the surplus falls below the initial level for the first time, given that this ever happens. The p.d.f. for L_1 is

$$f_{L_1}(y) = \frac{1}{p_1}[1 - P(y)], \quad y > 0, \tag{24}$$

where $P(y)$ is the d.f. of claim size random variable X, $p_1 = E[X]$.

4.1.4 The Maximal Aggregate Loss

Let

$$L = \max_{t \geq 0}\{S(t) - ct\} = \max_{t \geq 0}\{u - U(t)\} \tag{25}$$

denote the maximal aggregate loss random variable. By this definition, we know that $L \geq 0$ and

$$\Pr(L \leq u) = 1 - \psi(u), \quad u \geq 0, \tag{26}$$

$$\Pr(L = 0) = 1 - \psi(0). \tag{27}$$

If L_i denote the ith deficit and M denote the total number of deficits, then M has a geometric distribution with parameter $p = 1 - \psi(0)$ and $L_1, L_2, \cdots$ are i.i.d. with the common p.d.f. given by (24). We can represent L as follows:

$$L = L_1 + L_2 + \cdots + L_M. \tag{28}$$

4.2 The Relationship between the Order in Claims and the Order in Ruin Probabilities

From the following discussion we can finally see that the exponential ordered claim amounts induce the exponential ordered ruin probability functions.

Proposition 4.1 Let L_1^X and L_1^Y be random variables denoting the amounts by which the surpluses fall below the initial levels for the first time, given that these ever happen. If $X <_e Y$, and $E[X] = E[Y]$, then

$$L_1^X <_e L_1^Y.$$

Proof

From (24),we have

$$f_{L_1}(y) = \frac{1}{p_1}[1 - P(y)],$$

and by Theorem 3.12, we have for all $\alpha > 0$,

$$\begin{aligned} \int_0^\infty e^{\alpha x} f_{L_1^X}(x)\, dx &= -\frac{1}{E[X]} \int_0^\infty e^{\alpha x}\, d\Pi_X^{(1)}(x) \\ &\leq -\frac{1}{E[Y]} \int_0^\infty e^{\alpha x}\, d\Pi_Y^{(1)}(x) = \int_0^\infty e^{\alpha x} f_{L_1^Y}(x)\, dx. \quad \blacksquare \end{aligned}$$

Proposition 4.2 Let L_X and L_Y be the maximal aggregate losses related risks X and Y, θ_1 and θ_2 are security loadings related risks X and Y. If $X <_e Y$,

$E[X] = E[Y]$ and $\theta_1 = \theta_2$. then

$$L_X <_e L_Y.$$

Proof

From the former discussion in this section, we know that

$$L = L_1 + L_2 + \cdots + L_M,$$

where M is the total number of deficits and has a geometric distribution with

$$\Pr(M = n) = (1 - \psi(0))(\psi(0))^n = \theta \cdot (\frac{1}{1+\theta})^{n+1}.$$

Applying Proposition 4.1 and Proposition 3.4 on ${L_i}^X$, ${L_i}^Y$, M_1 and M_2 are identical distributed, we have

$$L_X <_e L_Y. \blacksquare$$

From (26) and (27) we know that $\psi(u) \geq 0$, monotonous decreasing and $\lim_{u \to 0} \psi(u) = 0$. Consequently, $\psi(u) \in \Omega$ and we can define the exponential order on the family of ruin probability functions as follows:

Definition 4.3 We say that ruin probability function $\psi_X(u)$ is less than $\psi_Y(u)$ in the meaning of the exponential order, denoted by $\psi_X(u) <_e \psi_Y(u)$, if for all $\alpha > 0$,

$$-\int_0^\infty e^{\alpha x}\, d\psi_X(x) \leq -\int_0^\infty e^{\alpha x}\, d\psi_Y(x) \quad \text{(can be } \infty\text{)}.$$

Theorem 4.4 If $X <_e Y$ and $E[X] = E[Y] < \infty$ and the security loading θ_1 and θ_2 are the same as in Proposition 4.2, then

$$\psi_X(u) <_e \psi_Y(u).$$

Proof

Note that

$$\psi_X(u) = \Pr(L_X > u) = 1 - F_{L_X}(u).$$

From Definition 4.3 and Proposition 4.2, we arrive at the conclusion. ∎

4.3 Moments of the Maximal Aggregate Loss Distribution

For some claim distributions it may be difficult to calculate adjustment coeffients and ruin probabilities. Approximation methods based on the moments of the maximal aggregate loss distribution may be easy to apply. In this section we develop theory which will provide more information about $E[L^i]$.

Theorem 4.5. If $E[X] = E[Y]$, $X <_e Y$ and the security loading θ_1 and θ_2 are the same as in Proposition 4.2, then

$$E[L_X] \leq E[L_Y].$$

Proof

According to Proposition 4 of Cheng and Pai (1999b) and Theorem 3.3, we have

$$\begin{aligned} E[L_X] &= \frac{E[X^2]}{2\theta_1 E[X]} \\ &\leq \frac{E[Y^2]}{2\theta_2 E[Y]} = E[L_Y]. \blacksquare \end{aligned}$$

Theorem 4.6. If $E[X^i] = E[Y^i]$, $i = 1, 2$,$X <_e Y$ and the security loading θ_1 and θ_2 are the same as in Proposition 4.2, then

$$E[(L_X)^2] \leq E[(L_Y)^2].$$

Proof

According to Proposition 4.2

$$X <_e Y$$

implies

$$L_X <_e L_Y,$$

and from Proposition 4 of Cheng and Pai (1999b)

$$E[X^i] = E[Y^i],\ i = 1, 2$$

implies

$$E[L_X] = E[L_Y].$$

Therefore by Theorem 3.3, we know that

$$E[{L_X}^2] \leq E[{L_Y}^2]. \blacksquare$$

Theorem 4.7 If $E[X^i] = E[Y^i]$, $i = 1, \cdots, k$ and $X <_e Y$, then

$$E[(L_1^X)^k \mid T_1^* < \infty] \leq E[(L_1^Y)^k \mid T_1^* < \infty],$$

where T_1^* is the first time at which a deficit occurs.

Proof

By Proposition 2 of Cheng and Pai (1999c) and Theorem 3.3

$$\begin{aligned} E[(L_1^X)^k \mid T_1^* < \infty] &= \frac{E[X^{k+1}]}{(k+1)E[X]} \\ &\leq \frac{E[Y^{k+1}]}{(k+1)E[Y]} = E[(L_1^Y)^k \mid T_1^* < \infty]. \blacksquare \end{aligned}$$

In Cai and Garrido (1998), a method was given to calculate $E[L^2]$, the purpose of proposition 4.8 is to give us a method that can also calculate $E[L^3]$.

Proposition 4.8 If $A_X > 0$, then

$$E[L^2] = \frac{1/3\theta p_1 \cdot p_3 + 1/2{p_2}^2}{\theta^2 {p_1}^2},$$

where $p_i = E[X^i]$.

Proof

From Theorem 13.6.1 of Bowers et al. (1997). we know that

$$M_L(r) = \frac{\theta p_1 r}{1 + (1+\theta)p_1 r - M_X(r)}.$$

Applying the formula

$$Var[L] = \lim_{r \to 0} \frac{d^2}{dr^2}[\ln M_L(r)],$$

we have

$$\frac{d}{dr}[\ln M_L(r)]' = \frac{1}{r} - \frac{(1+\theta)p_1 - M_X{}'(r)}{1 + (1+\theta)p_1 r - M_X(r)},$$

and

$$\begin{aligned} & \frac{d^2}{dr^2}[\ln M_L(r)] \qquad\qquad (29) \\ = & -\frac{1}{r^2} + \frac{M_X{}''(r)[1 + (1+\theta)p_1 r - M_X(r)] + [(1+\theta)p_1 - M_X{}'(r)]^2}{[1 + (1+\theta)p_1 r - M_X(r)]^2}. \end{aligned}$$

For $0 < r < \alpha < A_X$, from the proof of Proposition 3.9. we know that $M_X{}^{(i)}(r) < M_X{}^{(i)}(\alpha) < \infty$, $i = 1, \cdots, 5$. Hence

$$M_X(r) = 1 + rp_1 + \frac{r^2}{2}p_2 + \frac{r^3}{3!}p_3 + \int_0^\infty (\frac{r^4 x^4}{4!} + \frac{r^5 x^5}{5!} + \cdots) f(x)\, dx.$$

Since

$$\begin{aligned} & \int_0^\infty (\frac{r^4 x^4}{4!} + \frac{r^5 x^5}{5!} + \cdots) f(x)\, dx \\ = & \ r^4 \int_0^\infty \frac{x^4}{4!}(1 + \frac{rx}{5} + \frac{r^2 x^2}{6 \cdot 5} + \cdots) f(x)\, dx \\ \leq & \ r^4 \int_0^\infty x^4 e^{rx} f(x)\, dx \\ \leq & \ r^4 M_X{}^{(4)}(\alpha) = o(r^3), \end{aligned}$$

then

$$M_X(r) = 1 + rp_1 + \frac{r^2}{2}p_2 + \frac{r^3}{3!}p_3 + o(r^3).$$

Applying the same method we have

$$M_X{}'(r) = p_1 + rp_2 + \frac{r^2}{2}p_3 + O(r^3),$$

and

$$M_X{}''(r) = p_2 + rp_3 + \frac{r^2}{2}p_4 + O(r^3).$$

Let $k(r) = [1 + (1+\theta)p_1 r - M_X(r)]$, then (29) becomes

$$\begin{aligned}
&\frac{d^2}{dr^2}[\ln M_L(r)] \\
= \;& \frac{M_X{}''(r)k(r) + [(1+\theta)p_1 - M_X{}'(r)]^2 - 1/r^2k^2(r)}{k^2(r)} \\
= \;& \frac{1/3\theta p_1 p_3 r^2 + 1/4{p_2}^2 r^2 + o(r^2)}{[\theta p_1 r - r^2/2 + O(r^3)]^2} \\
= \;& \frac{1/3\theta p_1 p_3 + 1/4{p_2}^2}{\theta^2 {p_1}^2}.
\end{aligned}$$

From Proposition 4 of Cheng and Pai (1999b), we have

$$E[L] = \frac{p_2}{2\theta p_1},$$

hence

$$E[L^2] = \frac{1/3\theta p_1 p_3 + 1/2{p_2}^2}{\theta^2 {p_1}^2}. \;\blacksquare$$

Proposition 4.9 Suppose $E[X^i] = E[Y^i]$, $i = 1, 2, 3$. If $X <_e Y$, and the security loading θ_1 and θ_2 are the same as in Proposition 4.2, then

$$E[(L_X)^3] \leq E[(L_Y)^3].$$

Proof

From Proposition 4 of Cheng and Pai (1999b) and Proposition 4.8, we know that $E[L_X] = E[L_Y]$, $E[(L_X)^2] = E[(L_Y)^2]$, again by Theorem 3.3, we arrive at the conclusion. ∎

We give another example that apply Theorem 2.1 to estimate $E[L^n]$.

Example 2 Denote the random variable having an exponential distribution with parameter μ by e_μ. Suppose X and e_μ are two risks. If $E[X] = \frac{1}{\mu}$, $X <_{sl(n)} e_\mu$, security loadings θ_1 and θ_2 related to risks X and e_μ are equal, then $E[L_X^n] < \frac{n!}{(1+\theta)R^n}$, where $R = \frac{\theta\mu}{(1+\theta)}$.

Proof

From Lemma 1 of Cheng and Pai (1999c), we know that $L_X <_{sl(n-1)} L_{e_\mu}$, and by Theorem 2.1, we have $E[L_X^n] < E[L_{e_\mu}^n]$. We know that

$$1 - \Pr(L_{e_\mu} \leq u) = \psi_{e_\mu}(u) = \frac{1}{1+\theta}e^{-Ru},$$

consequently

$$E[L_{e^\mu}^n] = \frac{R}{1+\theta}\int_0^\infty u^n e^{-Ru}\, du = \frac{n!}{(1+\theta)R^n},$$

and we arrive at the conclusion. ∎

4.4 The Application in Risk Premium Principles

Now we cite some concepts of risk premium principles in Goovaerts et al. (1990).

The basis of insurance is the hypothesis that claims can be compensated by fixed payments called premiums. Premiums are calculated by a premium calculation principle. This is a rule π that assigns a real number $\pi[X]$, also written $\pi[F_X]$, to the distribution function F_X of risk X. Each premium principle induces a total order of all risks. ranking risk X with low premium $\pi[X]$ below risk Y with higher premium $\pi[Y]$. We make three assumptions.

1. If $X <_{sl(0)} Y$, then $\pi[X] \leq \pi[Y]$, with equality only if $F_X = F_Y$.
2. If $P[X = c] = 1$, $0 \leq c$, then $\pi[X] = c$.
3. Let X, X' be risks such that $\pi[X] = \pi[X']$, $p \in [0, 1]$, then

$$\pi[pF_X + (1-p)F_Y] = \pi[pF_{X'} + (1-p)F_Y].$$

These assumptions lead to a Mean Value Principle, where the premium is calculated from the formula

$$\pi[X] = f^{-1}(E[f(X)]),$$

for some suitable increasing continuous valuation function f. For example, $f(x) = -u(w-x)$ where $u(x)$ is a utility function and w is the wealth of the decision maker. We can narrow the class of premium principles even further by adding the fourth requirement of additivity.

A premium principle π is called additive if for independent risk X and Y, $\Pi(X+Y) = \Pi(X) + \Pi(Y)$. From Theorem 6.2.2 in Goovaerts (1990), we can see that by the four requirments mentioned above the set of feasible premium principles is reduced to the net premium principles $f(x) = x$ and the exponential principles $f(x) = e^{\alpha x}$.

For net premium principle, we can not distinguish the risk X and Y if $E[X] = E[Y]$ but $X <_{sl(1)} Y$(that is $Var(X) < Var(Y)$ by Theorem 2.1), the situation is different if we use exponential principle, from the following theorem we can see that the exponential premium principle can differentiate between losses more finely than the net premium principle under some conditions .

Theorem 4.10 Let X and Y be two risks. If $E[X^k] = E[Y^k]$, $k = 1, 2, \cdots, n-1$, and $X <_{sl(n)} Y$, then

$$\pi(X) < \pi(Y),$$

under the exponential premium principle for the same α.

Proof

From Theorem 2.1, we know that

$$E[X^n] < E[Y^n].$$

Consequently

$$\begin{aligned}
\pi(X) &= \frac{1}{\alpha}\ln[E[e^{\alpha X}]] \\
&= \frac{1}{\alpha}\ln(1+\alpha E[X]+\frac{\alpha^2}{2!}E[X^2]+\cdots+\frac{\alpha^n}{n!}E[X^n]+\cdots) \\
&< \frac{1}{\alpha}\ln(1+\alpha E[Y]+\frac{\alpha^2}{2!}E[Y^2]+\cdots+\frac{\alpha^n}{n!}E[Y^n]+\cdots) \\
&= \pi(Y). \blacksquare
\end{aligned}$$

Now that the exponential premium principle can differentiate between losses more finely than the net premium principle, maybe we can choose some proper form for c in the classical risk models, and discuss the properties of ruin probabilities. This topic is worth further study.

Proposition 4.11 Let X and Y be two risks. If $X <_e Y$, then

$$\pi(X) \leq \pi(Y)$$

under the exponential premium principles for the same α.

Proof

We arrive at the conclusion immediately from Definition 1.4. ∎

We know that if the aggregate claims process is not a compound Poisson process, sometimes it is difficult to calculate the exponential premium for $S(t)$ in a unit time interval, by using the maintenance properties of exponential order, we can compare the exponential premiums for two aggregate claims processes.

Proposition 4.12 Suppose the exponential principle is being used. Under the condition of Theorem 3.4 or Theorem 3.5, we have $\pi_\alpha[S_1] \le \pi_\alpha[S_2]$.

Proof

We can have $S_1 <_e S_2$ both from Theorem 3.4 and Theorem 3.5. Consequently

$$\pi_\alpha[S_1] = f^{-1}(E[f(S_1)]) = \frac{1}{\alpha}\ln(M_{S_1}(\alpha)) \tag{30}$$

$$\le \frac{1}{\alpha}\ln(M_{S_2}(\alpha)) = \pi_\alpha[S_2]. \blacksquare \tag{31}$$

4.5 The Concepts of Distance on Risk Sets

Now we introduce the concepts of nth stop-loss distances and ruin probability distances on the set of risks. From the discussion of Cheng and Pai (1999b), we can compare the ruin probability distances of different pairs of risks.

Definition 4.13 Suppose X and Y are two risks. We define the nth stop-loss distance as follows:

$$d_{s(n)}(X,Y) = \int_0^\infty \mid \Pi_X^{(n)}(u) - \Pi_Y^{(n)}(u) \mid \, du. \tag{32}$$

From Definition 4.13 and Theorem 1.1, we know that $d_{s(n)}(X,0) = \frac{1}{n+1}E[X^{n+1}]$, $d_{s(n)}(X,Y) > 0$ if X and Y are not identically distributed, and also we have following propositons.

Proposition 4.14 If $X <_{wsl(n)} Y$, then

$$d_{s(n)}(X,Y) = \frac{1}{n+1}\{E[Y^{n+1}] - E[X^{n+1}]\}. \tag{33}$$

Proof

From Definition 1.3, we know that

$$\Pi_X^{(n)}(u) \leq \Pi_Y^{(n)}(u),$$

by Theorem 1.1, we have

$$\begin{aligned} d_{s(n)}(X,Y) &= \int_0^\infty \Pi_Y^{(n)}(u)\, du - \int_0^\infty \Pi_X^{(n)}(u)\, du \\ &= \frac{1}{n+1}\{E[Y^{n+1}] - E[X^{n+1}]\}. \blacksquare \end{aligned}$$

Proposition 4.15 Suppose X. Y and Z are not identically distributed risks.

(1) If $X <_{sl(n)} Y <_{sl(n)} Z$, then

$$d_{s(n)}(X, Y) < d_{s(n)}(X, Z).$$

(2) If $X <_{sl(m)} Y$. $Y <_{sl(n)} Z$ and $l = \max\{m, n\}$, then

$$d_{s(l)}(X, Y) < d_{s(l)}(X, Z).$$

Proof

From Proposition 4.14 and Theorem 2.1, we can immediately get (1).

From Theorem 1.2, we know that

$$X <_{sl(l)} Y <_{sl(l)} Z,$$

and by (1), we can get (2). ∎

Definition 4.16 Suppose X and Y are two risks. We define the ruin probability distance between X and Y with parameter u as follows:

$$d_{\psi,u}(X, Y) = \int_u^\infty \mid \psi_X(x) - \psi_Y(x) \mid \, dx \tag{34}$$

From Definition 4.16, we know that $d_{\psi,u}(X, Y) > 0$ if X and Y are not identically distributed risks.

Proposition 4.17 Let X and Y be two risks. Suppose $\psi_X(x)$ and $\psi_Y(x)$ intersect at finite points, denoted by $x_1 < x_2 < \cdots < x_k$. If there is an integer $n \geq 0$ such that $\psi_X(x) <_{wsl(n)} \psi_Y(x)$, then

$$d_{\psi,u}(X,\ Y) = \Pi^{(1)}_{\psi_Y}(u) - \Pi^{(1)}_{\psi_X}(u), \quad \text{for all } u > x_k. \tag{35}$$

Proof

From Proposition 7 of Cheng and Pai (1999b), we know that

$$\psi_X(x) < \psi_Y(x), \quad \text{for all } u > x_k.$$

Consequently

$$\begin{aligned} d_{\psi,u}(X,\ Y) &= \int_u^\infty \psi_Y(x)\,dx - \int_u^\infty \psi_X(x)\,dx \\ &= \Pi^{(1)}_{\psi_Y}(u) - \Pi^{(1)}_{\psi_X}(u), \quad \text{for all } u > x_k. \ \blacksquare \end{aligned}$$

Example 3 Let X and e_μ be two risks. Suppose $\psi_X(x)$ and $\psi_{e_\mu}(x)$ intersect at finite points, denoted by $x_1 < x_2 < \cdots < x_k$. If there is an integer $n \geq 0$ such that $\psi_X(x) <_{wsl(n)} \psi_{e_\mu}(x)$, then

$$\Pi^{(1)}_{\psi_X}(u) < \frac{1}{\theta\mu} \exp\{\frac{-\theta\mu}{1+\theta}u\}, \quad \text{for all } u > x_k.$$

Proof

By Proposition 4.17, we have

$$d_{\psi,u}(X,\ e_\mu) = \Pi^{(1)}_{\psi_{e_\mu}}(u) - \Pi^{(1)}_{\psi_X}(u) > 0,$$

also from

$$\Pi^{(1)}_{\psi_{e_\mu}}(u) = \int_u^\infty \psi_{e_\mu}(x)\, dx = \frac{1}{\theta\mu} \exp\{\frac{-\theta\mu}{1+\theta} u\}$$

we arrive at the conclusion immediately. ∎

5 A Formula for Ruin Probabilities

Consider an insurance portfolio in which the distribution of the number of claims in a year is a conditional Poisson distribution with mean $\Lambda = \lambda$. For most classes of general insurance, many possible sources of heterogeneity of risk exist. For example, if Λ follows the gamma distribution representing heterogeneity of risk, then the number of claims follows the negative binomial distribution. At this time, the aggregate claims process $S(t)$ is complicated since $S(t)$ does not have both independent and stationary increments like compound Poisson process. In the following theorem we give a formula for ruin probabilities for heterogeneous portfolios.

Theorem 5.1. (Ruin Probability) For a heterogeneous portfolio, let $X_1, X_2, \cdots$ be independent, identically distributed claim amount random variables with common d.f. $P(x)$, Λ be a non-negative variable on $(\lambda_o,\ \lambda_1)$, $\lambda_o > 0$, $\lambda_1 < \infty$. $X_1, X_2, \cdots$ be independent of the process $N(t)$, $S(t) = \sum_{i=1}^{N(t)} X_i$, c be the premium rate for the portfolio, $c = (1+\theta)E[\Lambda]E[X]$. Given that $\Lambda = \lambda$, $N(t)$ is a Poisson process (λt). If $U(t) = u + ct - S(t)$, then for $u \geq 0$,

$$\psi(u) = \frac{e^{-Ru}}{E[e^{-RU(T)} \mid T < \infty]}, \tag{36}$$

where R is the smallest positive root of

$$1 + (1+\theta)E[X]r = M_X(r). \tag{37}$$

Proof

It is similar to the proof of Theorem 13.4.1 of Bowers et al. (1997). For $t > 0$ and $r > 0$,

$$\begin{aligned} & E[e^{-rU(t)}] \\ = \quad & E[e^{-rU(t)} \mid T \le t] \cdot \Pr(T \le t) + E[e^{-rU(t)} \mid T > t] \cdot \Pr(T > t). \end{aligned} \tag{38}$$

The term on the left-hand side is

$$\begin{aligned} & E[e^{-rU(t)}] = E_\Lambda[E[e^{-rU(t)} \mid \Lambda = \lambda]] \\ = \quad & e^{-ru} \cdot E_\Lambda[\exp\{[M_X(r) - 1 - r(1+\theta)E[X]]\lambda t\}]. \end{aligned} \tag{39}$$

The first term on the right-hand side is

$$\begin{aligned} & E[e^{-rU(t)} \mid T \le t] \cdot \Pr(T \le t) \\ = \quad & E_\Lambda[E[e^{-rU(t)} \mid T \le t,\ \Lambda = \lambda]] \cdot \Pr(T \le t) \\ = \quad & E_\Lambda[E[\exp\{-r[U(T) + c(t-T) - [S(t) - S(T)]]\} \mid T \le t,\ \Lambda = \lambda]] \cdot \end{aligned}$$

$$\begin{aligned}
& \Pr(T \le t) \\
= \; & E_\Lambda[E[e^{-rU(T)} \exp\{(M_X(r) - 1 - r(1+\theta))\lambda(t-T)\} \mid T \le t,\ \Lambda = \lambda]] \cdot \\
& \Pr(T \le t).
\end{aligned} \tag{40}$$

We choose r that satisfy (37), then (38) becomes

$$\begin{aligned}
e^{-Ru} &= E_\Lambda[E[e^{-RU(T)} \mid T \le t,\ \Lambda = \lambda]] \cdot \Pr(T \le t) + \\
& \quad E_\Lambda[E[e^{-RU(t)} \mid T > t,\ \Lambda = \lambda]] \cdot \Pr(T > t).
\end{aligned}$$

Let $t \to \infty$, if

$$\lim_{t\to\infty} E_\Lambda[E[e^{-RU(t)} \mid T > t,\ \Lambda = \lambda]] \cdot \Pr(T > t) = 0, \tag{41}$$

from (38), we can get

$$\begin{aligned}
e^{-Ru} &= E_\Lambda[E[e^{-RU(T)} \mid T < \infty,\ \Lambda = \lambda]] \cdot \psi(u) \\
&= E[e^{-RU(T)} \mid T < \infty] \cdot \psi(u).
\end{aligned}$$

Now we only need to show (41). From the proof of Theorem 13.4.1 of Bowers et al. (1997), we have

$$\begin{aligned}
& E[e^{-RU(t)} \mid T > t,\ \Lambda = \lambda] \cdot \Pr(T > t, \Lambda = \lambda) \\
\le \; & t^{-\frac{1}{3}} + \exp\{-R(u + \theta\lambda t p_1 - \sqrt{\lambda p_2} t^{\frac{2}{3}})\}.
\end{aligned} \tag{42}$$

Consequently

$$
\begin{aligned}
& E[e^{-RU(t)} \mid T > t] = E_{\Lambda}[E[e^{-RU(t)} \mid T > t,\ \Lambda = \lambda]] \\
= \quad & E_{\Lambda}[\frac{E[e^{-RU(t)} \mid T > t,\ \Lambda = \lambda] \cdot \Pr(T > t, \Lambda = \lambda)}{\Pr(T > t, \Lambda = \lambda)}] \\
\leq \quad & E_{\Lambda}[\frac{t^{-\frac{1}{3}} + \exp\{-R(u + \theta\lambda t p_1 - \sqrt{\lambda p_2} t^{\frac{2}{3}})\}}{(1 - \psi(u \mid \Lambda = \lambda)) \cdot f_{\Lambda}(\lambda)}] \\
\leq \quad & c_1 t^{-\frac{1}{3}} + c_2 \exp\{-R(\theta\lambda_{\circ} t p_1 - \sqrt{\lambda_1 p_2} t^{\frac{2}{3}})\} \\
\to \quad & 0 \text{ (when } t \to \infty),
\end{aligned}
$$

and we can have (41). ∎

In general, a closed form evaluation of the denominator of (36) is not possible. However, (36) can be used to derive inequalities. It is easy to see that $\psi(u) < e^{-Ru}$. Moreover, if the claim amount distribution is bounded so that $F(m) = 1$ for some finite m, then we have $\psi(u) > e^{-R(u+m)}$.

For the special case of $u = 0$ and the case where the claim amount distributions are mixtures of exponential distributions, the explicit expressions for $\psi(u)$ are worth further study.

6 Concluding Remarks

The theory of partial orders of risks is interesting and useful in many fields. This paper discussed the properties of nth stop-loss orders and exponential orders. The necessary condition and the sufficient condition for the nth stop-loss order are convenient tools to construct risk pairs that can have nth stop-loss orders. The relationship between a claim random variable and ruin probability was also established. This result is worth further study for finding upper bounds of ruin probabilities. The applications of these partial orders in evaluating existing risk premium principles and setting up new risk premium principles are worth further study.

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