Deepening Children's Understanding of Rational Numbers: A Developmental Model and Two Experimental Studies

by

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Abstract

The present thesis concerns the design, implementation and assessment of an experimental program to teach the difficult topic of rational numbers. Based on Case's theory of cognitive development, it was hypothesized that a core conceptual structure for a global understanding of rational number is formed by the coordination of children's intuitive understanding of proportion and their numerical splitting schemas. In order to support this coordination, the experimental curriculum introduced rational number through the teaching of percent in linear measurement. Props such as cylindrical beakers filled with water, allowed students to make proportional judgements of fullness of these containers relative to the whole using the language of percents. Thus, the traditional sequence of rational number instruction was altered so that decimals and fractions were taught later grounded in students understanding of percents.

Two formal teaching studies were conducted. In the first, the participants were an intact grade 4 class who were new to rational number and in the second the curriculum was implemented with grade 6 students who had 4 years of previous instruction in rational number prior to the intervention. Results showed that all of the students made large and statistically significant gains from pre to posttest.Qualitative analyses of this data revealed that the students had acquired a number of new competencies including the ability to 1) move flexibly among representations, 2) resist misleading cues, 3) order numbers by magnitude, and 4) invent their own procedures. In a further analysis, the posttest results of the experimental students were compared to the performance of normative groups on the same measure. These groups consisted of students from grades 4, 6, and 8 and from a postgraduate teacher training program. Both the Grade 4 and Grade 6 experimental students achieved better scores than the grade 8 students and equal scores to the preservice teachers. The experimental students also were less reliant on whole number strategies when solving novel problems, and made more frequent reference to proportional concepts in justifying their answers.

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Chapter 1

A Brief Introduction

Learning Difficulties in the Domain of Rational Number

1.1. Introduction

One area of mathematics that has always been a stumbling block to general mathematical competence, is the field of rational number—fractions, decimals, percent, and ratio. Introduced in most mathematics curricula as early as Grade 1, rational number is the most difficult topic that students encounter in their elementary education (Carpenter, Fennema, & Romberg, 1993). Unlike the whole number domain, which is eventually mastered and understood by all students to whom appropriate instruction is offered, (Ball, 1996; Cobb & Merkel, 1989; Fuson & Briars, 1990; Fennema, Carpenter, Franke, Levi, Jacobs, & Empson, 1996; Griffin, Case, & Siegler, 1994; Kamii, 1985; Resnick & Singer, 1993), competence in rational number is often elusive. Even among those students who can successfully perform a wide range of operations in this field, a majority show gaps in their understandings of the concepts that underlie these operations, and show very little principled knowledge of this number system as a whole (Behr, Harel, Post, & Lesh, 1993; Carpenter, Fennema, & Romberg, 1993; Keating & Crane, 1990; Kieren, 1993; Lamon, 1994; Moss, 1997; Moss & Case, 1999; Parker & Leinhardt, 1995; Resnick, 1989; Sowder, 1995). As rational number concepts are foundational to most areas in advanced mathematics such as algebra and geometry, and underpin advanced learning in chemistry, physics, and the biological sciences (Lesh, Post, & Behr, 1988; Lamon, 1999), the failure to gain a facility in this domain has always been considered problematic. However, recently these concerns have escalated. First, international comparisons of achievements in mathematics and science reveal that North American students and teachers perform well below their Asian counterparts (Ma, 1999; Stevenson & Stigler, 1992; Stigler, Fernandez, & Yoshida, 1996). Second, caveats come from contemporary analysts whose concerns are grounded in the implications of the information revolution. They warn that individuals who lack competence in mathematics will likely face severely restricted career opportunities; and societies with a citizenry lacking in mathematical literacy, will and do, find it difficult to compete in the global marketplace (Keating, 1993).

1.2. Interference of Whole Number Concepts in the Learning of Rational Number

The literature in mathematics education is replete with examples of the types of errors that students and adults make in performing rational number tasks. A major stumbling block that has been observed when students begin and continue their learning of rational number is that whole number concepts intrude on their ability to perform (Behr, Wachsmuth, Post, & Lesh, 1984; Hart, 1984, 1986; Hiebert & Behr, 1988; Lamon, 1995; Resnick,1994).

At a most basic level, whole number interference is considered to be at the root of students' inability to interpret the notation schemes for fractions and decimals because the notation system in both number systems are highly similar. Thus, for example, Hiebert and Wearne (1986) and Wearne and Hiebert (1988) report that most middle school students would assert that a number like .059 is a larger quantity than 0.2 because the number 59 is bigger than the number 2.

Similar problems exist for students in interpreting fraction symbols. Students often erroneously interpret a fraction as two independent and countable numbers (Kerslake, 1986). What is missing is the idea of how big the fraction is as a whole. Lacking this quantitative referent students' reasoning easily goes astray (Sophian & Wood, 1997). Data from the National Assessment of Educational Progress (#31) illustrate this point. For example, when asked to find the answer to 1/2 + 1/3 = ____, a majority of students in Grade 4 and 6 assert that the answer was 2/5. Moreover, 30% of the students in Grade 8 made the same error. Thus, even after a significant number of years learning fractions (approximately 7) many of these Grade 8 students still appeared to be counting the numerator and denominator as separate numbers and performing additive operations to get a sum. Furthermore Silver (1986) has shown that even college freshmen who are given remedial training still hold on to this misconception.

Another area where the intrusion of whole number concepts has been shown to be problematic is in the operations of multiplication and division of rational numbers. These operations are challenging to students of all ages (Armstrong & Bezuk, 1995). The most common misconception that is held is that the operation of multiplication will always result in a product that is bigger than the factors and that the operation of division will always result in a quotient that is smaller than the dividend and the divisor (Graeber and Tirosh, 1988). This misconception, labeled "mmbdms" (multiplication makes bigger, division makes smaller) ia based on an additive model of multiplication where this operation is conceptualized as repeated addition.

Finally, there are the well-known problems encountered when students attempt operations with ratios (Confrey, 1995; Hart, 1988; Karplus, Pulos, & Stage, 1983; Kramer, Post, & Currier, 1993; Lachance & Confrey, 1995; Lamon, 1993, 1994, 1995, 1999; Lawton, 1993; Noelting, 1980a, 1980b; Sophian & Wood, 1997). The inclination of students and adults alike to incorrectly make use of additive reasoning in situations that call for ratio and proportional reasoning is exemplified in the often replicated task "Mr. Tall and Mr. Short" (Karplus, Karplus, & Wollman, 1974). In this problem respondents are asked to determine the 4th term in a missing value task. The challenge that they are presented is to find the height of "Mr. Tall" when measured in paper clips. The problem concerns the following situation and mathematical relations. Mr. Short is 4 buttons tall or 6 paper clips tall, Mr. Tall is 6 buttons tall and x paper clips tall, i.e., 4:6 as 6 : X. This task has been implemented with diverse student populations within the US, Great Britain, and elsewhere. The results reveal a consistent tendency for students to use an "incorrect addition strategy" (Hart, 1988) in which the difference of 2, calculated by subtracting 4 from 6, is added on to 6 to get an answer of 8. It has been suggested that this additive strategy would be replaced by a correct multiplicative strategy as students get older and benefit from further instruction (Noelting, 1980a, 1980b). However, Hart's (1988) extensive longitudinal surveys reveal that 5% of students who use additive strategies when they first encounter ratio problems at 13 years of age are still using these strategies 3 years later. Thus additive approaches do not necessarily progress to proportional thinking with age (Hart, 1988, p. 208).

The forgoing list presents some concerns that have been expressed with regards to the types of errors that students make in rational number that are at least in part caused by the interference of whole number. However, many other concerns with students' learning and understanding of the rational number system have been voiced as well. These include 1) difficulty understanding the magnitude of rational number, 2) failure to understand the meanings of operations, or to apply them appropriately, and 3) inability to translate between representations. In sum, it appears that many students lack a conceptualization of the rational number system as a whole. Unfortunately, the problems listed above are not restricted to school-aged students. Many adults also lack a fluent understanding of this number system (Cramer, Post, & Behr, 1989; Post, Harel, Behr, & Lesh, 1991).

1.3. Failure of Traditional Teaching Practices

There is widespread acknowledgement that the failure of students to perform in rational number is at least partly attributable to deficits in traditional educational practice. Analyses of textbook units in rational number reveal substantial problems with content and coverage. 1) Topics are covered quickly and superficially. 2) Too much time is devoted to presenting procedures for manipulating rational numbers, and too little time to teaching their conceptual meaning (Hiebert & Wearne, 1986; Resnick, 1982; Schoenfeld, 1989). 3) Operations are taught in isolation and divorced from meaning. 4) Virtually no time is spent in relating the various representations, decimals, fractions, percents, to each other (Markovits & Sowder, 1991, 1994). 5) In contrast to units on whole number learning, textbook instruction in rational number does not support the development of students' informal knowledge. Furthermore, the style of presentation of the lessons denies students the opportunity to construct their own understandings (Armstrong & Bezuk, 1995; Ball, 1993; Hiebert & Wearne, 1986; Mack, 1990, 1993, 1995a, 1995b; Resnick, Nesher, Leonard, Magone, Omanson, & Peled, 1989; Sowder, 1995). 6) Finally, textbooks fail to connect the related topics of ratio and proportion with rational number instruction (Confrey, 1995; Moss & Case, 1999).

1.4. Lack Of Teacher Knowledge

The problems associated with traditional textbooks are compounded by the further problem that traditional teachers are ill-prepared to teach rational number (Ball, 1993; Carpenter & Lehrer, 1999; Post, Harel, Behr, & Lesh, 1991; Post, Cramer, Behr, Lesh, & Harel, 1993; Schifter & Fosnot, 1993). Like their students, these teachers have difficulty interpreting the symbol system, understanding the units and the referents, and using multiplicative or proportional strategies instead of additive ones. These findings are not really surprising, since the concept knowledge of these teachers has, for the most part, been built on the same limited textbook vision that has restricted the learning trajectories of the students in their classes (Confrey, 1994; Hiebert, 1992; Kieren, 1992; Mack, 1993).

1.5. Change in Teaching Approaches and Recommendations

In the last few years there have been a number of successful attempts to remedy the situation and many successful programs have been reported (Hiebert, Wearne, & Taber, 1991; Hiebert & Wearne, 1996; Kieren, 1994; Mack, 1990, 1993a; Markovits & Sowder, 1994; Streefland, 1991, 1993). It has been shown that, with revised conditions of instruction, children can be led to a deeper understanding of some aspects of the rational number system. Still, there is a growing concern that we may need to attack the problem in a broader and more integrated fashion. In their recommendations for curriculum reform, Post, Cramer, Behr, Lesh, and Harel (1993) suggested that "curriculum developers' attention should be directed away from the attainment of individual tasks toward the development of more global cognitive processes" (p.XX) . A similar point has recently been made by Sowder (1995) and by Markovits and Sowder (1991), who have suggested that children need to learn how to move among the various possible representations of rational number in a flexible manner. Although they retain a concern for deep conceptual understanding of the individual components of rational number, contemporary analysts are clearly urging us to create curricula that will help children develop more global conceptions of the rational number system as a whole and of the way its various components fit together.

This general goal is the overriding objective of this thesis. In the chapters that follow I outline a research program in which I have designed and assessed a curriculum which fosters such global understandings of the rational number system. The curriculum is based on a Case's theory of intellectual development (Case, 1985, 1992; Case & Okamoto, 1996) and on a developmental model for rational number learning that I have developed with him.

1.6. Outline of the Present Thesis

In the next chapter I review the research on the teaching and learning of rational number carried out by the mathematics education research community. In Chapter 3, I first present the general theoretical framework and the specific model that was proposed for the development of rational number. Then a detail of the curriculum that was designed based on the model is presented. Chapters 4 and 5 contain methods and results of two empirical studies where the rational number curriculum was implemented. In Chapter 6, I present a comparative analysis of these two studies and a third study that was reported earlier. Finally, in Chapter 7, I present my discussion of these findings and their potential contribution to the literature in rational number and multiplicative structures.

Chapter 2

Rational Number Research: An Overview

Old Paradigms and New Directions

2.1. What Is Rational Number?

A definition of rational number belies the complexity of the topic. Simply stated, a rational number is a number that can be expressed as a quotient or a ratio of two integers a and b, that is, a/b where b does not equal zero. However, despite its apparent simplicity, rooted in this elegant definition are particularly challenging axioms, properties, concepts, and constructs. Encompassing the representations of decimals, percents, fractions, and ratios, rational number is at once a system of numbers and a system of operations.

The history of rational number pedagogical research has been primarily concerned with the unraveling of these complexities. Researchers have conducted semantic, syntactic, and epistemological analyses of the rational number system, and the findings of this research have directly influenced the applied work in the field. As Carpenter et al. (1993) note, "Research on teaching, learning, curriculum and assessment of rational number concepts depends on the conception that the researchers hold of the nature of rational number" (p. 2). Thus, to understand the direction instructional reform has taken, it is important to examine the conceptions of rational number that predominate in the field and directly influence the research on teaching and learning.

2.2. The Subconstructs Theory of Rational Number

The cognitively-based research in rational number has looked at the different characteristics of this number system Freudenthal, 1983; Kieren, 1976, 1988; Ohlsson, 1988, 1987). Researchers have engaged in discussion concerning 1) the mathematical constructs that comprise the number system, 2) the different applications or characteristics of this number system, as well as 3) what a person can do when they know rational number (Kieren, 1993, 1995). These questions have led to the identification and analyses of special aspects of rational numbers which have come to be known as "subconstructs." Although there are some variations in the way these subconstructs are delineated, it is generally agreed that these interpretations include rational number as operator, rational number as ratio, rational number as quotient, rational number as measure and, with the exception of Kieren, rational number as part/whole.

The operator (or 'stretcher' or 'shrinker') subconstruct indicates the way that the number acts as a multiplicative transformer and operates on something else, as in a function that is applied to some number (e.g., 1/2 of 8). Rational number as ratio defines the situation in which two quantities are related to one another multiplicatively (e.g., there are 3 girls for every 4 boys). The measure subconstruct, most frequently accompanied by a number line or a picture of a measuring device identifies a situation in which the fraction 1/b is used repeatedly to determine a distance (e.g., 3/4 of an inch = 1/4, 1/4, 1/4). Rational number has the meaning of quotient, so that 3/4 is interpreted as 3 divided by 4 (for example, 3/4 indicates that 3 cookies are shared by 4 people). And finally there is what is known as the part/whole subconstruct which defines the act of partitioning an object into parts, and addresses how much there is of a quantity relative to a specified unit of that quantity [e.g., For the fraction 3/4 there are two part whole interpretations (a) 3 (1/4 units) and (b) 1(3/4 unit)]. Although all of the subconstructs represent separate interpretations, the operator and the ratio subconstructs refer to characteristics of rational number that are exclusively multiplicative, whereas the measure, quotient, and part/whole subconstructs incorporate ideas from additive structures as well as multiplicative ones.

2. 3. The Rational Number Project

The exploration of the subconstructs—both the identification and interpretation of these separate applications—has profoundly affected the style and content of rational number research. Even more importantly, the subconstruct theory has shaped the general paradigm for research and teaching studies in rational number. The subconstruct analysis, while initially conducted by Kieren (1976), became the focus of the work of a group of mathematics education scholars whose cooperative program of research is known as the Rational Number Project (RNP) (see, for example, Behr, Harel, Post, & Lesh, 1992, 1993; Bright, Behr, Post, & Wachsmuth, 1988; Harel, Post, & Lesh, 1993).

The extensive analyses of the subconstructs led the RNP researchers to conclude that each construct represents a distinct conception of rational number. Thus a comprehensive knowledge of this number system involves an ability to perform a variety of problems in all of these separate applications. In keeping with their analysis then, the subconstructs have become a focal point for their work on teaching and learning of this number system. And, although the overriding goal for rational number learning is that knowledge of the subconstructs be connected to form a unified scheme (Carpenter et al., 1993), nevertheless, the subconstructs have evolved as individual teaching strands for learning rational number.

2.3.1. The influence of the RNP

The legacy of the work of the RNP and the influence of that work cannot be overestimated. First is the contribution of fine-grained analyses of the subconstructs and the importance of their place in the field of rational number (Behr et al., 1992, 1993; Curcio & Bezuk, 1994; Harel, Post, & Lesh, 1993; Lesh & Landau, 1983; National Council of Teachers of Mathematics, 1989; Post, Cramer, Behr, Lesh, & Harel, 1993: Post, Harel, Behr, & Lesh, 1991). Second is the application of these subconstructs to teaching, learning and assessment. The strength of the influence of the RNP is evidenced by the longevity of its research program (which is still ongoing and has been funded consistently since 1979), by the significant number of publications that they have authored (approximately 85) and by the many other researchers who have modelled their own work on the RNP's analyses (e.g., Lamon, Harel, Mack, etc.). Over the years the research program of the RNP has included experimental studies (Cramer, Post, & Behr, 1989), surveys (Heller, Post, Behr, & Lesh, 1990), and teaching experiments (Behr & Post, 1992; Behr, Wachsmuth, Post & Lesh, 1984). The RNP has also developed curricular units and materials (Behr, Harel, Post, & Lesh, 1993), and made recommendations to influential organizations such as the National Council of Teachers of Mathematics (NCTM). Recently this team has also become involved in preservice teacher training and professional development.

2.3.2. Teaching Experiments

The RNP themselves have reported many successes in their teaching experiments based on the subconstructs, and they have also reported some shortcomings (Behr, Post, & Wachsmuth, 1984). While students seem to make gains in their understanding of, and competency in, isolated areas of this number system (i.e., understanding of the subconstructs and in the separate representations of decimals and fractions), they are not able to integrate these various individual proficiencies into a systematic conceptualization of this number system as a whole simply by learning the various meanings of the subconstructs in isolation. In their recent recommendations for curriculum development in rational number, the RNP observed that more instructional attention needs to be placed on students' integrated use of, and access to, the totality of the rational number domain (Post, Cramer, Behr, Lesh, & Harel, 1993). They concluded that the "curriculum developers'" attention should be directed away from the attainment of individual tasks toward the development of more global "cognitive processes" (p. 343).

2.3.3. Problems with the Research Paradigm of the RNP

While it is true that the analyses of the subconstructs give us a glimpse into the complexities of the rational number system, and serve as useful benchmarks for assessment, I concur with the critics (i.e., Carpenter et al., 1993; Carraher, 1996; Streefland, 1993) who all argue that there are some fundamental drawbacks to the RNP's approach to rational number learning, particularly at the introductory phase—drawbacks that I argue work against the possibilities of an integrated understanding of this domain. Further I argue that the RNP's approach, rather than promoting students' multiplicative intuitions, actually serves to reinforce students' whole number reasoning. These drawbacks can be subsumed under two separate but interrelated emphases. First, as already mentioned, is the focus on the individual subconstructs in the RNP's design for curriculum reform. Second, as suggested by Streefland (1993), is the hierarchical placement of the additive part/whole subconstruct as the organizing or core concept in the domain. In the sections that follow, I will elaborate these two points.

2.3.3.1. Problems with the Subconstructs as Learning Goals

The RNP's subconstruct theory can perhaps be better understood if we look at the mandates of the mathematics research community when the RNP first began its investigations 20 years ago. At that time, research on teaching emphasized rigorous and in-depth learning of individual mathematics topics;

they did not particularly focus on the promotion of the understanding of the interconnectedness of broad domains. Thus, the RNP's focus on and analyses of the individual subconstructs were very much in keeping with the prevailing methodologies and were respectful of the notion of discrete concepts being taught. However, the RNP's analytic approach is at odds with the goal of promoting an integrated understanding of the domain as a whole particularly of the interconnectedness among the various representations of the number system.

2.3.3.2. Problems with the Favouring of the Part/Whole Subconstruct

The second problematic focus in the RNP's approach is the prioritizing of the part/whole subconstruct. Streefland (1993), a researcher whose work has focused on the development of students' mathematical understandings particularly in fractions, has observed that partitioning and the part/whole subconstruct "... [are] the theoretical foundation and at the forefront of their [the RNP's] work." He goes on to observe that Behr et al. view part/whole construct as fundamental to all later interpretations of rational number. In the RNP's approach, although ratio and proportion concepts are embedded in partitioning work, ratio and proportion are proposed as subsequent to the part/whole subconstruct. This favouring of part/whole as a grounding for rational number has recently been questioned and there is growing consensus that ratio and multiplicative structures may be more fundamental concepts to building rational number concepts (Confrey, 1994, 1995; Confrey & Smith, 1995; Lachance & Confrey, 1995; Kieren, 1994a, 1995; Streefland, 1991, 1993; Vergnaud, 1983, 1988, 1996). In the following sections this position will be elaborated.

2.4. Dual Nature of the Numbers in the Rational Number System

The distinction that Streefland has pointed out, regarding the primary focus of instruction in rational number as being rooted in either additive representations (e.g., part/whole) on the one hand, or, multiplicative representations such as ratio/ proportion on the other, is a distinction that is inherent in the number system itself. Rational numbers are (at the same time) both numbers and ratios. For example, 1/4 can either be interpreted as a number as in the equation $1/8 + 1/8 = 1/4$ or, as a ratio for example if we say 1/4 of the selling price. This latter interpretation describes a relation between two numbers (a ratio) or a relation between two variables (a function). These distinctions will be further elaborated in the present chapter and throughout this thesis. For the present purposes it is important to note that there is a certain amount of disagreement amongst mathematicians as to whether rational numbers, particularly percents, are primarily numbers or operators.

Both Carraher (1996) and Davis (1988) have independently remarked on the disagreement: Both investigators have suggested that while this disagreement may be of little consequence in the field of mathematics, the prioritizing of these different interpretations may have serious implications for mathematics educators. To quote Davis, "While this distinction might seem merely like an academic quibble, our disagreement on these matters may contribute to our lack of success in teaching the concepts to children" (p. 299). He therefore admonishes curriculum developers and researchers to consider the impact of their choices for teaching and learning. Carraher is adamant on this point as well, and asserts that "the view that a fraction is simply a number [an additive concept] may make sense in discussions among mathematicians but it is pedagogically naive as well as historically and psychologically inaccurate" (p. 242).

Since students' learning in rational number is influenced by the representations that they use, particularly the initial ones (Kerslake, 1984; Silver, 1986; Sowder, 1995), it is clear that the favouring or privileging of additive part/whole structures over the multiplicative structures of ratio and proportion as an introduction or underpinning to the learning of rational number must be further considered.

2.5. The Part/Whole Interpretation of Rational Number: Additive Structures

It is useful to consider the NCTM's definition of the part/whole subconstruct. The following quote is the verbatim explanation that is provided to teachers in the NCTM curriculum document on rational number (Curcio & Bezuk, 1994). It is the first aspect of rational number to which teachers and their students are introduced.

"In the part/whole meaning a unit in the form of a continuous shape (e.g., a cake) or a discrete set (e.g., a number of cookies) is introduced. The unit is partitioned into equal sized parts or non-congruent shapes with equal area. For example, a [rectangular] cake is to be cut into eight equal-sized pieces. What will be the size of each piece? The cake will be partitioned into eight equal pieces, either congruent or pieces equal in area, each piece being 1/8 of the whole cake" (p. 2).

My reading of this explication of the "part whole subconstruct," is that the emphasis is placed on entirely additive notions; first, of the individual pieces of cake that have been cut (the parts) and, then, on the independent, countable units called "1/8's" that result from the cutting of the cake, and which yield a number 1/8. What is entirely absent conceptually in this definition is the idea that these 1/8ths are related to a whole, by a ratio 1:8. For the mathematician this may be entailed by the fact that the cutting was done in such a fashion as to make all eight pieces equal. But for the child it is not. One simply has a new kind of physical object—one generated by partition—and a new way of writing the result (total pieces on bottom, number "counted up" and taken away, and used for some purpose on top). Thus, there is not even an allusion to the multiplicative underpinnings of rational number.

This additive part/whole subconstruct is the most basic interpretation that can be given to rational number. This interpretation leads to classroom activities that are easily accessible to young students e.g., cutting pies, (partitioning circular regions), or dealing out, as in the sharing of candies. These exercises, drawn from experiences that children have engaged in outside of school, access children's many intuitions for counting and whole number. Marshall (1993) in her analysis of the "feature" and "constraint" knowledge associated with the subconstructs, asserts that "this knowledge for part/whole, is built directly on part/whole knowledge from another domain—that of whole numbers" (p. 272).

The RNP's privileging of this interpretation at the early stages of learning rational number can probably be understood from four separate vantage points: 1) as choosing the most basic form (a/b) of rational number; 2) as accessing children's previous school-learned mathematics; 3) as tapping into experiences in the world that are familiar to children from a very young age, and perhaps, 4) as permitting the teaching of rational number prior to the stage in cognitive development where abstract entities such as ratios can be conceptualized.

While there are benefits to this introduction, it has been noted that a continued reliance on this part-whole interpretation of children's informal partitioning schemes can place limitations on the robustness of their understanding of rational number (Confrey, 1995; Kerslake, 1986; Ohlsson, 1988). Sowder explains this position as follows: "Children may treat the individual parts that result from a partition as discrete objects—e.g., six pieces that a pizza is cut into, are simply six individual pieces. The children do not immediately recognize the consequences of the fact that each piece also represents one sixth of the whole pizza. Or in Kieren's words they still think in terms of "how many" instead of more appropriately, "how much?"

2.6. Multiplicative Reasoning and the Intrusion of Whole Number Concepts

As I have pointed out in Chapter 1, it has been acknowledged that a common problem, when students begin their learning of the various representations of rational number, is that their whole number conceptualizations interfere with or seem to dominate their thinking, thus causing them to confuse additive and multiplicative situations. Hiebert (1992) asserts that conceptualizations that become rote for students from their previous whole number learning interfere with their introduction to this number system. Given that students have spent many years mastering the whole number system prior to their introduction to rational number, some would argue that this situation is not surprising and can be interpreted as a temporary intrusion of ideas from one system onto another. However, while we know that some of the problems in learning rational number are eradicated as students progress through their school years, we also know that many of the problems that students encounter persist into adult life and appear to be robust and long-lasting (Armstrong & Bezuk, 1995; Ma, 1999; Parker & Leinhardt, 1995; Post et al., 1991; Silver, 1986; Thipkong & Davis, 1991). In fact, many adults do not master proportional reasoning (Lamon, 1999); a concept that we know to be important for a solid understanding of the rational numbers.

2.7. The Distinctiveness of Whole Numbers and Rational Numbers: Transition or Departure

While the notion of whole numbers interference is widely acknowledged, there are differences of opinion as to the nature of that interference, differences which in turn reflect curriculum decisions. While it is true that rational numbers share a language with whole numbers and use concepts from whole number and thus may be thought of an extension of whole numbers (Steffe, 1994), there is a growing understanding that Rational Number is a distinctive field that is rooted in very separate intuitions and precursor knowledge. Rational numbers are tightly interwoven with ratio and proportion concepts and thus there are fundamental differences that must be noted.

Harel and Confrey (1994) have analyzed some of the new conceptualizations that students must contend with in learning rational number: First of all, is the fundamental change that students will encounter in the nature of the unit. Whereas in whole number the unit is always explicit, in rational number, the unit is the context that gives meaning to that represented quantity, but often it is implicit. (e.g., "1/2 of 3/4" the notion of the unit "1" is not explicit; however, it is conceptually embedded in the construct). As well, the learner has to readjust to a new/different notion of quantity: whole number is based on the concept of discrete quantity, whereas rational numbers are "dense"—between any two rationals one can find another and hence an infinity of other numbers.

Finally, I argue that for the naive learner whole number and rational number function in ways that can be considered in Chi's terminology as "ontologically distinct" (Chi, 1992). Until students encounter rational number operations in school, although it is true that they have intuitions about proportions, their notion and experience of number is one that comes directly from whole number understandings. For these learners, numbers are symbols that refer to a discrete entity with a constant value. The number 3 has a particular and consistent meaning. In daily life, people use numbers as either "count words" which are subject to a host of systemic rules (Resnick, 1986), or as adjectives where they refer to objects. Numbers are related to the substance itself. If you have 3 objects they are inherently three as in a triad, as in the sides of a triangle, etc. Threeness is fixed and immutable and although it may be defined in relationship to other things like 2 and 1, as a cardinal property, 3 is fixed, regardless of the nature of the elements or the relation of the elements. Viewed in this way, one could say that people's standard use of number is grounded in discrete additive theory-like notions, and that the theory belongs to a "substance-based" ontological category (Chi, 1992).

But what of the rational numbers? First consider the numbers in fractions and ratios. These numbers behave entirely differently than whole numbers. The quantitative meaning of these numbers is neither fixed or immutable. These numbers do not behave as "substance-based" entities. Rather their meaning is based on interactions and relations. In Chi's terminology they can be called "process-based." Stated differently, in the rational number system, numbers are not associated with, and do not function as independent entities that have a constant value. The numbers in fractions, ratios, and proportions must be reinterpreted as dependent variables in a ratio relationship. The process by which the number is created must be considered and hence the ontological category is different. (Here numbers exist as entities in a process that is driven by an interaction). The quantity defined by the number "3" changes based on an immediate history and associations. (The numbers in the decimal system exist somewhere in and in-between category and can be interpreted in a more "substance-based" way, however their ratio underpinnings must be conceptualized in order to work with these representations with understanding).

Psychologically and ontologically then, these two systems behave very differently to naive thinkers. Students must move from understanding and using number as being associated with substances to being associated with a relationship of a process, one that is no longer at its core associated with the tightly held additive understandings but one that is now part of a multiplicative world.

Thus it can be seen that a reformulation of number is necessary to progress from whole number to rational number understanding as the two systems do not flow one into the other.

Kieren (1993, p. 56) also has suggested the distinctiveness of these two number systems. He points out that there are "fundamental new axioms and properties and fundamentally distinct actions for the knower." Thus knowledge of rational number is constituted in a fundamentally different way from that of whole number.

2.8. The Multiplicative Conceptual Field

Vergnaud, a French mathematician and theorist, provides insights worthy of consideration in looking at the distinctiveness of the rational numbers from whole numbers. He asserts that the rational number system may be considered in a broader context which he sees as a network of different but interconnected multiplicative concepts which he has named the Multiplicative Conceptual Field (MCF) (Vergnaud, 1983, 1988, 1994). Vergnaud postulates that along with multiplication, division, and linear functions, rational number encompasses all situations that can be analyzed as simple or multiple proportions.

Confrey's theoretical extension to Vergnaud's proposition of the MCF suggests that multiplicative action or reasoning occurs in a way that is independent of additive ideas. Confrey distinguishes between multiplicative and additive schema in the following way, "Whereas in additive operations, counting starts at 0 and the successor action is adding one with the unit being one, in a multiplicative splitting world, counting starts at 1 and the successor action is splitting by n" (Kieren, 1994, p. 395). She defines splitting as an action of creating simultaneous multiple versions of an original. The focus on splitting, based on the action of one-to-many, includes actions such as sharing, dividing symmetrically, growing, magnifying, and folding. By contrast, in additive situations the change is determined through identifying a unit and then counting consecutive instances of that unit. Further, the precursor actions are affixing, joining, annexing, and removing. Finally, Confrey (1994) asserts that the splitting schema is a precursor to an adequate concept of ratio and proportion as it provides a non-counting basis for multiplicative structures.

2.9. Working Hypothesis for this Thesis

This perspective raises the question as to how should we foster these multiplicative schemes and operations? Traditionally rational number concepts are developed through fractions and part/whole constructs. However, we have seen that this type of introduction fails to promote a conceptualization of the number system as a whole. Thus we need to find ways of introducing more complete conceptions. I concur with Kieren who points out (1994, p.89) that rather than relying on children's well-developed additive intuitions in our introduction to rational number, we must find children's intuitions and schemes that go beyond those that support counting. Thus, the goal is to develop a curriculum that can immerse the students in situations that are both multiplicative and are based on their current understandings and intuitions.

If success in rational number means that students must acquire new multiplicatively based intuitions, then we must provide a learning context where an immersion in multiplicative contexts is the foundation. Whole number understandings are carefully built over a number of years; now we must consider how rational number understanding develops and is fostered.

In the next chapter, I describe the curriculum for rational number that I have designed based on a developmental model for rational number that is underpinned by or grounded in multiplicative understandings. I will also describe the theoretical background of central conceptual structures that served as the framework for both the developmental model that was hypothesized as well as the curriculum that was developed. As will be seen, this rational number curriculum, like the other mathematics curricula that Case and his associates have developed (see Griffin & Case, 1998; Kalchman & Case, 1998) was designed with the goals of 1) fostering a deep understanding of the particular number system, and 2) promoting the kind of flexibility with the number system that has been characterized by Bereiter (1998) , Bereiter and Scardamalia (1997), Case (1998), Greeno (1992), and Sowder (1992) as "Number Sense."

Chapter 3

A Psychological Model and an Experimental Curriculum Designed to Foster Deep Mathematical Understanding of the Rational Number System

3.1. Case Number Sense Curricula

In the last several years Case and his associates have been working on a number of projects for the development of mathematical understanding in various domains. Case et al.'s early work on whole number development has resulted in two programs for young children; Rightstart and Number Worlds (Griffin & Case, 1996, 1997; Griffin, Case, & Seigler, 1994). More recently, Case and Kalchman have been involved in the design of a curriculum to foster a deep understanding for the difficult topic of mathematical functions (Kalchman & Case, 1998).

Not only have these curricula proven to be successful in promoting principled understanding of whole numbers and functions respectively, but they also appear to have promoted the types of understandings and competencies that have been characterized by Greeno (1991), Bereiter (1998), Bereiter and Scardamalia (1997), Sowder (1992), and Case (1998) as "number sense." These include: (1) Fluency in estimating and judging number magnitude; (2) ability to recognize unreasonable results; (3) flexibility with numbers when mentally computing; (4) ability to compose and decompose numbers; (5) ability to invent procedures for calculating, and to be flexible and creative in solving problems involving numbers; (6) ability to move among different representations of number and to use the most appropriate representation for a given situation, and (7) ability to represent the same number in multiple ways, depending on the context and purpose of this representation.

The whole number program and the functions program and the rational number program that will be reported in this thesis were designed for students at very different levels of their school careers (Kindergarten, Grade 1 and Grade 2 for Rightstart and Number Worlds, Grade 4-6 for rational number and middle and high school for functions): Nonetheless, all of these programs share many basic features. The core of their similarity is that they are all based on Case's theory of central conceptual structures.

3.2. Theoretical Background

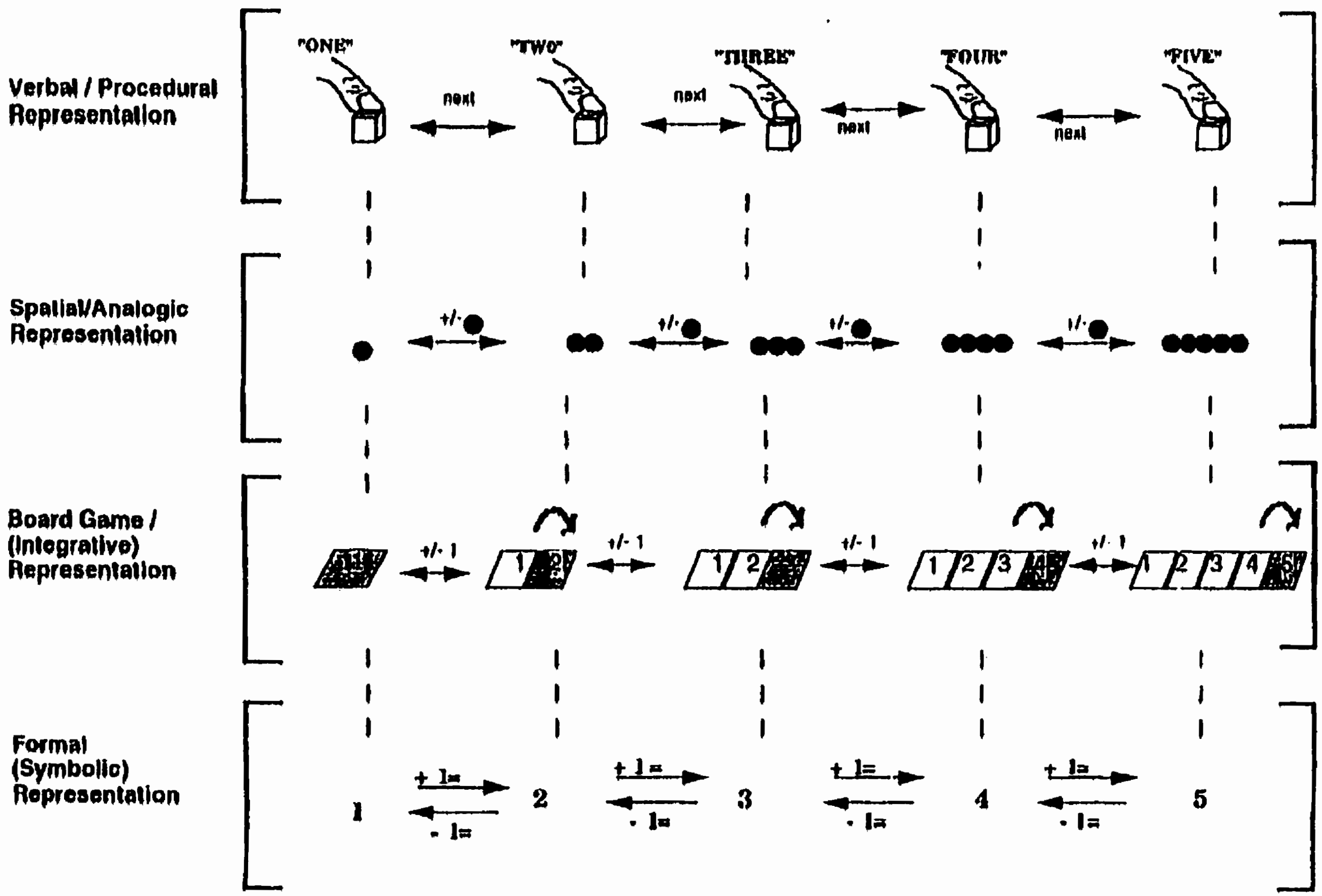
Case has proposed that children's number sense depends on the presence of powerful organizing schemata which he refers to as central conceptual structures. These structures may be defined as complex networks of semantic nodes, relations, and operators, which (1) represent the core content in a domain of knowledge, (2) help children to think about the problems that the domain presents, and (3) serve as a tool for the acquisition of higher order insights in the domain mastered (Case, 1998; Case & Okamoto, 1996; Case, 1998; Griffin & Case, 1996; Griffin, Case, & Siegler, 1994).

Case has proposed that central conceptual structures are assembled by the integration of two intuitive or "primitive" schemas. Furthermore, he has postulated that these two initial or precursor schemata differ from each other; One of these schemas is primarily spatial, analogic, and non-sequential and the other is primarily digital, verbal, and sequential (Case & Okamoto, 1996; Griffin & Case, 1996; Kalchman & Case, 1997; Moss & Case, 1999). Developmentally the central conceptual structures are built up in a series of phases. In the first phase of children's learning, these two core schemas are consolidated. Next, these two early schemas become more complex, while at the same time they are mapped on to each other. The result is that the student's understanding of the domain is transformed and a new psychological unit is constituted. It is this new unit that then constitutes the core central conceptual structure on which most of children's subsequent learning, in a numerical domain, will depend. During the next phases, students slowly begin to discriminate amongst the different contexts in which the new elements can be applied. Thus slightly different representations of the core structure are created. Finally it is hypothesized that students come to understand the domain in full and the differentiated representations are firmly embedded in the newly formed structure. In order to give a more detailed picture I will briefly describe this developmental sequence for whole number arithmetic.

3.3. Development Of Whole Number Understanding

To date the structure that has been most extensively analyzed is the CCS for the whole number system. According to the model proposed by Case and his colleagues (Griffin & Case, 1988; Okamoto & Case, 1996) the two primitive schemas on which an understanding of the whole number system depends are the schema for verbal counting (digital, sequential) on the one hand, and the schema for global quantity comparison (spatial, analogic) on the other. It has been shown that although young children have strong intuitions for both counting and global quantity comparisons, these two schemas initially develop separately. However as children make the transition to a higher level of thought, at about the age of 6, they gradually coordinate these two schemas resulting in the formation of the mental counting line: a structure which permits children to solve a wide variety of addition and subtraction problems with confidence, by moving forward and backward along the verbal counting sequence. Once children understand how mental counting works, they gradually form representations of multiple numberlines, such as those for counting by 2s, 5s, 10s, and 100s. The construction of these representations gives new meaning to problems such as double digit addition and subtraction, which can now be understood as involving component problems which require thinking in terms of different numberlines. Finally, as children become more familiar with these problems, they gradually develop a generalized understanding of the entire whole number system and the base-ten system on which it rests. Addition or subtraction with re-grouping, estimation problems using large numbers, and mental math problems involving compensation all are understood at a higher level, as this understanding gradually takes shape. The CCS for whole numbers is presented in Figure 3.1.

Central Conceptual Structure for Whole Number



This analysis for the development of whole number understanding has led to the creation of a highly successful curriculum called Rightstart which is designed to facilitate the underlying developmental integrations.

3.3.1. Rational Number Development

What about the central conceptual structure for rational number? Can a developmental sequence and a central conceptual structure be identified that is parallel to that for whole number?

In Chapter 1, I presented examples of the kinds of problems that students encounter in learning rational number. I also concurred with researchers who suggest that students' failure in rational number is primarily based on the resilience of their whole number schemas and the interference caused by these schemas. I then went on, in Chapter 2, to present a series of observations about the rational number system in an attempt to illustrate its distinctiveness from the whole number system. I stated that this distinctiveness was evident in the multiplicative foundation of this number system and the relational nature of both number and quantity. These arguments then lead to the suggestion that a solid understanding of the rational number system is founded upon schemas and intuitions separate from those of whole number. Again, this view is supported in the psychological literature on the development of mathematical understanding (Carraher, 1996; Confrey, 1994, 1995; Hatano, 1996; ; Kieren, 1993; Hunting, Davis, & Pearn, 1997; Nunes & Bryant, 1996; Schwartz, 1988; Vergnaud, 1988).

3.4. Schema for Rational Number

For the development of rational number, Case and I have proposed that the two primitive psychological units are 1) a global structure for proportional evaluation (Nunes & Bryant, 1996; Resnick & Singer, 1993; Spinillo & Bryant, 1991) and (b) a numerical structure for "splitting" or "doubling" (Case, 1985; Confrey, 1994; Kieren, 1992), both of which appear to be in place by about age 9 to 10 years. Coordination of these two structures at the age of 11 to 12 yields the first semi-abstract understanding of relative proportion and simple fractions (especially 1/2 and 1/4). As children grow older and receive further instruction, they learn about different forms of splits and the relationships among different sorts of fractions. They also learn about the relationship between fractional and decimal notation. Eventually (though often not until they have reached the end of hight school) they construct a generalized understanding of the entire rational number system.

3.6. Implications of the Model for Instruction: Instructional Program Design

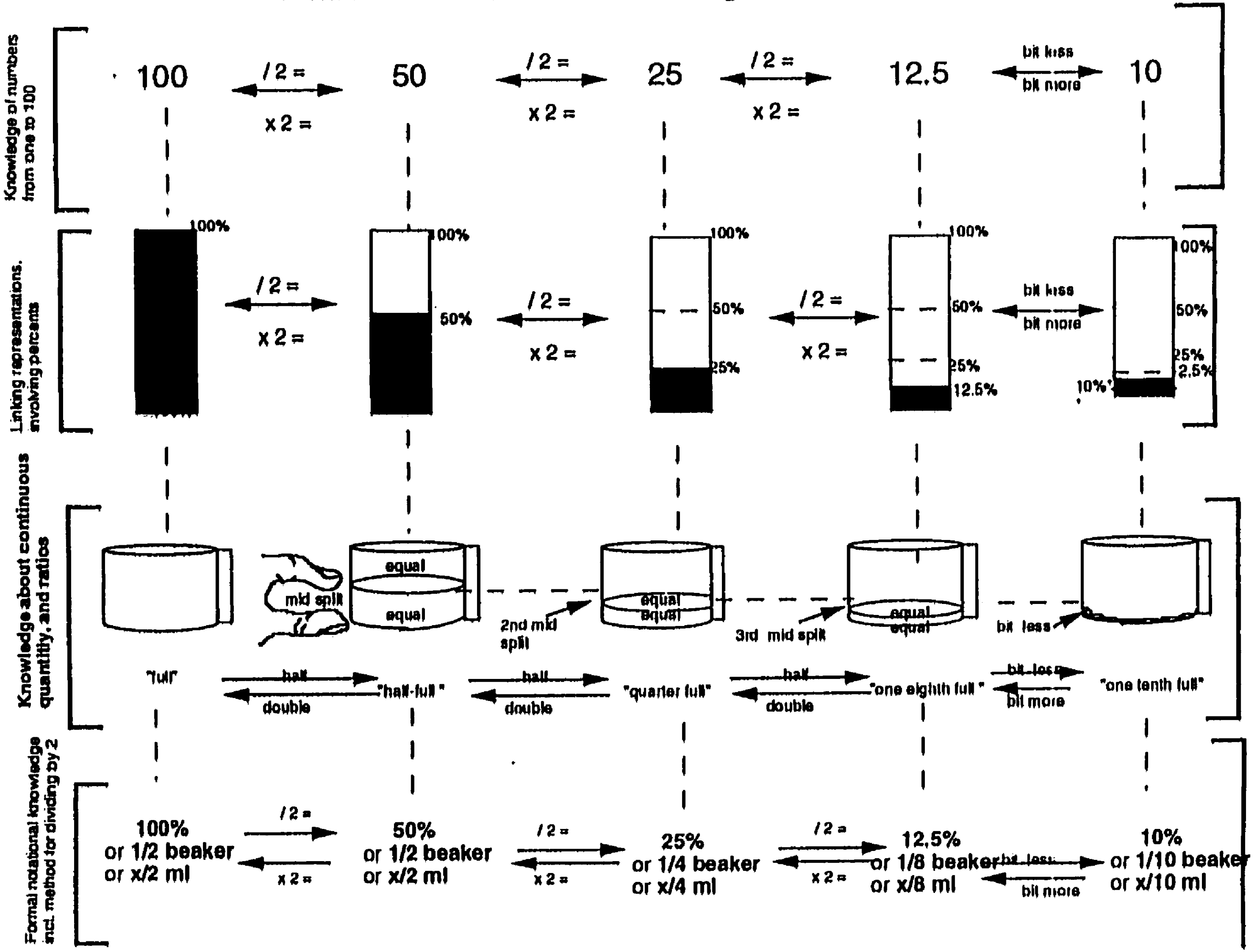
The goal for the rational number instructional program follows directly from this analysis and is consistent with the goal of the Rightstart program for whole number learning. Thus, we designed this program to provide students with activities that allow them to refine and extend their existing understandings in a natural fashion, and to use the resulting cognitive structure as a basis for conceptualizing the overall structure of the rational number system.

In keeping with the above goals, we presented children with a sequence of tasks that have the potential to maximize the connection between their original, intuitive understanding of proportion, their knowledge of numbers from 1 to 100, and their numerical procedures for 'splitting' numbers. In order to maximize the potential for achieving the forgoing connections, we introduced this number system in a measurement context and we chose percents as the first rational number to introduce. The curriculum was thus structured so that the other forms of representation, fractions, and decimals, were introduced later and were grounded in students' growing knowledge of percent. The general sequence of the program follows below.

The initial props that the children use are large drainage pipes of assorted lengths each partially covered by a moveable venting tube that can be raised and lowered over the pipes. Cylindrical beakers containing various amounts of water are also used in the introduction. Both of these props allow for direct evaluation of fullness relative to the whole. Both of these props also provide a "side view" that is easily represented by the students on paper using a narrow rectangular diagram showing the proportion of the total object that was covered. Accordingly, our first exercises are ones that asked children to think about relative height in terms of "fullness" and to describe the relationship using the language of percent. As the lessons progress, the children are encouraged to coordinate their estimates and intuitive understandings of percents in this context with their strategies for manipulating the numbers from 1 to 100. The two strategies that students spontaneously use and that we encourage are numerical halving (100, 50, 25, etc.), which corresponds to a sequence of visual/motor splits, and composition (e.g., 100 = 75 + 25), which corresponds to visual/motor addition of the results. Other percent activities that are used, for example model building and numberline games continue to use the measurement metaphor for representing percent. Once children understand how percent values can be computed numerically, in a fashion that corresponds directly to intuitively based visual/motor operations, the next step is to introduce them to two-place decimals. Again this introduction occurs in a measurement context. For this introduction we explain to the students that a two-place decimal number indicates the percent of the way between two adjacent whole number distances that an intermediate point lies (e.g., 5.25 is a distance that is 25% of the way between 5 and 6). This introduction to decimal is then gradually expanded to include multi-place decimals, using a transitional "double decimal" notation that the children spontaneously invented (Moss & Case, 1999). For example, they initially represented the number that lies 25% of the way between 5.25 and 5.26 as 5.25.25. Finally, the students are presented with exercises in which fractions, decimals, and percents are to be used interchangeably. Fraction teaching is never done independently of percents and decimals; rather, fractions are offered as an alternative form for representing these latter representations.

The psychological structure that we intend children to construct as a result of the above sequence is illustrated in Figure 3.2. The top line of the figure illustrates the perceptually based sequence of ratios that we hoped children would learn to recognize and to order at the outset of the program. The (left to right) arrows connecting the icons in this row indicate the operation by which we presumed children would move from one element to the next in the sequence. This operation, which might best be termed visual-motor halving, is most easily executed by putting one's forefinger beside an object—then moving it up and down until one finds the point at which the top and bottom halves of the object are symmetrical.

Alternative Rational Number Line Using Percents



In the second line of the figure appear the corresponding representations that we hope children will develop for benchmark percent values and the numerical operations that connect them. Once again, we presume that children will start with 100% and then calculate half of this value for each successive visual-motor split. We also presume that children can learn to compose and decompose percents that are calculated in this fashion (e.g., to determine the size of 75% by finding the sizes of 50% and 25% and then combining them).

Finally, the bottom row of the figure is meant to represent the corresponding set of measurement techniques and formal arithmetic procedures that we hope children will learn to use when the goal is to express a ratio in some standard set of units such as millilitres. For example, if one knows that the total volume a beaker can hold is 120 ml, one can determine what 75% of that volume must be by first computing half of 120 (60), then computing half of the resulting total (30), then adding these two values.

Case asserts that the psychological structure represented in Figure 3.2 contains a rich network of icons, symbols, and procedures that children can access and apply in a flexible fashion, to insure that their qualitative procedures for assessing and transforming continuous quantity, as well as their more formal, arithmetic procedures, will remain closely integrated. In this sense, the structure is directly parallel to the central conceptual structure for whole number that has been analyzed by Griffin and Case (1996).

We reasoned that, once children possess a ratio-measurement structure such as that diagrammed in Figure 3.2, they should be able to use this structure as a starting point for learning about decimals and fractions. The sequence of instruction that we employ in order to foster this development (percents, decimals, then fractions) is a reversal of the normal order for introducing these different representations and a significant departure from current "best practice." We believe that this is the optimal sequence for introducing rational number for a variety of reasons:

1) By the age of 10 or 11, children have well-developed qualitative intuitions regarding proportions (Case, 1985; Lamon, 1995, 1999; Noelting, 1980a; Resnick & Singer, 1993; Streefland, 1991); 2) they also have well-developed intuitions about the numbers from 1 to 100 (Okamoto & Case, 1996). By beginning with percents, we allow them to bring these two sets of intuitions together in a natural unidimensional fashion.

2) By beginning with percents rather than fractions or decimals, we postpone the problem of having to compare or manipulate ratios with different denominators, thus allowing children to concentrate on developing their own procedures for calculation, comparison, and composition and decomposition rather than requiring them to struggle to master a complex set of algorithms or procedures that might seem foreign to them.

3) Every percent value has a corresponding fractional or decimal equivalent that is easy to determine. The converse, however, is not true. Simple fractions such as 1/3 and 1/7 have no easily calculated equivalent as percents or decimals. By beginning
with percents, we allow children to make their first conversions among the different systems in a direct and intuitive fashion and thus to develop a better general understanding of how the three systems are related.

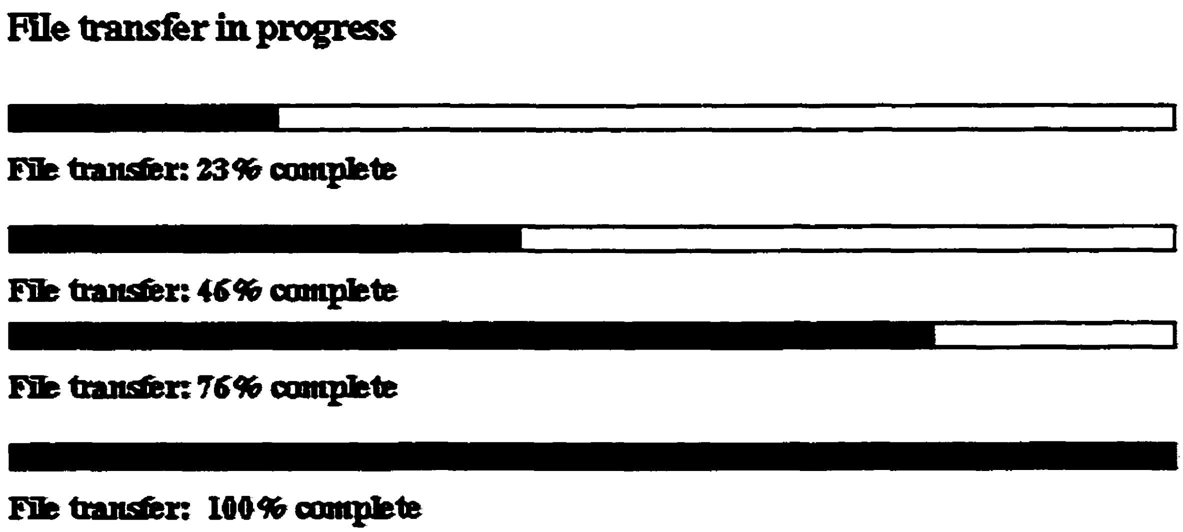


Figure 3.3—The "number ribbon" used on the Macintosh computer when a file is being transferred.

4) By beginning with percents, we were able to let children use a form of visual representation with which they were already familiar, namely, the "number ribbon" that is used on the Macintosh computer when a file is being transferred (See Figure 3.3). This representation further contributed to building a solid connection between children's intuitions about proportions and their intuitions about numbers.

5) Finally, although this was not central to our decision, it is worthwhile to mention that the children appeared already to know a good deal about percents from everyday experiences (Parker & Leinhardt, 1995). Before we began the instruction, we asked the children if they had every heard percent terminology used in their homes or daily lives. Not only were they able to volunteer a number of different contexts in which percents appeared (their siblings' school marks, price reductions in stores having sales, and tax on restaurant bills were the ones most frequently mentioned), they were able to indicate a good qualitative understanding of what different numerical values "meant," for example, that 100% meant "everything," 99% meant "almost everything," 50% meant "exactly half" and 1% meant "almost nothing." By beginning with percents rather than fractions or decimals, we were able to capitalize on children's pre-existing knowledge regarding the meanings of these numbers and the contexts in which they are important (see Lembke & Reys, 1994, for further discussion on this point).

3.6. A Review Of Study 1: Teaching Rational Number Sense to a Group of 16 High- Achieving Grade 4 Students

Drawing on the foregoing analysis for the development of rational number sense, I conducted an experimental study (reported in Moss & Case, 1999) in which this experimental curriculum was taught to 16 high-achieving Grade 4 students. The results were compared to the results of 13 similar children exposed to instruction of a more classic nature. To compare the two groups a detailed measure was designed (The Rational Number Interview) to assess children's conceptual understanding of fractions, decimals, percents, and the relationships among them. The 41-item Rational Number Interview was administered to both groups before instruction and the same measure with an additional 4 items was again administered immediately after the two groups had completed instruction. In all, the experimental group received twenty 40-minute instructional sessions (that I taught) at a rate of one per week over a 5-month period. The control group devoted a slightly longer time to the study of rational numbers and followed the program from the same text series that was used throughout the school from which the experimental classroom was drawn. The instructional sequence for the experimental group has been described in detail elsewhere (Moss, 1997). The times that are allotted for the various topics are as follows: 9 hours on percent, 4 hours on decimals, and 4 hours on mixed representations, including fractions.

The sequence for rational number instruction in the control group's text was more conventional: Fractions are the first topic to be covered. In this program, fractions are defined as numbers that describe parts of a whole and are illustrated with pie-chart diagrams. Exercises follow in which children are asked to determine fractions of a set, compare different fractions with regard to magnitude, and determine equivalent fractions. Decimals are taught next, using pie graphs, numberlines, and place value charts. Tenths are introduced first, and their relation to single-place decimals is shown. Finally, equivalent decimals are taught by a demonstration that numbers such as 0.3 and 0.30 are merely alternate representations of 3/10 and 30/100. Lessons involving operations with decimals were introduced next. The rules for addition and subtraction of decimals, as well as for multiplication of one- and two-place decimals are taught explicitly, with careful attention to the significance of place value. The use of a fraction as an operator and computations involving division of decimals are taught at the end of the sequence.

Both classrooms employed a variety of participation structures so that the students alternatively worked in small collaborative learning groups, in pairs, individually, as well as participating in whole group lessons.

The results of the study indicated that both the control and experimental groups showed some improvement from pre- to posttest; however the improvement of the treatment group was much greater. A two-way analysis of variance with repeated measures was conducted to assess the significance of this difference and the results showed a strong treatment by pre/post interaction in the predicted direction ($F(1,32) = 29.06; p < .001$). A similar interaction was present when we analyzed the results of the decimals, fractions, and percents questions separately. Further analyses of the results indicated that the students in the experimental group were much more successful than those in the control group in getting the correct answers on the posttest measure. Qualitative analyses showed that these two groups also had very different ways of reasoning and solving items: The explanations of the students in the control group were often based on additive reasoning, procedural knowledge, and often errorful rules, whereas the students in the experimental group demonstrated a response pattern that indicated a conceptual understanding grounded in ratio and multiplicative structures and a flexibility of operating in this number system that can be characterized as number sense.

For example, when asked to choose the larger of the two numbers .20 and .089, only 38% of these students indicated that they had an understanding of magnitude. The majority asserted that .089 was larger because 89 is a bigger number than 2. A second explanation they offered was that .089 was the smaller number because "The longer the number of digits after the decimal point the smaller is the number." This misconception has been noted many times in the literature (e.g., Resnick, Nesher, Leonard, Magone, Omanson, & Peled, 1989). By contrast, all of the students in the experimental group were able to correctly assert that .20 was the larger number and give a reasonable explanation for their choice.

Another set of items to which the control and experimental groups responded very differently were those that asked students to translate among representations. For example, at posttest, 93% of the students in the experimental group could correctly find the decimal notation for 6% as well as provide a sensible rationale for their answer, and 75% of these students were able to correctly respond to the question "What is 1/8 as a decimal?" Both of these questions elicited only a 17% correct response for the control group with the majority of the students asserting that 6% was equivalent to 0.6 and that 0.8 was the correct decimal representation for 1/8. Again the responses of the control group are typical of those reported in many other studies.

Another frequent problem for the control group was their inability to employ multiplicative strategies, rather than the additive strategies with which they were most familiar. One instance of this confusion was provided in the control's responses to a test item that required that the students shade 3/4 of a pizza that was partitioned into 8. Only 33% of the students in the control group were able to successfully answer this question. The majority of the students shaded three sections (3/8) asserting that, "3/4 means three parts of something." Not only does this response indicate a lack of a sense of magnitude but also shows the lack of consideration of the relation of the 3 (the part) to the 8 (the unit whole). By contrast all of the students in the experimental group were able to correctly answer this same question.

Finally, there was also a difference in response pattern between the two groups on items where students were asked to compute or perform operations with rational number. In the measure we included items that were presented in a standard form of computation as well as in a non-traditional form. An example of a question in the latter category was as follows: "Fifteen blocks spilled out of this bag. These 15 blocks represent 75% of all of the blocks that were in the bag to start with, how many blocks were in the bag to start with?" In response to this question, 88% of the Experimental students provided the correct answer compared to____ in the control group. Since the Experimental Group had not been taught any procedures for performing standard algorithms in their program, we expected that these sorts of questions would be difficult for them. However, we were surprised to find that the experimental students were better able to tackle computation problems posed in a standard format than the control group, even though the latter group had frequently encountered these problems in their instructional program. For example, in answer to " What is 3 1/4 –2 1/2?," a student from the control group answered as follows: "We need a common denominator which would be 4. So we have to find the answer to 3 1/4 – 2 2/4? So the answer is 1 0/4." This lack of sense making was also majority of the control group students asserted that .43 was the sum of .38 + .5. Sixty-one percent of this group maintained that 1/4 was the product of 1/2 of 1/8.

In summary, the posttest analyses showed that the students in the experimental group had gained a principled, ratio-based understanding of rational number, as well as a fluency and flexibility in applying that understanding. In strong contrast, the type of reasoning that the control group demonstrated in their posttest protocol displayed the limitations of their knowledge of the rational number system as a whole and their reliance on additive structures; misconceptions that are frequently reported in the mathematics education literature. Thus we believe that this approach to teaching rational number did foster the type of understanding that we had hoped for.

3.7. Further Questions

In the first study the experimental curriculum was tried out with a very special group of high achieving students. Thus, many questions regarding the robustness of the curriculum remained:

- How would this curriculum work for different populations of students, such as less able students, and those who have had previous instruction in rational number?
- Would these two groups also show the same type of flexibility (number sense) as the first group?
- Would the program still be successful if students had already received a more conventional (pie chart) introduction to the rational numbers?

As well there were questions about the assessment measure:

- What might children's understanding and competencies look like at posttest, when examined in greater detail and across a broader range of test items?
- Would students be able to use their invented strategies based on halving and doubling when working with more complex problems?
- Could students perform as successfully on tests that included more standard computation items?

Finally questions remained about the curriculum design:

- What particular aspects of the curriculum made the most powerful contributions to the outcomes? Specifically, what is the benefit of using percents as an introduction to this number system?
- How useful are representations based on measurement in teaching rational number?
- What are the unique contributions this research can make, both towards establishing a different psychological model for rational number development, and providing a curriculum for teachers to use that is grounded in, and explores, the development of students' intuitions.

In the next chapters of this thesis, I will present two further studies that begin to answer some of these questions. Chapter 4 describes a study with a mixed-ability group of Grade 4 students. Chapter 5 presents a study with a group of

Grade 6 students of mixed-ability, who had all received several years of conventional rational number teaching prior to the intervention.

Chapter 4

The Development of Rational Number Understanding: An Intervention Study with an Intact Group of Grade Four Students

4.1. Introduction

The results of the original study (Study 1) that was previously reported as a Masters thesis, (Moss.1997, and in Moss & Case, 1999) clearly demonstrate that the particular group of students who participated made substantially more gains in their ability to perform in the area of rational number than did a control group. Furthermore, as I have already mentioned, these students displayed the characteristics of number sense that have been outlined: an overall understanding of the number system, a sense of the magnitude of these numbers, and a flexibility that allowed them to use the representations of rational number interchangeably and to invent procedures for operations. However, there were limitations to the first study that left many questions unanswered. The study that I report in this chapter was designed to address some of these questions. This time the students came from an intact class where there were several students who had particular difficulties with mathematics and others who were less able than the students in the first study. Thus I was able to consider a number of new questions:

Question 1: Effectiveness of the Curriculum for Mixed-Ability Students

The first question concerned the effect of the curriculum on mixed-ability students. I was interested to discover, a) if the class as a whole would make significant gains on the measure from pretest to posttest, and b) whether the instructional intervention would foster the same kinds of number sense with these mixed-ability students that had been achieved by the high-ability students of the initial intervention.

Question 2: Differences for High- and Low-Ability Students

Secondly, I was interested to discover if students who are initially high or low in their mathematics achievement would show a different pattern of change from pre- to posttest as a result of the experimental curriculum. Would they achieve differently on the curriculum as a whole and would their performance on the three subtests of percent, decimals, and fractions show differences based on their mathematics ability?

Question 3: Performance on Standard Computation Tasks

The third question concerned the ability of the experimental students to perform standard algorithms. In Study 1, I assessed the students on a variety of types of tasks. Some of these, 20%, were direct measure items, and were congruent with the context of the curriculum. Others, 80%, were transfer items. Included in the transfer items were a very small number of standard computation questions. The findings from Study 1 indicated that some of the students were able to provide correct answers to these standard computation tasks. Their methods of calculating, however, were not standard. My analysis of the control group's performance on these same items indicated that, even for Grade 4 students who have been taught to perform standard algorithms for addition and subtraction of fractions and decimals, these types of problems are still very challenging. In fact, we know these to be very challenging even for adults. Typically, students try to apply procedures that they have been taught, and reveal their lack of understanding by making mistakes that are grossly inaccurate. In the present study I wanted to examine the performance of these students on an increased number of standard computation problems in decimals and fractions in order to determine what types of problems they are and are not able to solve as well as to determine what strategies the students employ in the absence of having learned standard procedures.

4.2. Method

4.2.1. Subjects:

Twenty-one Grade 4 students participated in the study. As in the first study, these children attended the Laboratory School at the University of Toronto. However, in this study the children were from an intact Grade 4 class. The class was composed of 9 girls and 12 boys. Five of the students in the class had been identified as having learning problems. Each of these five students received individualized help from a school resource teacher two or three times per week. None of the students in the class had received any classroom instruction in rational number prior to this study. As well, although three of the five special education students received some individualized help in mathematics during the duration of the study, at no time did they cover rational number topics.

4.2.2. Design:

The 49-item pretest measure was administered on an individual basis to all of the students in the class in January, 1997, prior to the instructional unit. The same measure was re-administered in early June, three weeks after the unit had been completed. The experimental rational number curriculum consisted of 20, 45-minute lessons, all of which were taught by the researcher. These lessons were taught at a rate of two or three per week over a three month period. During the year in which this experiment took place, I was employed as a half-time teacher for this particular group of students. In this role, my duties included sharing in the teaching of all Grade 4 curricular areas.

4.2.3 Procedure:

Pretest interviews were carried out by two specially trained graduate students who were teacher candidates. As in the first study, these pretests were administered to all of the students in the class on an individual basis. Each student was taken to a quiet room by one of the interviewers where the student could freely answer the questions that were posed. These pretests took between 15 to 30 minutes to complete. Because the posttest interview took 1 1/2 hours, this interview was administered to the students in two parts over a two-day period.

4.2.4 Experimental Program:

Instruction schedule:

The experimental sessions were approximately 45 minutes to 1 hour in length with the exception of the final three review lessons, which were 1 to 1 1/2 hours in length. All of the lessons were documented; selected classes were videotaped. Each lesson was reviewed, and, although the lessons followed the sequence described in the preceding chapter, the specific plans for each lesson were revised on the basis of class progress.

A breakdown of the topics covered in the lessons and the time allotted for these topics is presented in Table 4.1:

Table 4.1
Lessons, Topics, and Hours of Experimental Curriculum

Subject	Lessons	Hours
Percent	Lessons 1 - 6	4 hours
Decimals with percent	Lessons 7 - 13	5 hours
Fractions and mixed representation	Lessons 14 - 17	5 hours
Review of percent lessons	Lessons 18 - 20	4 hours
Total hours spent		18 hours

Below is a brief account of activities presented to the children.

Estimating Percents (Lessons 1- 3):

The lessons started with an introduction to percents. To begin the unit, the students were challenged to think about all of the instances where percents occurred in their daily lives and to report these instances to the class as a whole. A definition of four key benchmark points (100%, 50%, 25%, 0%) was discussed. Next, large drainage pipes of varying lengths covered with specially fitted sleeves were presented. These sleeves were pieces of flexible venting tube that fit around the pipes and could be pulled up and down and set to various levels. The children were invited to demonstrate their percent understanding using the tubes and to consider how they would use the tubes to teach percent to a younger child. The children were also challenged to estimate the percentage of the pipe that was covered. The objective of the first few lessons was to encourage children to think of strategies for making reliable estimates. The perceptual halving strategy was encouraged. "Percent full" estimations were also made using beakers and vials filled with sand or water. These estimation exercises were designed to allow the students to integrate their natural halving strategies with percent terminology. The children were then introduced to a standard numerical form of notation for labelling percents. Standard notation for the writing of fractions for benchmarks was also introduced so that the students would be comfortable moving between representations. These lessons also included a variety of other measurement situations where students could operate with percentages and discover/construct methods for operating with percents.

Computing Percents (Lessons 4-6):

The visual estimation exercises using vials and beakers were continued with a new focus on computation and measurement. Children were instructed to compare visual estimates with estimates based on measurement and computation. For example, if a vial is 20 mm tall, 50% of that should be 10 mm. The children then began to estimate and mentally compute percentage of volume, for example, this vial holds 60 ml of water, 50% full should be 30 ml, 25% full should be 15ml. Other challenges included measuring objects in the classroom and then estimating and calculating different benchmark points such as 50%, 25%, 12 1/2%, and 75%. The children were not given any standard rules to perform these calculations. An example of a method that was commonly used is as follows; 75% of 80 cm (the length of the desk) should be 60 cm because 50% of 80 cm is 40 cm and half of that (25%) is 20 cm and together they equal 75%. Other exercises included comparing heights of, for example, children to teacher and then assigning an estimated numerical value using the language of percents. For example, "Peter's height is what percent of Joan's?" A series of specially made laminated cut-out dolls ranging in height from 5 cm to 25 cm provided additional practice at comparing heights. Percent lessons were concluded with the students planning and teaching a percent lesson to a child from a lower grade.

Introduction to Decimals using Stopwatches (Lessons 7-8):

In these two lessons children were introduced to decimals as an extension of their work on percents. The lessons started with discussions of decimals and how they permit more precise measurement than whole numbers. Two–place decimals were introduced as a way of indicating what "percent" of the distance between two whole numbers a particular quantity occupies. LCD stopwatches with screens that displayed seconds and hundredths of seconds (hundredths of seconds were indicated by two small digits to the right of the numbers) were used as the introduction to decimals. After lengthy discussions of what these small numbers represented quantitatively, the students came to refer to these hundredths of seconds as centi-seconds. The stopwatch activities served to build up children's intuitive sense of small time intervals, and to give students experience of the magnitude of centi-seconds. More importantly, use of these stopwatches provided the students with the opportunity to represent these intervals as decimal numbers. In the stopwatch activities, centi-seconds indicated the percentage of time that had passed between any two whole seconds; they came to represent the temporal analogs of distance. Many activities and games were devised for the purpose of helping the students to actively manipulate the decimal numbers in order to illuminate the conceptually difficult concepts of magnitude and order. The first challenge that was presented to the students was "The Stop/Start Challenge." In this exercise, students attempted to start and stop the watch as quickly as possible, several times in succession. They then compared their personal quickest reaction time with those of their classmates. In this exercise, they had the opportunity to experience the ordering of decimal numbers as well as to have an informal look at computing differences in decimal numbers (scores). Another difficult initial aspect of using decimal symbols is the ordering of decimals when the numbers move from 0.09 to 0.10, for example. Some students were able to respond quickly enough to the challenge to achieve a score of .09 seconds. Therefore, such traditionally difficult rational number tasks such as, what is bigger, .09 or .40? could be naturally introduced. Another stopwatch game that offered active participation in the understanding of magnitude was "Stop The Watch Between." The object of this game was for the student to decide which decimal numbers come in between two given decimal numbers and then to stop the watch somewhere in that span of decimal numbers. In the game "Crack the Code," the students moved between representations, as they were challenged to stop the watch at the decimal equivalent of 1/2 (.50), for example. As an extension to these exercises the students were encouraged to invent variations on these games to use as challenges for their classmates.

Learning about Decimals on Numberlines (Lesson 9):

A second approach to decimals was through the use of metre-long, laminated numberlines that were calibrated in centimetres. This approach was based on students' work with percents using numberlines. The first activities served as a review. Each child was given a small numberline and asked to find designated percents of the whole line by placing a unit block on the appropriate spot ("Please place a unit block on the line that indicates 44% of your numberline"). The students were then told that these percent quantities could also be expressed as a decimal number; thus, for example, 44% could also be shown as 0.44. Other activities included "Percent/Decimal Walks" where several numberlines (which were referred to as "sidewalks" by the students) were lined up end to end on the classroom floor with small gaps between each. Students walked a given indicated distance on the numberlines, e.g., "Can you please walk 3.67 sidewalks." This game was also played in such a way that a single student walked a "mystery distance" and the other students had to determine what distance they had walked. In keeping with the preceding exercises they expressed this distance as a mixed number, i.e., a whole number and a decimal number.

Playing and Inventing Decimal Board Games (Lessons 10-13):

A board game "The Dragon Game"was devised with the intention of giving the students the opportunity to learn about the magnitude of decimal numbers, as well as to add and subtract decimal numbers. The game board was approximately 60 cm x 90 cm and was composed of 20 individual laminated 10 cm numberlines that were arranged as a winding path. Each number line was marked as a ruler: ten black thick lines indicated cm measures, ten slightly shorter blue lines highlighted the .5 cm measures and 100 red lines provided the mm measures. This game directly followed on from the "sidewalk" excercises mentioned above. The object of the game was to get from the beginning (the first sidewalk) to the end (the 20th sidewalk) before the other players. At each turn, a child picked two cards; an "Add" or "Subtract" card and a "Number" card. Each Number card had two digits written on it. The rule was that before making a move on the board, the player had to expand the two digits on the card by adding both a zero and a decimal point strategically so as to optimize the distance that the player travelled. For example, if a child picked a card with the numbers 1 2, they had the options of calling that card .120, 1.20, 12.0, or 120. The game also had other appealing features, for example, good luck bonuses and hazard areas. Three players could play at once. The rest of the group worked with the teacher to practice adding and subtracting on their own numberlines. Three lessons followed where the students invented and planned their own rational number board game and then played each other's games.

Fractions (Lessons 14- 17):

In keeping with the curriculum focus of translating among representations, fraction lessons were taught in relationship to decimals and percents. In these lessons, the children were challenged to, for example, represent the fraction 1/4 in as many ways as they could, using a variety of shaded geometric shapes as well as formal fraction, decimal, and percent representations. They also worked on problems and invented their own challenges for solving mixed-representation equations involving decimals, percents, and fractions.

Review (Lessons 18- 20):

Games were played where the students had to add and subtract decimals, fractions, and percents by creating their own hands-on concrete materials. For example, students invented card games with mixed-representations and challenged their classmates to solve a variety of problems that were posed. As a final culminating project, students were invited to either, (a) invent their own rational number teaching strategies and lessons that could be taught to another group, or (b) to design a game or video presentation that incorporated specific rational number teaching objectives.

4.2.5. Measure:

The original rational number measure of 41 items was refined and expanded to 49 items for the present study. Thirty items were retained from the original measure, and 19 items were added. The new items were similar in structure to those on the first measure, except for the computation items that were deliberately presented in a more standard form.

The 49-item measure was subdivided into three sections: Percents (16 items), Decimals (17 items), and Fractions (16 items). The items in each subtest were arranged in order of difficulty. As in the previous study, the whole test was constructed so that a percentage (approximately 20%) of the tasks were direct measure questions related to the experimental curriculum, while the remaining questions were transfer items. The direct measure questions assessed the students' ability to perform rational number tasks in a familiar context. Some examples of these included: asking the subjects to estimate, for example, 25% of the height of a container, or to translate "benchmark" quantities such as 50% to decimals (.5) and fractions, (1/2). The majority of the measure was composed of transfer items that required the students to work in novel contexts. Some of these items were chosen because they reflected a broadened conceptualization and required extrapolation from one domain to another, e.g., shade 0.3 of a circle that is partitioned into 5 sections. This question asked students to overcome both a misleading physical feature, as well as to translate an unfamiliar decimal representation (.3) into the more familiar form of 30%. Another such complex type of question was: Can these be the same amount, .06 of a tenth and .6 of a hundredth?

As well as including items that were either direct or transfer, the measure was also designed to include a variety of types of tasks. Thus we included items where students were required to estimate distances along number lines; shade specified fractional quantities in standard geometric shapes; or, respond to rational number questions that were posed as word problems. Still other questions were designed as visual distractors intended to divert subjects from a straightforward solution. The entire measure is presented in Appendix B.

4.2.6. Scoring Procedures:

In administering both the pretests and posttests, interviewers read the questions and asked the students to respond out loud. Although the interviewers did not give assistance to the students in interpreting the questions, they did repeat the questions as many times as the students requested. The interviewers asked the students all of the questions in the order that they were presented on the measure and continued to the end of the measure regardless of the students' performance on earlier tasks. The students were provided with pencil and paper and told that they could write or draw anything that might help them to work out the answer. These notes were kept by the interviewers to provide additional information about students' reasoning. At no time in the interview did the interviewers indicate whether or not the students had been successful in their response. This practice was instituted so that the students would not be influenced in their responses at posttest.

The items were scored dichotomously with one point allocated for each correct answer. For those questions which required that subjects explain or justify their responses, one point was allocated only if the correct answer and the correct or logical corresponding explanation were both provided. The items on the Percents, Fractions, and Decimals subtests were added separately for the three subtest scores. The composite score provided a grand total for the rational number test.To analyze the data, I assigned 35 of the questions to four number sense subcategories: (a) Interchangeability of Representations, (e.g., What is 1/3 as a percent?), (b) Compare and Order (e.g., Draw a picture to show which is greater, 2/3 or 3/4?), (c) Misleading Appearance (e.g., Can you construct the number 23.5 with base 10 blocks using the longs (10 cm sticks) as ones?), (d) Nonstandard Computation (e.g., Another student told me that 7 is 3/4 of 10, is it?). A final category that was analysed was Standard Computation, composed of eight items, five in fractions (e.g., "What is 4 3/4 + 6 6/8?") and three in decimals (e.g., "What is 3.46 - .8?). Individual scores were obtained for the four number sense categories as well as for the final category of Standard Computation.

4.3. Results and Discussion

The Results and Discussion section is broken down into three parts reflecting the questions that I posed at the outset of this chapter. First, I present the overall results of the measure for the class as a whole as well as the scores for the whole group on the individual subtests Percents, Decimals, and Fractions. Next, I look at the differences between high- and low-achieving students. Next, I analyze students' ability to perform standard computation. The effects of particular aspects of the curriculum on learning outcomes is considered throughout the results section.

4.3.1. Overall Results of the Rational Number Measure:

An analysis of the pre- and posttest results of the measure as a whole revealed that the students made significant gains. Table 4.2 shows the means and standard deviations, and also reports the breakdown of these scores on the individual subtests of Percents, Fractions, and Decimals. As can be seen, there was significant change on all three of the individual subtests. When two tailed t-tests were performed on all three of the individual subtests the differences were all found to be highly significant with scores $t = 9.28$ $p < .0001$, $t = 6.01$ $p < .0001$ and $t = 6.77$ $p < .0001$ for the Percents, Fractions, and Decimals tests respectively. Since these 3 subtests are related, a Bonferroni approach was used for significance testing. The resulting alpha (.05/3) was .017. All of these results far exceeded this level. The Bonferroni correction was used throughout the analysis.

Table 4.2
Total Scores on the Rational Number Test, and the Individual Subtests, Before and After Instruction

	Mean score on pretest (max = 49)	Mean score on posttest (max = 49)
Entire Measure (max = 49)	9.71 (7.82)	24.57 (10.18) ****
Percents (max = 16)	3.81(2.83)	8.67 (2.79)****
Fractions (max = 16)	3.90 (3.14)	8.71 (4.18)****
Decimals (max = 17)	2.00 (2.34)	7.19 (3.84)****

** $p < .01$; *** $p < .001$; **** $p < .0001$; ns = not significant

4.3.2. High and Low Mathematics Achievement:

In order to evaluate differences for high and low achieving students, I made a median split on the combined mean scores of the Concepts and Computation subtests of the Canadian Test of Basic Skills. This split provided a group of 10 high-achieving and 11 low-achieving students. Both of the classroom teachers were in agreement with this designation.

Table 4.3 shows the mean scores for the pre- and posttest results for the entire measure comparing high- and low-achieving students. In Table 4.4, and Figure 4.2, I present the mean scores for these high- and low-achieving students on the three subtests of percents fractions and decimals. As can be seen, both groups made substantial improvement.

Table 4.3
Total Score on Rational Number Test for High- and Low-Achieving Students, Before and After Instruction

	Mean Score on Pretest (max = 49)	Mean Score on Posttest (max = 49)
Low-achieving (n = 11)	6.64 (3.11)	16.91 (5.14)****
High-achieving (n = 10)	13.1 (10.05)	33 (7.07)****

** p < .01; *** p < .001; **** p < .0001; ns = not significant

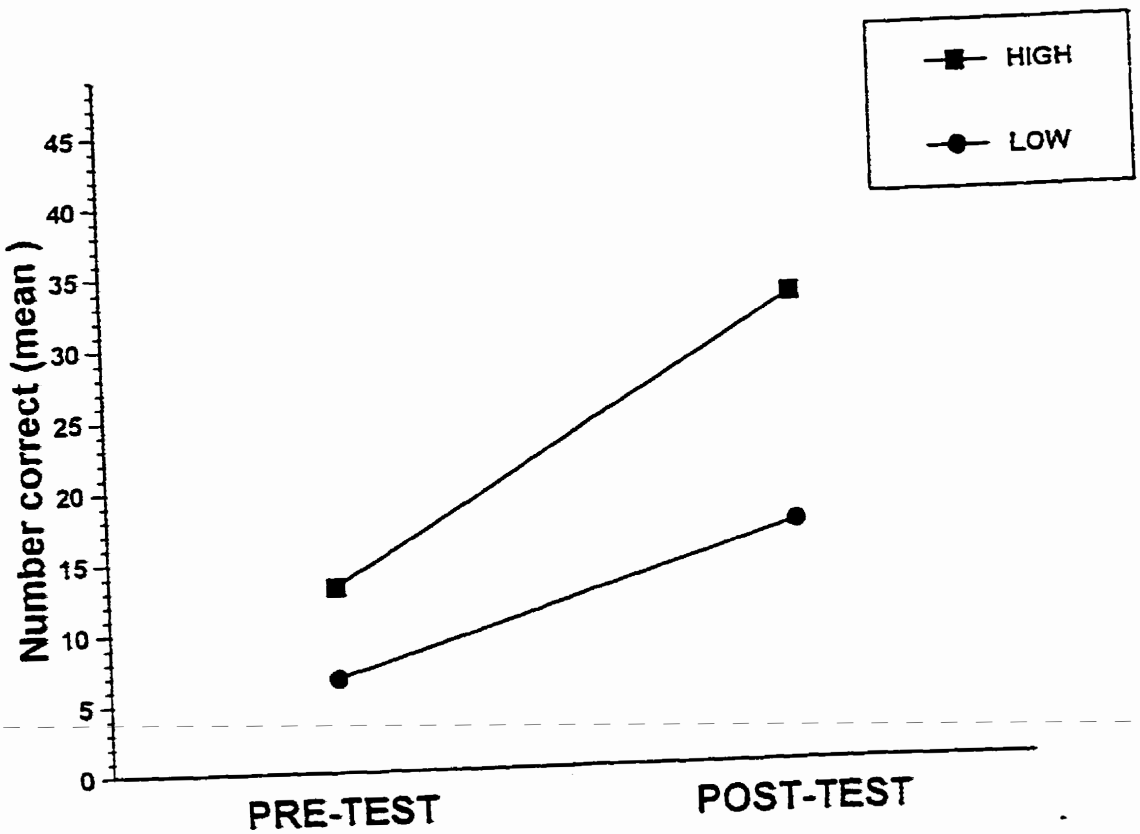

HIGH
LOW
Number correct (mean)
45
40
35
30
25
20
15
10
5
0
PRE-TEST
POST-TEST

Figure 4.1—

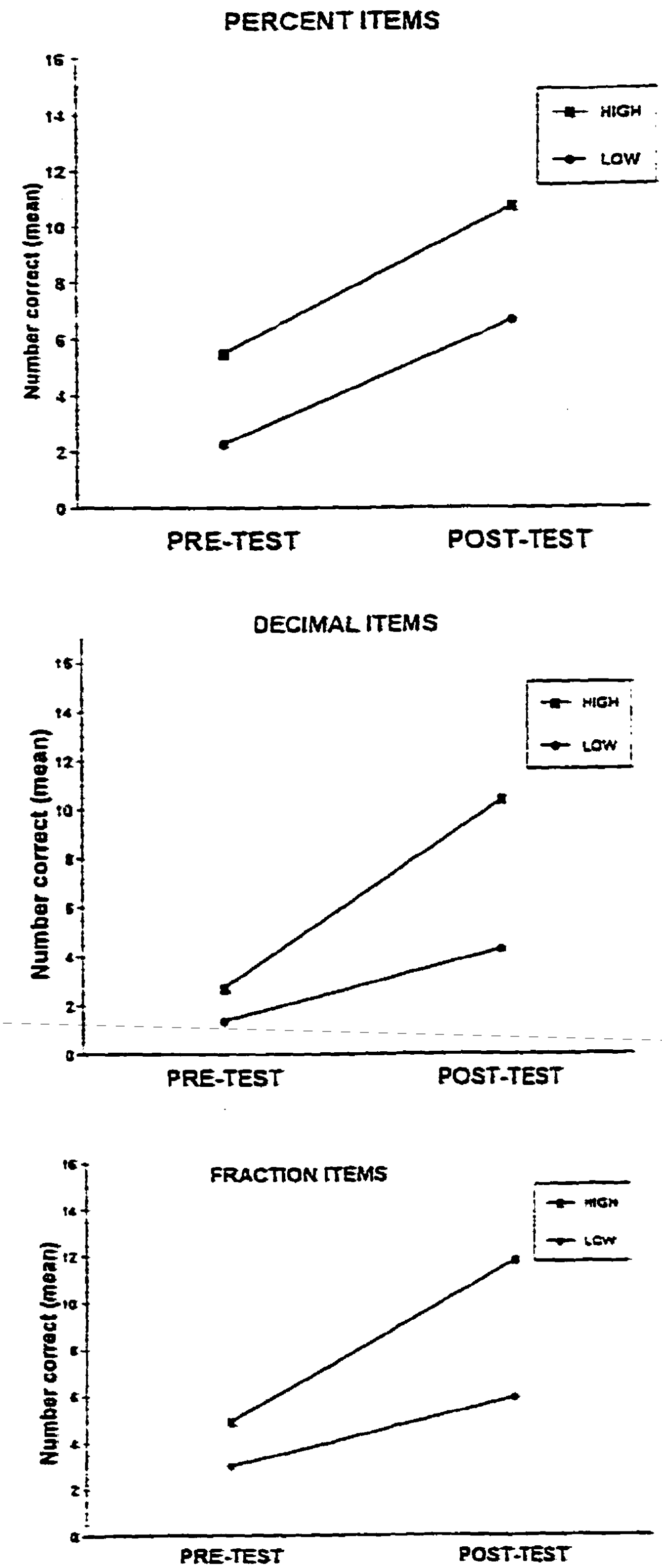

PERCENT ITEMS
Number correct (mean)
HIGH
LOW
PRE-TEST
POST-TEST
DECIMAL ITEMS
Number correct (mean)
HIGH
LOW
PRE-TEST
POST-TEST
FRACTION ITEMS
Number correct (mean)
HIGH
LOW
PRE-TEST
POST-TEST

Figure 4.2—

Table 4.4
Total Scores on Rational Number Test for High- and Low-Achieving Students on the Individual Subtests, Before and After Instruction

Subtests	High-Achieving Students		Low-Achieving Students	
	Pretest	Posttest	Pretest	Posttest
Percent (max= 16)	5.5 (3.27)	10.8 (1.93)****	2.27 (1.00)	6.72 (1.9)****
Fraction (max= 16)	4.9 (4.12)	11.8 (3.76)****	3 (1.61)	5.9 (2.02)****
Decimal (max = 17)	2.7 (3.12)	10.4 (2.59)****	1.36 (1.12)	4.27 (1.95)****

** $p < .01$; *** $p < .001$; **** $p < .0001$; ns = not significant

In order to further evaluate the differences between the high- and low-achieving students, I conducted a two- way analysis of variance with repeated measures [high- by low-achievement (group)], x [pre- and post (time)]. The results revealed that there was a significant group by time interaction, and that the high achievers improved significantly more than the students in the lower half of the class, $F(1,19) = 12.4$ $p < .005$). As well, the effects of levels of achievement and the effects of time were also significant at $F(1,19) = 118.5$ $p < .0001$ and $F(1,19) = 18.8$, $p < .001$ respectively.

When ANOVAS were also performed on each of the subtests (Percents, Decimals, Fractions), a similar interaction was found for the Fractions and Decimal subtests, again showing greater improvement for the high-achieving students. However, the result of the ANOVA for the Percent subtest showed no interaction, thus showing that the high-achieving group improved significantly more than the low-achieving students on Decimals and Fractions but not on Percents. Table 4.5 presents the results of the of the anovas for the individual subtests, Percent, Decimals, and Fractions. Figure 4.1, indicates the significant interaction on the measure as a whole and Figure 4.2 reflects greater learning by the high-achieving group on two of the three subtests.

Table 4.5
Results of Analyses of Variance with Repeated Measures on the Individual Subtests for High- and Low-Achieving Students

Subtests	F tests	F (1,19)
Percents	Achievement Level x Time	.97 ns
	Time Effects	129.3 ****
	Achievement Level Effects	18.9 ***
Fractions	Achievement Level x Time	8.5 **
	Time Effects	49.7 ****
	Achievement Level effects	11.9 ***
Decimals	Achievement Level x time	17.9 ***
	Time Effects	84.6 ****
	Achievement Level effects	20.4 ***

As the analysis of these results indicates, there was in fact a significant interaction shown for math ability by time on the measure as a whole. On the other hand, it was also evident from that results that the pre/post difference was high for both the high- as well as the low-achieving students. Therefore, in the following analyses of the number sense subcategories, I present quantitative and qualitative results for the class as a whole.

4.3.3. Analysis of Number Sense Subcategories:

4.3.3.1. Interchangeability of Representations

Flexibility in moving among symbolic representations in the domain of rational number is considered to be a good indication of rational number understanding as well as an important factor in rational number sense (Sowder, 1994). Using multiple representations for quantities allows students to work conceptually and to "transform problems on the basis of useful equivalencies." In fact, Lesh, Post, & Behr (1987, see p. 320) make the claim that recognizing or constructing correspondences between different representational systems is at the heart of knowing or understanding mathematics.

Eight items from the rational number measure assess the ability of students to use rational number representations interchangeably. Table 4.6 shows the pre- and posttest scores of the children on these items. The items are arranged in order of the difficulty that they posed to the students at pretest. The mean score on the posttest was 3.05 (1.46) out of 7, compared to .81 (1.28) for the pretest. A paired two-tailed t-test revealed that the difference in scores were highly significant. $t = 7.27$ $p = .0001$. As well as obtaining t-test scores for this and the other number sense categories, I also calculated effect scores, by dividing the difference of the pre- and posttest scores by the standard deviation of that difference. Thus the effect size of the group as a whole was 1.53.

The following two protocol examples provide illustrations of student reasoning in translating among representation of rational number.

Interviewer:	How would you write 6% as a decimal?
Student:	Lets see... I guess it is point 6 .. No that can't be, because point six is 60% so that can't be right...Oh I know it is point 06 because when we did the number line sidewalk thing in class we walked 5% (of the number line sidewalk) for .05.

(This student is referring to the activity where the class was challenged to walk particular distances given as decimals on meter-long laminated number lines).

This student was able to find the correct solution based on her reasoning about magnitude of the powers of ten and her understanding of percents. In eventually providing a correct answer, she also showed that she was able to overcome the distractor embedded in the question and could revise her initial assertion that .6 would be 6% as a decimal; an assertion that is highly representative of the kinds of answers that a majority of high school students give for questions of this sort Another feature of the protocol is that this student made a direct connection between this item, presented in standard format and the numberline representation that had been presented as part of the experimental curriculum.

The students' response to the next item that I report in this category is also representative of how the students' explanations are closely tied to their experience of the curriculum.

Interviewer:	Do you know what 1/8 is as a decimal?
Student:	Well one eighth is half of one fourth. And one quarter is 25%, so half of that is 12 1/2 %. So as a decimal, that would have to be point 12 and a half. So that is point 12 point five, so that means that it is point one two five.

The protocol above illustrates three strategies that most students in this class typically employed in their reasoning in rational number. Note the use of percents as a guide (intermediate step) even when the problem does not contain the percent representation. In solving this item, the student used the familiar 25% benchmark as an transitional step in moving from a fraction to a decimal. This same protocol reveals another bridging step that many students found useful; a double decimal representation. As I have already indicated, when these students first encountered decimal notation, they worked exclusively with two-place decimals as an alternative representation for percent. As the lessons progressed and multi-place (e.g., three place) decimals were introduced, the students referenced the third place to the double decimal notion. For example, they reasoned that .12 1/2 is .5 of the way between .12 and .13 and should thus be represented as .12.5. A final strategy that this student used that is also central to the curriculum is the operation of halving and doubling. In order to find the decimal equivalent this student first doubled 1/8 to produce the very familiar fraction 1/4 and its percent counterpart of 25%. Then he again halved that quantity to get the desired amount of 12 1/2%.

Although the students were able to perform successfully on some of the items in this category of interchangeability, there were a number of questions that were clearly difficult. An examination of the table below reveals that the students had much more difficulty when the numbers that they were asked to translate were unfamiliar. As can be seen, when asked to find the percent representation for 1/3 and 1/5, the students scored only 42% and 19% respectively.

Table 4.6
Percentage of Students Succeeding on Items Requiring Movement among Different Rational Number Representations, Before and After Instruction

ITEMS	PRE	POST
A package of blocks contains 10 yellow blocks and 10 blue blocks. Do you think the yellow blocks are .5 of all the blocks?	57	95
How would you write 6% as a decimal?	5	81
What is 1/8 as a decimal?	0	48
What is 1/3 as a percent	5	43
What is 1/5 as a percent?	5	19
How should you write seventy-five thousandths as a	0	19
What is 6% as a fraction?	0	10

4.3.3.2. Compare and Order Numbers

Closely associated with the seamless movement among representations and the ability to use different rational number symbolic representations interchangeably is the ability to compare and order rational numbers. Tests that require students to order a series of fractional numbers reveal that students have difficulty attaching a quantitative referent to decimal symbols, even when the decimal symbol notation is familiar (Carpenter et al., 1981; Hiebert, Wearne, & Taber, 1991).

Eight items were included on the measure in which the students were either required to compare quantities or to find a third quantity that fit between two others. When the pre- and posttest scores were analyzed for these 8 "Compare and Order" items it was revealed that the students showed even more improvement on this class of item than on those items in the previous category; their pretest mean score was 1.66 (1.53) (out of 8) and their posttest score was 4.33 (1.90). The difference score was 2.66 (1.59) yielding an effect size of 1.67. When a paired t-test was performed the significance was high $t = 7.678$ $p = .0001$.

The students made substantial gains on the item that asked them to choose the largest of two decimal numbers. Middle school students and adults often use incorrect rules based on considerations of either the number of digits after the decimal point or the size of the numbers without regard to their decimal place value (Resnick et al., 1989). Thus, it was encouraging that all of the students were able to correctly answer this item:

Interviewer:	Which is bigger, decimal 20 or decimal 089?
Student:	Well, it is decimal 20 because that's like 20% and the other is just like 8 and something percent.

This response and variations of this response are highly representative of the strategies that many of the students employed at posttest, thus again demonstrating the usefulness of the percent representation in appreciating magnitude differences of decimals.

Another conceptual difficulty that students encounter in learning rational number involves the density property of the rationals, i.e., that a third number can be inserted between any two numbers. The response given by a student below is typical of the answers given by most students in the class and illustrates that an understanding of density was gained by posttest.

Interviewer: Can you think of a number that lies between point 3 and point 4?

Student: Well lets see, there is point three five but there are also numbers like point three zero nine.

Although the majority of the students found suitable answers to this item at posttest, their assertions at pretest were very different. Most of the students, prior to their lessons, believed that it was not possible to insert a number between these numbers. This commonly held misconception is clearly based on interference from, or the tenacity of, students' whole number concepts. Although this student does not indicate how he derived his answer, the exercises of recursive halving in which the students regularly participated did lead to the insight of the property of infinite "smallness" characteristic of rational number. I also conjecture that insights of the density property that students acquired were also supported by exercises using the stopwatches as well as the number lines: The stopwatches gave the students visual evidence of the idea of numbers fitting between other numbers; in playing the number line games, students regularly encountered the challenge of requiring a smaller unit than the one available.

Again as in the interchangeability of rational number category, there were some items in this category that the students were less able to perform. Interestingly these items contained similar characteristics to those in the previous category. For example, the students again revealed their difficulty in working with thirds. Thus, when asked to find the larger of 1/2 and 1/3 they were only able to achieve a score of 61%, and further when asked to find a number that falls between 1/2 and 1/3, they achieved a score of only 29%.

Table 4.7 contains a complete list of the items in the Compare and Order category as well as the percentage of students that succeeded on each item.

Table 4.7
Percentage of Students Succeeding on Items Requiring Comparison and Ordering of Numbers, Before and After Instruction

ITEMS	PRE	POST
Which is bigger, .20 or .089?	33	3
Order from smallest to largest, 1/2, 1, 1/3	48	76
Can you tell me a number that comes between .3 and .4?	14	76
Draw a picture to show which is greater 2/3 or 3/4.	14	62
Which is less, 1/3 or 1/2 of the blocks?	48	62
Which is bigger; tenths, hundredths, or thousandths?	5	33
Is there a number between 1/2 and 1/3?	0	29
Could these be the same amount, .06 of a tenth and .6 of a hundredth?	5	24

4.3.3.3. Visual Distractors

Piaget (1970) believed that in order to assess children's conceptual understanding, they should be presented with tasks that contained misleading features, thus minimizing the opportunity for students to merely parrot what they had been taught. Behr, Lesh, Post, & Silver (1983) held a similar view and suggested that misleading items are particularly revealing of conceptual understanding in the domain of rational number. As predicted, the students' performances on these items improved significantly. Their mean score on the posttest was 5.81 (2.18) (out of 11) as opposed to 3.29 (2.23)at pretest ($\underline{t}$ = 4.90; $\underline{p}$ < .0001). Difference = 2.52 (2.35) effect size 1. 2.

One of the items in this category required students use base-10 manipulative blocks (Dienes blocks) that were customarily used for whole number exercises to construct a decimal number (23.5). The students were given a box containing 10 of each type of block. They had 10 "flats"(square blocks partitioned in a hundreds) and 10 "longs"(stick like blocks 1 x cm partitioned into 10 cm) and 10 "cubes" that were each 1 cm square. The instruction that they were given was that they should use the "longs" to represent "ones." (It must be noted, that the long sticks are commonly used to represent "tens" in whole number exercises). Thus, in order to succeed at this task the students had to extrapolate that the other blocks would have new identities as well based on the proportion of powers of 10. In order to complete this task, then, the "flat blocks" needed to be considered as tens and the centicubes (which are the standard representation of ones) would in this case be transformed into tenths.

Interviewer: Can you construct the number 23.5 with base-10 blocks, using the long ten sticks as ones?

Student: I get it, if this is one (points to the long tens stick) then this (points to the square hundreds board) has to be ten. So these (points to the centicubes) become tenths.

When this same item was included in the first study, the control group found it very difficult. Their responses showed that they could not easily transform these blocks and use them in a coherent fashion. Rather they used an assortment of random strategies. For example, some used the centimetre cubes as decimal points and the "longs" for the "ones" and the "tens" (see Moss, 1997; Moss & Case, 1999). The reasoning of the experimental students in solving this problem shows by contrast the flexibility of their understanding and illustrates their knowledge of the proportional and multiplicative nature of the task; an idea that is consistently reinforced in the rational number curriculum.

Another item from this category asked the students to shade three quarters of a pizza that was partitioned into eight parts. At pretest the students used additive strategies, and responded that three pieces of the pizza represented three quarters. Their posttest responses showed that they considered the proportion of the pizza rather than the pieces. The protocol below illustrates this.

Interviewer: Can you shade three quarters of this pizza?

Student: Well, three-quarters is like 75%. So this part (points to 1/2 of pizza) is 50% and this part (points to 1/4 section) is half of fifty so it is 25%. So the whole thing is 75%.

Finally, I present an item from this category in which the students were required to shade decimal three (0.3) of a circle that was partitioned in 5. Although this was a very difficult item and was only passed by a few students in the upper half of the class, nonetheless, I include the strategy to illustrate the potential of the experimental curriculum to provide useful strategies for students to solve this challenging problem.

Interviewer: Shade point three (0.3) of this circle. (This circle is equally divided into 5 sections).

Student: These pie things must be 20% because there are 5 of them 'cause it is like 20, 40, 60, 80, 100. (He touches the five fingers of his one hand as he counts). If you want to get 30,you should shade in one of the pieces and half of another.

This student in assigning the value of 20% to the pie segments not only revealed an exceptional ability to make sense of a difficult situation and overcome a distractor in a proportion that he is unfamiliar with, but also reveals a sophisticated ability to understand quotient division.

While the illustrations above point to many students' successful strategies and improved ability, still there were items in this category of Misleading Visual Features that were difficult. One surprising finding was the difficulty that many students had in locating .05 (45% passed) and .29 (24% passed) on number lines. This finding, while highly consistent within rational number literature (Behr et al., 1984), was nonetheless disappointing as the curriculum featured a variety of numberline games.

A complete analysis of the passing rates for items in this category is presented in Table 4.8.

Table 4.8
Percentage of Students Succeeding on Items with Misleading Visual Features, Before and After Instruction

ITEMS	PRE	POST
(Divide 10 blocks into three groups of 3, 5, and 2. Shift group of 5 blocks ahead.) Is this half the blocks?	86	100
Where would you put the number 3 1/2 on a number line from 0 to 4?	86	95
Can you construct the number 23.5 with base 10 blocks using the long, 10-unit blocks as ones?	48	81
Can you shade 3/4 of this pizza? A pie sectioned into 8 pieces.	38	76
How about 1 1/3 (on a number line from 0 to 4)?	29	71
What number is marked by the letter A (.05) on a number line?	10	45
How about the number 1/4 (on a number line from 0 to 4)	5	33
Two cartons of chocolate milk mixed in the same vat. One carton is 300ml and the other 200ml. The percentage of chocolate syrup in the larger carton is 60%. What is the percentage of syrup in the smaller carton?	14	24
What number is marked by the letter B (.29) on a number line?	5	24
Shade .3 of the circle (divided into 5 pieces).	5	23
What fraction of the distance has Mary travelled from home to school?	5	5

4.3.3.4. Non–standard Computation:

The ability to invent procedures to solve standard and non-standard computation problems is generally seen as an important feature of number sense. The types of errors that are consistently shown in the rational number literature demonstrate that students are overly dependent on the use of procedures. Even when uncertain of the rules, they will misuse a procedure, preferring to accept an improbable answer rather than to invent an alternate strategy. Hatano distinguishes between two types of expertise, routine, and adaptive (Sowder, 1995). People who demonstrate routine expertise are able to perform standard problems with speed and accuracy. It is the adaptive expert that is able to use idiosyncratic and modified procedures to adapt to the constraints of a problem. It is this kind of adaptive ability that allows the problem solver to invent personal strategies to solve mathematical problems.

On the items requiring nonstandard computation, the mean score for the group at pretest was 1.9 (1.81) out of 9 and at posttest the group was able to score 5.85 (2.56). When a paired two-tailed t-test was performed, the difference was highly significant $t = 10.08$ $p = .0001$, with an effect size of 1.84. An example of this kind of invented procedure follows below:

Interviewer:	If a beaker holds 80 millilitres of water, how many millilitres of water would there be if you filled it 75% full?
Student A:	I know. It's 60 mls because 20 mls is 25%.

Student B: Well, 50% of 80 is 40 and 25% (of 80) is 20, so you have to add the 20 to the 40 and you get 60. So the answer is 60 millilitres.

This question is directly related to the types of activities that the children engaged in as part of the experimental curriculum. As can be seen, these students were very comfortable in their reasoning and quickly determined that units of 25% percent (20 mls) would be a useful quantity for her calculation.

The reasoning that the students use in the next example is similar to that of the previous one. However, this item,"Is 7 three-quarters of 10?" is one that students would not have encountered in the teaching sequence. For this problem, the students also used benchmark fractional units. The first student chose 1/2 as a starting point for her reasoning whereas the second student chose 25% respectively.

Interviewer: Another student told me that 7 is 3/4 of 10. Is it?

Student 1: No, it can't be 7, it's 7 and a half. I tried to figure out one half of ten. One half of 10 is 5. Then half of that is 2 and a half. And I added that to 5.

Student 2: No because 25% of 10 is 2 and a half. You need three 2 and a halves to get three fourths so 2 and a half and 2 and a half and 2 and a half make 7 and a half so that's 3 quarters.

And finally I present the reasoning of a student on a more difficult non-standard item.

Interviewer:	What is 65% of 160?
Student:	Oh yah, I can figure that out. The answer is one hundred and four. First I did 50% which was 80. Then I did 10% of 160 which was 16 then I did 5% of it which was 8. I added them (16 + 8) together to get 24. And added that to 80 to get 104.

The reasoning in the above examples clearly indicates that using benchmark quantities and translating among decimals, fractions, and percents was are effective strategies for solving non-routine problems.

Table 4.9
Percentage of Students Succeeding on Items Requiring Some Form on Non-standard Computation, Before and After Instruction

ITEMS	PRE	POST
How much is 50% of $8.00?	81	100
How much is 10% of 90 cents?	10	91
6 blocks spilled out of a bag. This was 25% of the total number of blocks. How many blocks were in the bag to begin with?	33	81
If a beaker holds 80 ml of water, how many mls of water would there be if you filled it 75% full?	24	76
Another student told me that 7 is 3/4 of 10. Is it?	5	52
What is 65% of 160?	5	52
The school went on a trip to hear Ani DiFranco. The total number of students in the school is 814. 70% of the students attended. How many students would that be?	0	33
What is .05 of 20 candies?	5	29
How much is 1% of $4?	5	24

4.3.4. Standard Computation: Differences Between High- and Low-Achieving Students

As well as assessing students' performance on number sense competencies, one of the questions of this study was how the students would perform on standard computation items. Eight items were included in the measure that assessed the students' ability to perform standard computation.These items were not easy; in fact many of them have been cited in the literature as being particularly difficult.

When the pre- and posttest results for these group of items were analyzed it was revealed that the students scored only 0.5 (.17) out of 8 on the pretest, 4.0 (3.4) at posttest. Although these scores were low there still was a significant difference from pre- to posttest with a t value of 5.19 ($p = .0001$ paired 2 tail t-test).

Given that the class as a whole did poorly on these standard computation items, I was interested to see if there would be a significant difference in the way that the higher-achieving students were able to perform non-standard items compared to the lower. When these scores were analyzed it was revealed that the difference was substantial. Students in the lower half of the class were unable to answer any of these questions at pretest and achieved only a posttest score of 1.8 (1.7). The high-achieving group also experienced difficulty on the pretest achieving a score of 2.3 (2.1). By contrast with the low-achieving students, the posttest score was high, 6.3 (3.2). When a repeated measures ANOVA was conducted comparing the gains of the two groups from pre- to posttest it was discovered that there were highly significant differences both within and between groups $f = 13.86$ $p = .001$ 40.992 $p = .0001$ and of subjects and $f = 10.44$ $p = .0001$.

In Table 4.11, I present the items that comprised the category of Standard Computation. The data that is presented on this table is of two kinds. In the first columns I present the pre- and posttest scores of the group as a whole. And in the righthand columns I include the separate posttest scores on all of the items that were achieved first by the upper half of the class and then the lower half.

Table 4.11
Percentage of Students Succeeding on Items Requiring Standard Computation, Before and After Instruction, and Posttest Scores Comparing High and Low-Achieving Students on these Items

ITEMS	Total Group Pre	Total Group Post	High-Achieve Post	Low-Achieve Post
How much is 1/2 x 1/8?	5	71	80	60
How much is .5 + .38?	5	53	80	27
How much is 3.64 - .8?	5	45	70	20
What is 2 1/4 + 3	0	38	70	9
What is 3 x .4	0	35	50	20
What is 4 3/4 + 6	5	33	70	0
How much is 3 1/4 - 2 1/2?	5	33	70	0
How much is 2/3 of 6/8?	0	9	20	0
Total Mean score	.5	4.0	6.3	2.3
Standard Deviation	(.17)	(3.4)	(3.2)	(1.)

4.3.4.1. Reasoning Strategies for Standard Computation Items

As can be seen in the table above, when the scores were broken down, the upper half have were far more able than the lower half of the class to perform the standard computation problems. In fact, the lower half was only successful on the one item that directly reflected the instruction in the experimental curriculum, i.e., "How much is 1/2 X 1/8?"

In the following section I present examples of the reasoning strategies that the high-achieving students used to solve these standard computation problems. For each of these protocols I have analyzed how the students response relates to the instruction. Because these students had not been taught any formal algorithms, there were a number of interesting ways that they approached these tasks.

Interviewer: What is 4 3/4 + 6 6/8?

Student: Easy, cause six eighths is three quarters so 3/4 + 3/4 equals one and a half. So one and a half plus six, plus four, equals eleven and a half.

A second student used a different approach.

Interviewer: What is 4 3/4 + 6 6/8??

Student: Ok, so 4 and 6 equals 10. And 3/4 is 75%.

So 6/8...let's see. Two eighths is 25%, so, if I times three, it is 75% So 75% and 75% is 1 point 50, so that is 10 and 1 point 50, so it is 11 point 50.

As can be seen, these students were able to solve these standard problems in a non-standard way, namely, by translating among the representations of rational number. A similar strategy was used by a another student on a different computation item:

Interviewer: What is 3 1/4 - 2 1/2 ?

Student: Um..3 and a quarter is 3 point 25 and 2 and a half is 2 point fifty. (Then she took her pencil and wrote horizontally) 3.25 -2.50 = .75.

Finally I present the reasoning of a student on a particular challenging item who showed her successful understanding of traction multiplication and highlighted by the operator subconstruct .

Interviewer: What is 2/3 of 6/8 and how would you explain your method for answering this question?

Student: (First the student drew a circle partitioned into eight and shaded 6 parts). Well 6/8 is this, which is the same as three quarters or 75%. Well 25%

> goes into 75% 3 times. So 1/3 of 75% is 25. But
> you need *two* thirds so it is 50%, and that's a half.

4.4. Summary and Conclusions

The study reported in this chapter was designed to answer four questions. The first question was whether the experimental curriculum would be effective with a mixed-ability class. The overall results of 49-item measure indicated that these mixed-ability Grade 4 students made substantial gains at posttest in their ability to perform a wide variety of rational number tasks. While a further analysis of the three separate subtests, Percents, Decimals, and Fractions further confirmed this finding, the gains on these subtests were uneven. In each of the subtests the students improved significantly. However, the improvement on the percent subtest was more substantial with the students achieving an effect size or 2.1 on this subtest compared to 1.4, and 1.2 for the Decimals and Fractions subtests respectively. This finding is probably linked to the strong emphasis placed on percent teaching in this curriculum and is perhaps to be expected. However, at the same time, it must be noted that the results indicated that Grade 4 students can achieve success in learning percents, a topic that is considered to be so difficult that it is typically not introduced until middle school.

Another question concerned the performance of these students on the four number sense categories; Compare and Order, Interchangeability, Nonstandard Computation, and Misleading Visual Features. The students' performance on these categories also improved greatly from pre- to posttest. However, similar to the way they performed on the subtests of Decimals, Fractions, and Percents, there were uneven gains at posttest on these number sense categories. The strongest areas of improvement were in the categories of Compare and Order and Nonstandard Computation, where students' number sense flexibility seemed most apparent. Although there was significant improvement on the other two number sense categories, Interchangeability and Misleading Features, the gains from pre- to posttest were more variable. Although students were able to perform tasks that were related to the curriculum, they were less successful on items where the numbers were unfamiliar (e.g., questions incorporating fractions such as 1/5, 11/12, etc.). They showed similar problems when solving operations with numbers that were not easily halved. These findings were consistent with our hypothesized developmental model where we anticipated that students' ability to translate among representations would follow their work with halving and doubling and comparisons within representations. Still, I anticipate if the curriculum had been longer and I had incorporated more exercises that featured interchangeability, particularly with more cahllenging numbers, the students may have been more able to perform these tasks.

Another question concerned the differences in performance of the high- and low-ability students. Since the students who participated the first study were all high achievers this study was designed to assess whether low-achieving mathematics students would also benefit. When the effects of instruction were looked at for high- and low-achieving students separately, it was clear that the high- and low-achievers benefitted equally from instruction in their understanding of percent. However, although there was significant gain from pre- to posttest on the Decimals and Fractions subtests for both the high- and low-achieving students, the upper half of the class gained significantly more on these subtests. Thus while the high-achievers can extrapolate their learning of percents to perform successfully in fractions and decimals, the low-achievers appeared to be relatively poor at this. Thus, according to these results a conjecture is that the low-achievers were not yet developmentally ready for this more advanced instruction.

A third question in this study concerned the performance of students on standard computation items. Although not a feature of number sense, the importance of performing standard computation can not be overlooked. It is this ability that validates research and is generally used as the standard to judge competence. The analysis of the items of standard computation revealed that it was only the high-achieving students who were able to pass these items. Quantitative analyses revealed that the strategies they used were based on the halving benchmark quantities that they had learned as well as on their ability to translate among the representations. Thus I conclude that this curriculum was effective in helping these more advanced students to perform these standard operations. It is hoped that the lower-achieving students would, with time, be competent in this regard as well.

In the next chapter I report findings of a subsequent study in which the curriculum was presented to an older group of students who had already received previous instruction in rational number. As will be seen, I was able to consider some of the questions that arose in the present study as well as address several new areas of interest.

Chapter 5

An Intervention Study with a Traditionally Trained Mixed-Ability Grade 6 Class: Comparison to Normative Groups

5.1. Introduction

In this chapter I report a study that I conducted to teach the experimental rational number curriculum to a group of mixed-ability Grade 6 students who had all received several years of previous instruction in this number system. By selecting older students for the present study I hoped to further assess the robustness of the curriculum. In particular, I hoped to investigate: 1) the efficacy of the curriculum for students who have had previous traditional instruction; 2) the potential differences of learning outcomes when the curriculum was shortened and modified with older students in mind; and 3) differences in learning outcomes of high- and low-achieving students. I also had questions of a developmental nature as I was interested in comparing the posttest performance of these Grade 6 students with students at other grade levels. Thus, I conducted a further investigation, or an "auxiliary study," in which I administered the Rational Number Test to four other normative groups of students at several other grade levels, in order to investigate some of these questions. All of these questions will be elaborated in the sections that follow.

5.2. Question 1: Effectiveness of the Curriculum for Traditionally Trained Students

The first question was how the experimental curriculum could work for students who had already had several years of learning about rational numbers and had formed their understandings based on the traditional teaching sequence with the learning of fractions as a foundation. Which types of misconceptions would be more robust and which would be most easily changed?

It is well known that when students encounter reform curricula that require a greater depth of mathematical understanding, it is difficult for them to abandon old thinking patterns that they have developed as a result of traditional teaching. Bad habits, reliance on rote and often faulty calculations, resilience of initial representations and images (Kerslake, 1986; Silver, 1986: Sowder, 1992, 1995) and a lack of disposition to meaning-making, all contribute to the difficulties they encounter when remediation or reform is attempted (Kamii, 1994). Thus, given that to date all of the students who have participated in the experimental intervention had not had previous training in rational number, I considered that they had an advantage over those with previous training. A primary goal, therefore, of this study, was to evaluate the effectiveness of the experimental curriculum when the participating students were not new to the topic of rational number but, on the contrary, had received several years of traditional teaching and formed their understandings based on the conventional sequence (fractions, decimals, percents), with learning fractions and the part/whole subconstruct as a foundation. Assuming that change is possible at all, an important analysis would be to consider which types of misconceptions would be more resistant to change, and which would be most easily restructured, or transformed.

5.2.1. Traditional Training and Experimental Curricula—a Review

Three researchers who have dealt have dealt directly with evaluating change in rational number understanding of traditionally trained students are the team of Heibert and Wearne, who have done extensive work in students' understanding and performance in decimals, and mathematics education researcher Nancy Mack, who has paid particular attention to students' informal knowledge of rational number, particularly fractions. In the next section, I report the findings from studies that were conducted by Heibert and Wearne and by Mack in which the learning gains of previously instructed students were monitored during experimental interventions, and then gains were assessed after the instruction was completed.

5.2.2. Hiebert and Wearne: An Intervention Study with Decimals

In a training study involving students in Grades 4, 5, and 6, Heibert and Wearne instructed small groups in semantic processes for solving decimal tasks (Wearne & Hiebert, 1988). The students were taught to use base-10 blocks, traditionally used for whole number addition and subtraction, as an alternative representation to written decimal symbols. Their curriculum was designed to first help students make connections between the symbolic representation of decimals and these physical referents and then to support a conceptual understanding of procedures such as the addition and subtractions of decimals. A series of nine instructional lessons was designed to help students to create meaning to solve problems that were posed symbolically. The older group of students (n = 15) who participated in this experiment had received previous instruction in decimals. The younger students (n = 14) had not had any instruction in decimal prior to this teaching unit. After two weeks of instruction, they found that although all of the students were able to perform tasks that were directly related the instruction, the results on the transfer tasks at posttest were different. On these tasks, 11 of the 14 students who had not received prior instruction used semantic rather than syntactic explanations. By contrast, only 5 out of 15 students who had received prior instruction were able to make similar gains.

5.2.3. Mack: Intervention Studies with Fractions

Similar findings have also been reported by Mack (1990, 1993, 1995), who has done extensive work investigating students' informal knowledge of fractions. In her experimental studies, Mack worked with Grade 6 students to help them to connect formal fractions symbols and procedures with concrete, everyday representations of fractions. The central foci of her training studies were to promote the understanding of concepts such as 1) the more partitions, the smaller the part; 2) a fraction represented symbolically is a single number with a specific value, rather than two whole numbers; 3) the addition and subtraction of fractions requires the same denominator; and, 4) fraction knowledge is underpinned by an understanding of equivalencies. In short, her goals in these studies involved the basic concepts of fractions of the sort that are expected in most Grade 5 and 6 curricula. In monitoring their progress during the training period, she noted that the students tended to continue to use algorithmic solutions, even when they were uncertain of their correct application. More disturbingly, she noted that, if the answers that the students derived using standard algorithms were different from correct answers that they had found through invented procedures for solving the same problem, the students either chose the incorrect answer or suggested that both answers were correct.

Although the students did make gains at posttest, their previously taught algorithmic solutions continued to interfere with their solutions. Mack concluded from this (and other studies that she had conducted) that overcoming misconceptions and faulty procedures based on prior learning was very difficult for students to accomplish, and required a significant effort on the part of both students and teachers.

The findings of these researchers are certainly compelling, and would lead to a prediction that a rational number experimental intervention—certainly one that is a brief as ours—would not easily achieve our ambitious goals of changed conceptualizations and rational number sense. One might predict that the Grade 6 students in this present study would have difficulty abandoning the misconceptions that they had developed due to previous instruction, and thus would have trouble acquiring an overall conceptualization of the rational number system.

However, it is my contention that, while the Mack and Hiebert and Wearne studies are exemplary in both the structure of the lessons and in their detailed analyses of children's understandings, the activities that they have created, are grounded in concepts from the whole number domain. Thus, I propose that these two separate interventions both tend to reinforce additive reasoning.

In the study conducted by Hiebert and Wearne the concrete referents that they used were Deines Base-10 blocks. I argue that while base-10 blocks have the potential for illuminating the power of 10 relationship in adjacent numerals, they serve to reinforce elements of the symbol system for whole number as well as to ground students reasoning exclusively in discrete (rather than continuous) quantities. Mack's instructional sequence is firmly rooted in students' intuitions about partitioning and fair sharing; thus she has chosen to use pizza pies as the central representation. My central thesis argues against building up children's understanding of rational number either from the symbol system of the whole number domain, on the one hand, or, as Mack does, from students' readily available store of partitioning intuitions. What is missing in both of these approaches, and what our program attempts to provide are learning contexts where children can explore their intuitions for ratio and proportion as well as their intuitions for halving.

Given these considerations, I hypothesized that the posttest performance of the students participating in the present study would be different from those reported in the literature and would show that the children had, in fact, abandoned many of their misconceptions, in favour of an understanding that is grounded in the multiplicative nature of rational number. I anticipated that this finding would be instantiated by quantitative analyses, where there would be strong changes in scores on the measure from pre- to posttest. I also predicted that qualitative analyses would reveal that the students had adopted new methods of reasoning and strategizing and that these methods would fall under the general rubric of number sense.

5.3. Question 2: . Differences for High-Achieving and Low-Achieving Students

A second question that drove this present study concerned the differences in performance of high- versus low-ability students. As in the previous study, I wondered if students who are designated as "higher-achieving" or "lower-achieving" would gain differentially on the measure from pretest to posttest. Recall that in the previous study, the experimental curriculum was shown to be most effective for the most advanced students in Grade 4. As mentioned before, one possible explanation was that the lower half of the class was not developmentally "ready" for this curriculum and therefore could not benefit as much from the instruction. Thus, I wanted to assess whether the approach would be more useful for students who were two years older and thus more uniformly capable of the complexities posed by this number system; would low-ability students who are two years older benefit more evenly than had the low-ability Grade 4s? My conjecture was that these older students would in fact, benefit more evenly than the younger students, as the developmental literature indicates that they are at a more suitable age for rational number learning (e.g., Case, 1985; Hart, 1988; Lamon, 1993, 1994; Noelting, 1980a, 1980b; Resnick & Singer, 1993). Thus, I anticipated that the rate of gain for these Grade 6 students would be equal for students in both the high- and low-ability group. This question is of some theoretical interest, in view of the controversy in the developmental literature concerning the first emergence of ratio thought and the effect of context and task performance (Lawton, 1993; Sophian & Wood, 1997; Spinillo & Bryant, 1991). However, it is also of practical interest since it bears on the question of at what age or grade a program like ours should first be first introduced into the mainstream curriculum.

5.4. Question 3: Developmental Questions

The final questions were of a developmental nature. The first of these concerned the magnitude of improvement that can be expected from the program, in developmental terms. It is generally acknowledged that children's understanding of ratio and proportion continues to develop, long past the years when these topics are taught in school. The same is true for many other aspects of rational number understanding, at least in middle-class populations (Case,1985; Cramer, Post, & Currier, 1993; Vergnaud, 1988; Watson, Collis, & Campbell, 1995). Given that this sort of development continues to take place, under conditions where the part/whole subconstruct of fractions is used as the core organizing device, a developmental question that naturally arises with any new program is whether the children who receive it are being enabled to construct understandings that they would never have constructed under existing curriculum conditions, or, whether the usefulness of the curriculum was that it accelerated students' construction of understandings that they would have achieved anyway, but at a later point in their schooling. Stated in more quantitative terms: what magnitude of improvement can be produced by our new program, not in terms of standard deviations, but in terms of developmental advance? Is it an advance of one year, two years, three years? Or do our children attain understandings that they would *never* attain under the standard rational number curriculum? In order to answer these questions I conducted a developmental study in which students at a variety of ages were interviewed using the measure that was designed for the Grade 6 intervention study. I then was able to compare the results of the Grade 6 students performance to that of the various groups in this developmental sample.

5.5. Method

Since this study was comprised of two separate strands, (the experimental intervention study and the normative testing) reporting of each subsection of the methods will be divided into two parts.

5.5.1. Design

Experimental Study

In the spring of 1997, one week prior to the start of the experimental instruction, the 20-item Rational Number Test was administered as a pretest interview to each of the Grade 6 subjects, on an in individual basis. Immediately following this interview process, the students were instructed over a four week period. Finally, three weeks after the instructional sequence the same measure was re-administered as a posttest to evaluate the effectiveness of the intervention.

Normative Study

The same measure was used for the normative study. At the beginning of May 1997, one week after the experimental Grade 6 study was completed, a team of interviewers administered the same 20-item measure to students first in an elementary and a middle school and then to students in a two-year MA program in elementary teaching at the University of Toronto.

5.5.2. Subjects

Subjects for Experimental Study

Sixteen Grade 6 students participated in this study. These children comprised the entire class of students at a private school located near the University of Toronto. This school is known for its strong academic programs, small classes, individualized attention, and a strong commitment to quality instruction in mathematics. The students all come from high SES backgrounds. According to their classroom teacher, most of these children were performing at grade-level, and in some cases approximately a year above grade level. Four of the students in this class ranged from one to two years below grade level; each of these students received extra tutoring from the school's special education teacher. Because the school is very small and there is only one class per grade, all of the students had been in the same class since Grade 1 and thus had all received the same previous instruction in rational number prior to the intervention. This instruction was based on a widely used Canadian text series. Although the teachers in the school used other resources for teaching rational number as well as other mathematics topics, the basic sequence and the core concepts were based on the series. The sequence is as follows:

The first topic in the text was fractions which were defined as numbers that describe parts of a whole and which were illustrated with pie chart diagrams. Exercises followed in which children were to determine fractions of a set, compare different fractions with regard to magnitude, and determine equivalent fractions. Decimals were taught next, using pie graphs, numberlines, and place value charts. Tenths were introduced first, and their relation to single-place decimals was shown. Finally, equivalent decimals were taught by showing that numbers such as 0.3 and 0.30 are merely alternate representations of 3/10 and 30/100. Lessons involving operations with decimals were introduced next. The rules for addition and subtraction of decimals, as well as for multiplication of one- and two-place decimals were taught explicitly, with careful attention to the significance of place value. The use of a fraction as an operator and computations involving division of decimals were taught at the end of the sequence.

Subjects for the Normative Study

(a) Grade 4 Students (n = 21)

Twenty-one Grade 4 students were interviewed, using the 20-item measure. All of these students attended a public elementary school that caters to a predominantly Caucasian population, with a small percentage of second-generation Asian students. This particular school was selected because it had received the highest standings in the city on the Grade 3 Provincial Tests in Mathematics and Language. All of these students spoke English as their first language. The Grade 4 students came from two different classes and all participated on a voluntary basis—only students who returned consent forms signed by their parents were allowed to participate. Although the students came from different classes, they all had received the same number of rational number lessons, and had been exposed to fractions in Grade 2 and decimals in Grade 3 and continued to learn these rational number representations in Grade 4. This school used the same text series that was used by the experimental group.

(b) Grade 6 Students (n = 45)

Two different groups of students comprised the Grade 6 sample. The first, n = 31 were drawn from the same school as the Grade 4s. These 31 students were drawn from 3 different classrooms. The conditions for participation were the same as those mentioned above. The students in these Grade 6 classes had received a substantial amount of rational number instruction prior to the test, and had already covered fractions, decimals, percent, and ratio. The textbook that was used by these students was the same one that the experimental groups in the preceding studies had been using. The second group of Grade 6 students (n = 15) were the participants in the experimental intervention study. More detail of this group will be provided in the following section of this chapter.

(c) Grade 8 Students (n = 20)

The students in the Grade 8 sample came from the junior high school fed by our sample elementary school. These students also participated on a voluntary basis and were also required to present signed consent forms. At the time of the interviews, the students were working on percent computation problems and ratio in their mathematics classrooms, and had covered all of the topics on the measure.

(d) Pre-Service Education Students (n = 31)

This group was comprised of 31 postgraduate students enrolled in a two-year elementary school teacher training program. At the time of the testing the students were in their second month of the first year of their program. All of these students had completed mathematics in high school, approximately 80% of this group had taken statistics as part of their undergraduate programs and five of these students had taken university mathematics courses. It must be noted that the program in which these students participated had very high academic criteria for acceptance. Thus, each student in the program had a minimum average of B+ on completing their undergraduate programs, and all had taken high school mathematics.

5.5.3. Testing Procedure

Experimental Students

In the spring of 1997, one week prior to the start of the experimental instruction, the 20-item Rational Number Test was administered as a pretest interview to each of the Grade 6 subjects, on an individual basis. The researcher administered one half of the tests, and a graduate student who was trained to perform the interview, administered the rest. The interviews were standardized, and all the students answers were recorded verbatim by the interviewers.

As in the preceding studies, the children were withdrawn from their regular class and brought to a quiet room. Administration time for the pretest varied from 25 minutes to 45 minutes, according to the knowledge level of the student. The test in its entirety was read aloud to the students; the interviewers encouraged the students to respond to all of the items and praised them for attempting each item. However, at no time did they indicate whether the student had responded correctly, nor did they ever reveal the correct answer. The researcher taught the experimental rational number curriculum to the students, in their regular classroom, over a four-week period. The homeroom teacher observed most of the lessons but did not participate in any of the experimental teaching. This classroom teacher taught all of the other mathematics on days when the researcher did not come to the class. In all, there were 12 rational number classes that were approximately 45 minutes in duration. A breakdown of the lessons is presented in the section below.

Two weeks after the experimental program had been completed, the pretest measure was administered again as a posttest. These interviews ranged from 25 minutes to 70 minutes according to the needs of the individual subjects.

Normative Sample

The procedure for interviewing the students from the normative sample was exactly the same as it had been for the participants of the experimental study. Each student was escorted from their classroom by a trained interviewer, to a quiet testing room in their own school. Two of the three interviewers for this group were also the interviewers of the Grade 4 study. The third interviewer was a different graduate student who was also trained to administer the tests. The interviewer read the questions out loud and recorded the students' answers verbatim. The interviewers informed the students that they would be able to re-read the questions as many times as requested; however, they also indicated to the subjects that they would not be able to further elaborate on or clarify a question. Students were provided with paper and pencil to make notes or work out their solutions. These were kept by the interviewer for later analysis. Regardless of a student's success on the items, the interviewer provided him or her with an opportunity to try all the questions on the measure.

5.5.4. The Experimental Rational Number Curriculum

Instruction Schedule

The experimental sessions were approximately 40 minutes in length. All of the lessons were documented and selected classes were videotaped. Each lesson was reviewed at the end of each instructional day for the purpose of assessing student thinking and subsequent lessons planned to build on students' developing understandings.

Special Features of the Grade 6 Experimental Curriculum

The sequence and style of the experimental Grade 6 curriculum was exactly the same as it had been for the Grade 4 intervention. Percents in a measurement context served as the introduction as well as the reference point for all subsequent learning of the other representations. Hands-on activities that focussed on measurement were also featured. As well, in the Grade 6 curriculum, there was no teaching of formal algorithms. Finally, an endpoint for the students was a focus on mixed-representations of fractions, decimals, and percents.

There were, however, differences in the curriculum. Because these students were in Grade 6 and the topics that needed to be covered at that grade level were more extensive, and included a component on formal ratio, I expanded the curriculum and covered ratio concepts through the incorporation of scaling activities. Scaling activities such as enlarging rectangular regions promote ratio thinking in a measurement context, which is consistent with the measurement objectives I had for the Grade 4 students. The demands of the Grade 6 classroom schedule, and the limited time that could be allotted to this rational number teaching, meant that although the general content of the curriculum was similar, and the starting point of percent in measurement was the same, I had to reduce the total number of lessons. The sequence that I taught to the Grade 6 class was as follows:

1. Introductory lessons which included percents in measurement contexts, visual estimation of relative quantities, and invented procedures for calculating with halving and doubling.
2. Scaling activities which provided links to their formal ratio and proportion background.
3. Translation among mixed representations.

The lessons ranged from 45 mins to 1 hour, with a final lesson of 1 1/2 hours.

Table 5.1 shows the breakdown of time devoted to the different representations.

Table 5.1 Breakdown of Topics and Teaching Hours of the Experimental Rational Number Curriculum for the Grade 6 Students

Subject	Lesson #	Hours
Percent	1, 3, 5, 6	4
Decimal	4	1
Fraction/Mixed-representation	2, 7, 8, 9, 10	4
Review	11, 12	4
Total no. of hours		13

The modes of participation that I employed in the classroom included the same structures as the previous two experiments: teacher-directed whole group discussions, work in pairs and small groups, and opportunities for students to instruct their colleagues either through games or videos which they had created. Since there were fewer Grade 6 lessons than Grade 4 lessons, and since only the first and fourth lessons in the Grade 6 sequence contained the very same activities as those in the Grade 4 lessons, I include here a brief description of each lesson:

Day 1. Introduction to Percents

The intent of this first lesson was to assess students' informal knowledge of percents, and gain insight into the ways that they would spontaneously operate with percents in a measurement context. The activities closely followed the structure of the first lessons in the experimental Grade 4 classes. However, in the context of this study, the activities of Lessons 1-3 were incorporated into a single session. These lessons started with visual estimation and carried on with calculations using the halving strategy. The students used pipes and tubes to demonstrate their informal knowledge of percent. They completed a variety of measurement tasks which included, 1) finding and calculating percentages of water in beakers, and, 2) cutting lengths of string to represent various percent measurements of objects in the classroom (i.e., "The length of this string represents 75% of the length of this table"). At the end of the lesson, students were asked to consider the relation between percent and decimals and were encouraged to represent quantities or relations in both percent and decimal modes.

Day 2. How Could You Show and Teach Percents?

In order to consolidate their recently acquired insights into percent quantities and operations, students were challenged to design their own props to teach percents. As in the Grade 4 curriculum, I provided the students with a variety of materials including coloured sand, jars, string, paper tubes, and laminated numberlines.

Day 3. Paper Folding Horizontal Fraction Strips

This lesson was designed to use the familiar context of halving and doubling to explore the formal symbolic representations of decimals, fractions, and percents and their interrelatedness (e.g., 1/16 = 6.25%). The folding activities were adapted from classroom lessons that had been devised by Kieren (1992, 1995) to provide his students with concrete representations to help them recognize and learn about fraction equivalencies, and which they could then use to perform simple addition and subtraction operations on fractions. My intention was different—the focus was on formal representations of quantities that were established through halving and doubling. Students folded rectangular paper strips that were 24 cm x 8 cm. Each successive fold resulted in the creation of new quantities that the students in turn labelled using the three representations—percents, decimals, fractions. Thus, for example, after performing a single symmetrical fold, the students created two equal parts, which they were instructed to label as 1/2, .50 and 50%—all familiar symbols—or, after three folds, they established and labelled portions as 1/8, .125 or 12 1/2 %—much less familiar symbols In this lesson, students gained experience with recursive halving and were able to review standard notation for the three representations.

Day 4. Stopwatches and Centi-seconds

As in the preceding study with Grade 4 mixed-ability students, I once again introduced stopwatches to promote the notion of a temporal analog to linear measurement. As the Grade 6 students in this experiment had already received several years of decimal teaching, the stopwatches provided them with opportunities to add and subtract decimals in meaningful contexts where they did not need to rely on rules such as "lining up the decimal before calculating." By using stopwatches, the students were able to work with these decimals from an understanding of their magnitude and thus were much less likely to make the typical kinds of computation errors that these students demonstrated in their pretest interviews.

Day 5. Percent/Proportion of Body Parts to Height

In this lesson, students worked in pairs, using percent language to compare length of body parts to height. The students were provided with a blank table with two columns: the first column was headed "body parts" and this column was followed by 4 blank columns to the right. The students were first instructed to record their estimates of the proportion of the body part to the height based only on visual cues. Next, they performed and recorded a more accurate estimate which they derived by folding a piece of string that had been pre-cut to the measure of their height. Following this, using the string and rulers or measuring tape, they recorded both their height as well as the length of their body parts in centimetres. Finally, using calculators if they desired, they computed the actual percentages of the individual parts to their heights. A lively discussion ensued where students shared their calculations and discussed and compared their findings.

Day 6. The 50 cm Man: A 1:2 Ratio

This lesson directly followed the previous day's lesson of comparing of the length of body parts to heights using percents. However, although one of the purposes of this lesson was to continue to estimate percents and calculate averages this lesson had an additional goal, which was to give students the opportunity to experience a 1:2 scale drawing. Students first collected data from all of their classmates on the various percent calculations that they had performed in the previous lesson. For example, it was established after averaging all the percent measures that the students had generated, that length of circumference of the head to the height was 22%. (The percentages for this comparison ranged from 24% to 18%). Students were then given 1 cm graph paper and asked to draw a "proportion person" with a height of 50 cm using the data from the class as a whole as their guide.

Days 7 & 8. Find the Odd Item: Compare and Order Mixed-Representations

In the next few lessons students worked with equivalencies in mixed-representations. These types of exercises had been done in the Grade 4 curriculum. However, for the Grade 6 students, I extended the fraction learning and included the symbolic representations of ratio. To start with, I presented students with lists of mixed-representations, all but one of which were equivalent quantities. For example, the students were presented with the following row of numbers and relationships (32/64, 0.5, 1:2, .05, 50%, 3/6, 4:8) where all of the quantities represented 1/2 except for .05. The students were asked to determine which quantity was not equivalent and then to justify their choice to their classmates. After the students responded in a whole class format to a number of these exercises that I had designed, they were invited to create their own similar series of numbers where they presented equivalencies and inserted one or two anomalous items.

Day 9. Rational Number Equations

In this lesson the students were presented with written versions of operations that previously they had invented and solved mentally. For example, True of False?

1. Does 45% of 80 = [4 x (10% of 80)] +(5% of 80) ? ________

2. Does 45% of 80 = .45 x 80 ________

3. Does 45% of 80 = (50% of 80) - 5 ______

(Example 3 represents a type of error that students typically made in the early lessons when they first began to invent procedures for calculating percentages).

Day 10. Playing and Inventing Games of Multiple Representations and Computation

This lesson started with an ordering activity in which the students were challenged to order a deck of specially created cards on which rational number quantities using multiple representations (fractions, decimals, percents, ratio, and geometric regions with portions that were shaded) had been written. These cards were similar, although more complex than those that students had created in Study 2 (see Lesson 14 of the Grade 4 curriculum), except for these Grade 6 students. In the first activity the entire pack was dealt out to the students (about four cards each) and they were challenged to place them in order of increasing quantity face up on the classroom floor. The game ended when all the cards were lined up along the floor and all of the students agreed that they were correctly placed in ascending order. Since the rule was that consensus amongst all the students in the class, this game promoted a great deal of debate as the goal was that consensus would be reached amongst all the students in the class. In the second card game that the students were instructed to play, the same deck of cards was used as well as LCD stopwatches. The structure of this game was similar to the popular game of War. Two children sat opposite each other with a small pile of face-down cards. Both students revealed the top card in their face down pack at the same time. The first player, Student A, was required first to declare which of the two cards was higher or if they had the same value. If the two cards were not of the same value, Student A had to mentally subtract the higher from the lower and then use the stopwatch centi-seconds to indicate that difference. If the two cards that the students drew were the same quantity, then the player whose turn it was, had to find the sum of the two quantities and stop the watch as close to that sum as possible. The game ended when there were no more cards left in the pile. Once the students had played this particular game they were instructed to invent their own games, using cards and stopwatches or other materials such as numberlines, that they felt could be good teaching tools for teaching multiple representations and calculations. Finally the groups of students who worked on these games presented them to the class.

Days 11 & 12. Wrap Up and Review

The students continued to design their games and then to teach them to their classmates. As well, a number of the students videotaped the sessions in which their own games were being played.

Summary of Lessons

In summary then, we maintained the basic sequence and tenets of the curriculum as it was taught to the Grade 4s so that students would be able to relate both their previous and newly gained knowledge of fractions and decimals to the relational construct of percents.

5.5.5. Assessment Measure

The measure that was designed for this study was to serve several purposes. As part of the experimental intervention study I wanted to continue to evaluate students' pre- and postconceptual understanding and number sense abilities across the three representations of rational number; Percents, Decimals, and Fractions. To this end I retained a total of ten items that had appeared on the previous measures. As I have mentioned, however, I also wanted to use this same measure to interview subjects across a wide age range. Thus, I included items on this measure that spanned a broader range of difficulty than on the previous measures and I shortened the test to 20 items to make it more viable for use with a much larger target group. The most difficult (Level 3 items) were multistep items that required conceptual understanding and number sense flexibility (e.g., "Order from largest to smallest: .48, 5/8, 14/13, .99, 1.03"). In all there were 8 items in this category. The least difficult of the items (Level 1) assessed computational halving and proportional computations based on 1/2: ("One can holds 1 quart of oil which is the same as 2 pints. The other can holds 3 quarts of oil. How many pints will it hold?") The "middle" items (Level 2 items) required that students display rational number understanding or number sense, but use only a single operation, e.g., "What is 6% as a decimal?" Finally, I also included two items from previous studies in the standard literature, both of which have proven difficult: Mr. Tall and Mr. Short (Karplus et al., 1974) and an item from a National Assessment of Educational Progress study which is "Estimate the sum of 11/12 + 13/14" (Lindquist, 1989).

Immediately following the lessons, I administered half of the posttests and a student teacher administered the other half. As there was no difference in scores based on tester bias, I proceeded to analyze the data.

5.6. Results and Discussion: The Grade 6 Intervention Study

These results are presented in three sections which are ordered in the same way as the questions that were posed at the beginning of the chapter.

5.6.1 Pretest Misconceptions of Traditionally Trained Students

Because I was interested in understanding students' misconceptions at pretest (that they presumably held due to previous instruction), I examined the pretest results to determine if some pattern might be found. Below, in Table 5.2 I present the entire measure with the pretest scores that the students' achieved. The measure is presented in the order that it was designed, according to the three hypothesized levels of difficulty, with the simplest questions first, concluding with the most difficult items. As can be seen, the students were able to correctly answer a little more than 1/2 of the questions at pretest (mean score 10.33, out of 20). However, it also must be noted that the measure contained very easy items that were included to investigate basic understandings that we would have considered to be present in much younger students (five Level 1 items); these were all passed by the Grade 6s. If we exclude these items from the Grade 6 results, the success rate is closer to 30%. Thus it is evident that the Level 2 and 3 items on the measure presented substantial problems for the students at the outset of the program. Recall that these students had all received sufficient instruction in rational number prior to the intervention so that none of the content of these items should have been unfamiliar to them.

Table 5.2
Percentage of Students Succeeding on Levels 1, 2, and 3 Items at Pretest

Level One Items	
Draw a line on beaker to show it is approximately 1/2 full?	100
Now, draw a line where 1/4 full would be.	100
What is 50% of $8.00?	100
What is half of 84?	93
One can holds 1 quart of oil which is the same as 2 pints. The other can holds 3 quarts of oil. How many pints will it hold?	86
Total Mean	4.9
(Standard Deviation)	(.41
Level Two Items	
Can you shade in 3/4 of the pizza (divided in 8 pieces)?	73
Which is bigger, .20 or .089?	67
Tell me a number that comes between .3 and .4?	60
If a beaker holds a total of 80 ml of water, how many mls of water would there be if you filled it 75% full?	60

Write 6% as a decimal.	40
Find 1/4 on this numberline. 	40
How much is .5 + .38?	46
Total Mean (Standard Deviation)	3.9 (2.0)
Level Three items	
Estimate the answer to 12/13 + 7/8.	33
Order from largest to smallest: .48, 5/8, 14/13, .99, 1.03	33
Mr. Short's height is 4 matchsticks. Mr. Tall's height is 6 matchsticks. When we measure their height with paperclips Mr. Short's height is 6 paperclips. How many paperclips are needed for Mr. Tall's height?	33
Is 7 3/4 of 10? Explain your answer.	26
What is 1/8 as a decimal?	13
Is there a fraction between 1/4 and 2/4?	13
What is 65% of 160? Explain your answer.	6
Mrs. Cheever is 50% taller than her daughter. Her daughter's height is ___% of Mrs. Cheever's?	0
Total Mean (Standard Deviation)	1.6 (1.4)
Total Score On all 3 Levels (Standard Deviation)	10.3 (3.33)

5.6.1.1. Problems with Magnitude, Symbols, and Operations with Fractions at Pretest

Not surprisingly, the errors that the students made in the pretest interviews were typical of the types of errors that often result from traditional teaching and that have been mentioned elsewhere in this thesis. Although the students had difficulty with many sorts of items on the pretest, the most striking problems were with fractions; students demonstrated problems both in interpreting fraction symbols as well as in determining their magnitude. In fact, they were unable to perform any but the most rudimentary fractions operations.

One question that proved to be very difficult for almost all of the students was "Can you think of a fraction that comes between 1/4 and 2/4?" As can be seen on Table 5.2 only 13% of the students were able to correctly answer this question. While it is true that it is a difficult question, the range of misconceptions that was discovered was surprising. Recall that these students had received four years of fractions instruction prior to this intervention. First, when attempting to answer this question the majority of the students merely asserted that there was no such number. However, when students did make an attempt, their deficiencies with fractions were revealed.

Below I present two different sets of erroneous explanations in answer to the an ittem that requested that the students find a number between 1/2 and 2/4.

Interviewer:	Is there a fraction between 1/4 and 2/4?
Student (1):	Hum.... What number comes between 1/4 and 2/4? I'll take a guess and say 1/3 .
Interviewer:	What made you think it was 1/3?
Student (1):	Well 1 + 2 = 3 so that why I think it is 1/3.

Interviewer:	Is there is a number that goes between 1/4 and 2/4?
Student (2):	Well there is no fraction that goes in between but there is a decimal.
Interviewer:	Excellent, a decimal will do just fine. What decimal
were	you thinking of?
Student (2):	1 point 5.
Interviewer:	Why do you think that it is 1 point 5?
Student (2):	Because you need a number between 1 and 2 so that
is 1	point 5.

Both students based their invented strategies in calculations involving the numerator. Neither of these students included the denominator in their reasoning, thus indicating their lack of understanding of fractions and their reliance on their whole number learning.

A similar lack of understanding was shown in the standard question that was included: "Estimate the sum of 11/12 and 7/8?" Just as reported in the literature, the majority of these students were not able to consider these fractions as quantities, thus realizing that "2" would be the closest estimate. Rather they selected either "19" or "21" as the answer to this question from the multiple choice list, indicating again that they saw the denominator and the numerator as two separate numbers that could be manipulated independently of one another.

Finally, when we requested that students order the following series of numbers .48, 5/8, 14/13, .99, 1.03 we uncovered several misconceptions that we had not hitherto encountered. For example, most of the students subscribed to one of two opposing (both erroneous) ideas. On the one hand some asserted "that fractions are always smaller than "decimals because they are tiny numbers" (one student argued that 99/100 was a "very very tiny number" and that 48/100 was "quite a bit bigger"). The other typical error was the assertion that fractions are always larger than decimals because they have two numbers. As might be expected there were also students in this group who lacking an understanding of the interchangeability of decimals and fractions suggested that it was not possible to order these numbers at all.

Students' performance on the decimal items at pretest were generally better. For example, students were more adept at converting decimals to fractions than they were at performing the reverse operation: thus, in answer to "What is 1/8th as a decimal?" The majority of students asserted that the answer was .8, .08, or 1.8. This same problem was even evident when the students attemted to convert a more familiar fraction, 1/4. As in the above example, students asserted that the answer was .4 or .41.

Another indication of these students difficulties was the confusion they showed in selecting and performing operations. The most obvious problem of this sort was revealed in their answers to the item: "Is 7 three quarters of 10?" As in the earlier studies, students attempted addition, (3 + 4 = 7; so the answer is "yes") or faulty division ("No, because 3 doesn't go evenly into 10"). Finally, it was apparent that students had difficulty working with proportional relations. On the item "Mr Tall and Mr Short" the majority of the students in this class used additive reasoning to find the incorrect solution of 8 (see Chapter 3 for further explanation).

In summary, it was observed that the Grade 6 students had many misconceptions and difficulties with fraction symbols, conversions between representations, ideas of magnitude to select the correct operation to solve them.

5.6.2 . Posttest Results: Changed Understandings

When the posttest scores were analyzed there was a significant improvement found. Table 5.3 presents the pretest and posttest scores for the measure as a whole.

Table 5.3
Total Scores on the Rational Number Test, Before and After Instruction

	Mean Score on Pretest (max = 20)	Mean Score on Posttest (max = 20)
Mean score	10.33	15.53
Standard deviation	(3.33)	(2.16) ***

** p < .01; *** p < .001; **** p < .0001; ns = not significant

When a t-test was conducted it was evident that the posttest gains were highly significant, t = 10.46; p < 0001 with the mean score improving from 10.33 (3.33) for the pretest to 15.53 (2.16) at posttest. An effect score was calculated by dividing the difference 5.2 by the mean standard deviation of the pre- and post scores (2.7) thus producing an Effect Score = 1.9. Analyses of the data revealed that these gains included: 1) an improved understanding of magnitude, symbols, and interrelationships of rational number; 2) the ability to compute with percents and to use percent to represent fractions and decimals; and 3) a fundamental change in the ability to recognize the proportional nature of rational number. Evidence for these changes can be seen in the improvement of students' scores on the individual items from pre- to posttest, changes in the strategies they used at posttest, as well as in excerpts from classroom lessons. In the following sections, I will consider the kinds of new understandings that the students developed, first by looking at these changes in the test items and then by considering data from classroom lessons. Following this analysis of the improvement I will include a section on items that the students continued to find difficult even after the intervention.

5.6.2.1. Improved Understandings of Magnitude, Interchangeability, and Fraction Symbols After Instruction

One of the most striking changes in these students' understandings at posttest was the improvement in their ability to work with fractions. The results revealed that the students could correctly evaluate the magnitude of fractions, interpret symbols, and move between representations. This was evident in their response at posttest to the items such as, "What is 1/8th as a decimal?" "Write 6% as a decimal," and "Find a fraction between 1/4 and 2/4" where the scores were 93, 96, 58, respectively. In order to solve the latter item, the students re-interpreted these two fractions as percents and/or decimals asserting for example, that 1/4 = 25% and 2/4 is twice that, or 50%. In this way they were able to find a large variety of answers including for example 40/100 "because that is the same as 40%" or 3/8, "because 1/4 = 2/8 and twice that is 4/8 , so 3/8 comes in between." As well, it is worth noting that in their response to "What is 1/8th as a decimal?" these students demonstrated the use of the halving strategy in a very similar manner used by the previous Grade 4 students (see Chapter 4).

A further illustration of the students' newly acquired ability to interpret symbols and judge magnitude can be seen in the following excerpt from a classroom lesson on equivalencies that took place on the eighth day of instruction. In previous classes, students had been asked to invent challenges for their classmates. On this day, one of these challenges was taken up with the class as a whole in a teacher directed discussion. The students had invented a list of quantities in mixed-representations that were sometimes equivalent and sometimes slightly anomalous:

1600/3200 .05 250/500 .5 4/7 8/16 12 : 18 X:XX .529 50% 5/9

The teacher wrote this list on the chalkboard and then began the lesson by asking for volunteers to comment on the numbers on this list that they thought were not equivalent to 1/2.

The first student to comment chose 5/9 as an anomaly. "Five ninths is not the same as 1/2. It can't be, because 5/10 is one half, so, it can't be." When the teacher questioned the student as to whether 5/9 was more or less than 1/2, the student hesitated and said that he didn't really know. Another student then came up to the front of the class and said, "in order to get 1/2 you would have to have 4 point 5 ninths (she proceeded to write 4.5/9 on the board). "That,"she said pointing to the fraction, "is less than 5/9 so, 5/9 is larger than 1/2." The students were satisfied with this explanation but before moving on to the next representation, a third student made a related observation. She offered that she could now see a pattern in regards to another fraction on the list, 4/7. "Now you see that 4/7 was also larger than 1/2 'cause it would take four eighths or 3 point 5 sevenths to make a half so it was like a pattern with 5/9ths." These kinds of discriminations had entirely eluded the students at pretest.

The next number from the list that was pointed out as anomalous was .529. All of the students agreed that 1/2 was the same as .5 and thus different than .529. "I know that .529 is larger than 1/2 because the difference between .5 and .529 is that .5 is just five tenths but .529 also has 29 thousandths as well." The explanations provided by the students indicated that they had acquired an understanding of tenths and hundredths that had not been evident in the interviews and early classroom lessons. The following explanation typifies the kinds of reasoning that the students displayed: In fact, at pretest 40 % of these students had voiced an erroneous conceptualization that many children hold, i.e., that "the shorter the number, the larger the decimal." This same misconception was evident in the reasoning that many students offered at pretest as an explanation for why they believed that .2 is larger than .089. Clearly, the students were no longer reasoning with this misconception.

Finally, another illustration of students' changed understandings occurred in the discussion that focussed on the function of the zero in decimals. The discussion began when two students disagreed with each other as to whether .05 was equal to 1/2. Clarification was offered by a third student who pointed out that .5 was 50% but that .05 only means 5%.

As can be seen, the students' sense of magnitude across representations showed increased awareness and precision. Generally what the classroom lesson revealed was that the students had developed tools for making magnitude judgements and that they appeared to find that using percents as a reference was helpful in this regard.

5.6.2.2. Students Ability to Compute with Percents After Instruction

The forgoing protocol from the classroom revealed that these Grade 6 students became adept with percents as a referent or an intermediary step for comparing and ordering both fractions and decimals. This finding was very much in keeping with those of the other two studies. It was particularly encouraging that these results were obtained as less time was devoted to the teaching of percents in this Grade 6 curriculum (see Table 5.1). On the other hand, the posttest results also reveal that students' were less proficient in performing percent calculations than the students in the previous interventions. Even though all of the students at posttest could calculate an answer to the simple item "If a beaker holds 80 mls of water, how many mls would there be if you filled it 75% full?"only 47% of the students could find the correct answer to what is 65% of 160? In attempting the latter item all of the students first successfully computed 50% of 160. However it was in the second step (i.e., calculating 15% of 160) that the students encountered difficulties: Some students merely added 15 to 80, achieving 95; others had difficulty performing the computation at all. While there was good improvement on this item (students moved from 6% to 47%), the mean score on this item was lower than the means that were achieved by the Grade 4 students in Study 1 and the high-achieving Grade 4s in Study 2. A conjecture is that these students, because of their previous traditional teaching, were not as disposed to or as able to invent procedures. Therefore, it is hoped that a longer time spent on percent exercise might well have been valuable and would have made a difference in these students' ability to compute with percents.

5.6.2.3. Multiplicative/Ratio Reasoning After Instruction

Finally, in line with our hypothesis, another area that showed improvement was in proportionally and multiplicative reasoning. The item on the measure that most directly assessed ratio understanding was the Karplus et al. (1974) item, Mr. Tall and Mr. Short. As can be seen, there was substantial improvement on this item at posttest and 80% of the students were able to reason multiplicatively and solve the problem.

Students' ability to think of ratios and relative proportion can also be seen in another excerpt from that same classroom lesson. Again, the list of equivalencies was revealing in this regard: One of the challenges included in the list was to prove whether X:XX might be equivalent to 1/2. The first student to address this challenge asserted that X:XX could not possibly equal 1/2. "Lets say that X is equal to 2. Well then X:XX is 2:22. Or let's say you call X equals 5, then it would be 5:55. So it can't be half because 5 :55 is not the same as 1/2 or 1:2."Another student then noticed that 5:55 and 2:22 represented the same relationship as 1:11. This observation became a subject of interest and other students then noticed and commented on the pattern of the constancy of the ratio.

Another student then volunteered a different interpretation of X:XX: "I thought about the XX as meaning X times X so if you make X to mean 5, then you would have 5 : 25 or if X is 6 , then it is 6: 36 and 7 would be 7: 49." There was general excitement at this discovery and then one student thought out loud "but if X = 2, then you would have 2:4 which is the same as 1:2 so it can be the same as one half." Finally the students who had proposed the challenge in the first place explained their intended meaning of X:XX . "We made X:XX Roman numerals, so then X = 10 and XX = 20, so that just like a half." This discussion highlights students' interest in and ability to consider the ratio construct of rational number.

Table 5.4 presents items from the measure that are ordered according to gains that students made from pre- to posttest. Since the Level 1 items were all at ceiling to start with, these are not included on the following table:

Table 5.4
Percentage of Students Succeeding on All Items Excluding Level 1 Items, Before and After Instruction

Items	Pre	Post
What is 1/8 as a decimal?	13	93
Write 6% as a decimal.	40	96
Is 7 3/4 of 10? Explain your answer.	24	73
Mr. Short's height is 4 matchsticks. Mr. Tall's height is 6 matchsticks. When we measure their height with paperclips Mr. Short's height is 6 paperclips. How many paperclips are needed for Mr. Tall's height?	33	75
What is 65% of 160? Explain your answer.	6	47
How much is .5 + .38?	38	80
What is 75% of 80ml of water	60	93
Is there a fraction between 1/4 and 2/4?	19	58
Find 1/4 on a numberline	38	67
Shade 3/4 of this Pizza	75	100
Tell me a number that comes between .3 and .4	60	67
Order from largest to smallest: .48, 5/8, 14/13, .99, 1.03	38	58
Mrs. Cheever is 50% taller than her daughter. Her daughter's height is what % of Mrs. Cheevers?	0	13
Estimate the answer to 12/13 + 7/8	38	44

5.6.2.4 . Limitations of the Curriculum—Items With Little Change

While there was substantial improvement on many of the items, there were still some that remained difficult for the students at posttest. As can be seen from Table 5.4, there were items where the gains were limited. Although most of these items appeared to generate no changed conceptualization from the pretest, there was one item where the students, while still unable to achieve the correct answer, were at least able to answer part of the question successfully. This item asked students to order the numbers: .48, 5/8, 14/13, .99, and 1/03. In answering this question all but two of the students were able to correctly order three of the numbers: .48, 5/8, and .99. However, when it came to finding the larger of 14/13 or 1.03, the students ran into difficulty. Thus, although the score for this item at posttest was disappointing, still the students had improved their strategy use and had lost their misconceptions about comparing decimals and fractions.

Although the previous item showed at least a broadened understanding, there were three items on which there was no change in conceptualization *or* understanding between pre- and posttest. On the first, the standard item "Estimate the sum of 12/13 + 7/8," the students were not able to consider the quantities represented by these symbols. They responded to this item exactly as they had at the pretest, and gave the incorrect answers of 19 or 21. It is believed that when confronted with difficult numbers, students often regress in their reasoning and rely on rules and procedures rather than access to the conceptual knowledge that they have acquired I believe that this item is sufficiently difficult that the students showed exactly this pattern of behaviour. Nonetheless, their lack of flexibility in thinking on this item is an area of concern in our results and perhaps demonstrates that more classroom time needs to be devoted to fraction activities or that they're not appropriate for this grade level.

Another item of little change required that students find a number between .3 and .4. In answer to this question the students all agreed that such a number existed however, when they attempted to find an answer they often chose numbers that were incorrect. Finally, the most difficult item which asked students to derive the percent height of Mrs. Cheever's daughter to her mother, was as frequently missed after instruction as before. Students asserted most frequently that the daughter's height was 50% of her mother's, or else that the daughter was 0% of her mother. This is a very difficult item as the students are not able to understand the referent for the percent. Thus when asked, they could not even draw a picture that represents this question. Although there were a great many activities that dealt with percent measurements and percent comparisons of height in the Grade 6 curriculum, the concepts embedded in this problem were not addressed.

5.6.3. Pre- and Posttest Performance Differential in High-Achieving and Low-Achieving Students

In the previous study, the more able Grade 4 students improved significantly more than the students in the lower half of the class. A hypothesis for this study was that this finding would be different for older students. Table 5.5 shows the pre- and posttest scores after they were split for high and low. I derived the designation of high and low as I did in the preceding study, by finding a median split on the raw scores for the Canadian Test of Basic Skills, Mathematics Concepts Subscale. Again, I solicited ratings from the classroom teacher. Her ratings were in agreement with the division of the class suggested by the test.

Table 5.5
Total Scores on the Rational Number Test, High- and Low-achieving Students , Before and After Instruction

Items	High-achieving		Low-achieving	
	Pre	Post	Pre	Post
Mean Score (max = 20)	13.43 (2.44)	16.57 **** (2.15)****	8.12 (1.35)	14.12 (1.59)

** p < .01; *** p < .001; **** p < .0001; ns = not significant

When t-tests were performed to evaluate change for each of the two groups from pre- to posttest, it was discovered that both the upper half and the lower half improved significantly, with the high achievers t = 6.181, p = .0008 and the low achievers t = 7.64, p = .0001. Effect sizes were calculated and it was discovered that the effect size for the high-achieving students was 1.4 for the low achievers this was 1.7.

In order to further evaluate the differences between the high- and low-achieving students, and to test the conjecture that both groups of students would make equal gains from pre- to posttest, I conducted a two-way analysis of variance with repeated measures [(group) high- by low-achievement level], x [(time) pre- and posttest]. The results revealed that the gains were not equivalent for the two groups. But rather what was found was that there was a significant group by time interaction, and that the low-achievers improved significantly more than the students in the upper half of the class, $F(1,13) = 4.802$ $p < .0471$. As well, the effects of group and the effects of time were also significant at $F(1,13) = 12.89$ $p < .003$ and $F(1,13) = 95.57$ $p < .0001$ respectively.

5.6.3.1. Differences for High and Low-Achieving Students on the Items in the Three Levels Of Difficulty

Given that there was a difference found in the performance of the two groups, a question that arose was how the groups compared at pre- and posttest on the items when they are broken down into the three levels of difficulty. Thus I calculated the means and standard deviations for both the high- and low-achieving students on the different levels..Table 5.6 shows a breakdown of the mean scores for the three levels for both the high- and low-achieving students.

Table 5.6
Percentage of Students From Low- and High-Achieving Groups Succeeding on Level 1, 2, and 3 Items, Before and After Instruction

	Low Achieving		High Achieving	
	Pre	Post	Pre	Pos
Level 1 Items (max = 5)	4.75 1.30	5 .98	5 1.90	5 1.88
Level 2 Items (max = 7)	2.65 (1.06)	5.37 (.52)	5.57 (1.13)	6.57 (.78)
Level 3 Items (max = 8)	.75 (.70)	3.75) 1.48	2.85 (1.57)	5.85 (1.57)

As has already been reported all of the students were successful on the Level 1 items even at pretest.

On the Level 2 items however what was revealed was that the high-achieving students were able to answer most of these items before instruction attaining a mean score of 5.57 (1.13) out of 7. At posttest this group performed at ceiling achieving a score of 6.57 (.78). The scores for the low-achieving students on the Level 2 items were different. At pretest this group did poorly, achieving a mean score of 2.65 out of 7 (1.06). However they did make substantial improvement on these items as a result of instruction and scored 5.37 (.52) at posttest. Finally on Level 3 items both groups had difficulty on the pretest; the low-achieving students scored .75 (.70) out of 8 and the high-achieving scored 2.85 (1.57).

However as the table indicates each group improved their score by three points on these Level 3 items.

When a repeated measure ANOVA was conducted on Level 2 items ; [group (high- and low-achieving)] X [time, (pre and post)] a significant interaction was found for group and time $F = 97.1280$ $p < .000$. Similar interactions were found for the effect of time as well as for the effect of levels. Post hoc analyses (Scheffe = 1.68) revealed that there was a highly significant difference from pre- to posttest on the Level 2 items (Scheffe = 27.03). By contrast when an ANOVA was performed on the Level 3 items [group (high- and low-achieving)] X [time, (pre and post)] the results were different. There was no interaction found for group and time $F = 1.75$ $p < .207$. Thus it is apparent from these results that the curriculum was particularly effective for the low-achieving students as they made gains in both levels of difficulty.

5.7. Results and Discussion of the Performance of The Normative Groups: A Comparison with the Experimental Grade 6 Students

5.7.1. Comparison of the Experimental Students at Posttest to the Four Normative Groups

The final group of questions addressed in the study concerned the improvement that the curriculum produced for these Grade 6 students measured in developmental, rather than absolute (percent gain) terms. Following the adminstration of the 20-item measure, mean scores and standard deviations were obtained for all of the students in the normative groups. As expected, the scores increased by age of the students. The mean scores for the students in Grades 4, 6, and 8, were 6.75 (3.22), 11.5 (4.04) and 13.5 (3.87) for Grade 4, 6, and 8 respectively. The adult students in the MA preservice teaching program attained a score on 15.33 (2.35) on the measure. These scores along with the posttest scores of the Grade 6 experimental class are presented in Table 5.7.

Table 5.7

Total Scores on the Rational Number Test for the Normative Groups and the Experimental Students at Posttest

Test Max = 20	Grade 4 n = 20	Grade 6 n = 30	Grade 8 n = 20	Preserv n = 32	Grade 6 Experiment Posttest
All items max = 20	6.75 (3.22)	11.5 (4.04)	13.5 (3.87)	15.2 (3.02)	15.33 (2.35)

When a one-way ANOVA was performed it was revealed that the differences in score were, in fact, highly significant: $F(4, 122) = 23.3$ $p < .0001$. A Scheffe post hoc was also conducted to further explore these differences. What was revealed was that the mean score obtained by the experimental group was significantly different than those of the Grades 4, 6 and 8, but not of the preservice teachers.

While this finding was of interest, still, there remained the question as to which test items contributed to that result. Did the similarity of scores of the preservice teachers and experimental group reflect the passing of similar items or were the items that comprised the mean scores more random in nature? To answer this question I conducted a breakdown of the means scores for each group by level of difficulty. These scores are presented in Table 5.8. As can be seen the experimental students were able to score even slightly higher than the graduate students on the Level 3 items attaining a mean score on these items at posttest of 4.53 (1.72) out of 8 compared to the preservice teachers who had a mean score on these Level 3 items of 4.03 (2.49). On these same Level 3 items the Grades 4, 6 and 8 normative groups attained means scores of .75 (1.2) 2.6 (1.86), and 3.0 (2.58) respectively. On the Level 2 items, on the other hand, the preservice teachers were more successful than the students in the experimental group scoring 6.28 (1.14) compared to 5.73 (1.03) respectively. As can be seen, the Grade 8 students also attained a similar score 5.42 (1.50). Table 5.8 presents these scores.

Table 5.8

Percentage of Students From the Normative Groups and the Experimental Grade Six Class at Posttest Succeeding on Level 1, 2, and 3 Items

Grade Levels	Grade 4 n = 20	Grade 6 n = 30	Grade 8 n = 20	Preservice n= 32	Exp Post G 6 n = 15
Dev. Level 1 max = 5	4.45 (.88)	4.81 (.39)	4.94 (.22)	5.00 (0)	5.00 (0)
Dev. Level 2 max = 7	1.6 1.78	4.41 2.00	5.42 1.50	6.28 1.14	5.73 1.03
Dev. Level 3 max = 8	.75 (1.2)	2.6 (1.86)	3.0 (2.58)	4.03 (2.49)	4.53 (1.72)
TOTALS	6.75 (3.22)	11.5 (4.04)	13.5 (3.87)	15.2 (3.02)	15.53 (2.35)

Finally, I wanted to compare the posttest scores of the high- and low-achieving Grade 6 students to those of the normative groups. Since the Level 1 items were passed by all of the groups, I exclude these items from the following table and present the results of the Level 2 and 3 items for all of the groups.Table 5.9 re-presents the results for Level 2 and 3 items for the normative groups together with the results obtained by both high- and low-achieving experimental students. As can be seen the high achievers attained a superior score on both levels of items compared to the preservice teachers and the low achievers' scores were similar to those of the Grade 8 normative group.

Table 5.9
Percentage of Students From the Normative Groups and the Experimental Grade Six Class at Posttest Split for Low- and High-Achievers Succeeding on Level 1, 2, and 3 Items

	Low Post	High Post	Grade 4 n=20	Grade 6 n=30	Grade 8 n=20	Preserv. n=32
Level 2 items max = 7	5.37 (.518)	6.57 (.78)	1.6 1.78	4.41 2.00	5.42 1.50	6.28 1.14
Level 3 items max = 8	3.75 1.48	5.85 (1.57)	.75 (1.2)	2.6 (1.86)	3.0 2.58	4.03 (2.49)
TOTALS	14.12 (1.59)	16.57 (2.15)	6.75 (3.22)	11.5 (4.04)	13.5 (3.87)	15.2 (3.02)

5.8 . Summary and Conclusions

In this study, I examined the learning gains of a mixed-ability Grade 6 class who had all received previous traditional instruction in all aspects of rational number. In the two previous studies that were implemented to assess the benefits of the experimental rational number curriculum, the students who participated had not had any previous instruction. Thus, by working with this particular group I was able to ask a series of new questions that had hitherto not been addressed. We hoped that these questions might further our understanding of the application of this curriculum to broader contexts.

The first questions concerned the potential of this curriculum to generate changed understandings for students who had received prior traditional training. Since it has been reported that students have a difficult time adopting new thinking strategies and considering new approaches to topics they have previously learned by rote, I wondered if students who had received traditional instruction in fractions and decimals based on additive part/whole notions would be able to reorient their understanding to include much broader based ideas. My hypothesis was that these Grade 6 students would in fact be able to gain new understanding since the present curriculum was so different. This is an important consideration as any implementation of this curriculum (or any other curriculum) at the Grade 6 level would necessarily involve the intrusion of previous learning. This hypothesis was confirmed.

Another question of interest was whether the curriculum could be effective when presented in less time. As I have reported, the Grade 6 experimental students had only thirteen hours of instruction compared to twenty and seventeen hours of the first and second study respectively. In order to reduce the total teaching hours, less time was given to percent teaching at the opening of the sequence.

Although the students came to use percents as a referent in performing invented solutions particularly on items involving fractions, they were not as successful as the Grade 4 students in computing with percents. This was evident in the lower scores that they received for the item that asked them to calculate 65% of 160.

A third question concerned the differences in learning outcomes of the high-achieving versus the low-achieving students. The results of the Grade 4 study revealed that the upper half of the class improved significantly more than the lower half. My hypotheses for this study was that, because the students were two years older and more experienced with multiplication and division, the gains of the two groups would be the same. In fact, what the analysis revealed was that the lower ability students benefited more from instruction tthan their higher ability counterparts. Interestingly, these gains were primarily on the mid-level tasks where the Grade 6 students in the higher ability group were already competent. As well, these gains were on very difficult problems (Level 3 items). Although both groups of students started with differing prescores on these items, and ended with different postscores, both groups improved significantly and equally. Two interpretations seem possible. The first is that there was a ceiling effect for the high-achieving students. The second (not incompatible) is that the main impact of the program because of the content it contains, is on children just making the transition to formal (Piaget), abstract (Fischer), or vectorial (Case) thought, i.e., advanced Grade 4 or 5 students, and average or below average Grade 6 students. In either case, one might conjecture that this curriculum would be best introduced at the Grade 5 or early Grade 6 level.

A final question considered the magnitude of the improvement in developmental terms. From the results of the normative study and the comparison of these results to the performance of the experimental Grade 6 students, I learned that the Grade 6 students performed as well as the students in an M.A. teacher teaching program, and at a higher level than the students in Grade 8. Indeed, the high-achieving Grade 6 students, who are probably the most appropriate comparison group for the M.A. students, performed higher on the Level 3 items than they did.

In the next chapter I will examine the results that the students obtained across all three studies to discover more about the curriculum and to see where the similarities and differences are to be found.

Chapter 6

Assessment of the Curriculum Across Three Studies

6.1 Introduction

Each time the curriculum was taught, although the approach to the rational number content and the classroom teaching structures remained consistent, there were significant changes made to further our understanding of students learning of rational number and to refine and test aspects of the curriculum.

For example, there were changes made in the population of students who participated. In the first two studies the students were in Grade 4; the group from study 1, reported in my MA thesis (Moss, 1997) were high-achieving mathematics students who had been especially selected, whereas the second group, an intact Grade 4 class, was more academically diverse. In Study 3 the students who participated came from an intact mixed-ability Grade 6 class thus providing an opportunity to assess the effectiveness of the intervention with older students who had already had several years of traditional instruction in rational number and had formed their understandings based on this prior training.

Changes were also made to the curriculum that was delivered. Significantly, the number of teaching hours for the Grade 6 students was shortened. As well, variations on exercises and tasks were substituted for the original, prioritizing different rational number representations.

Finally, the measures that were used for the pre- and posttest evaluations were substantially different at each iteration of the program. All of the measures shared common purposes: to probe for students' conceptual understanding, to assess their ability to perform standard tasks and to evaluate the extent of their rational number sense. Nevertheless, changes were made to the number of items on the measures as well as to the content of items. In the first two studies, which comprised 41 and 49 items respectively, the measures were divided into three separate subtests (Percents, Decimals, and Fractions). Several items in the second measure were modelled on those of the first, but in the second the numbers used were more challenging. Also, on this second measure, more standard computation items were included. The third measure was substantially different, including only 20 items that encompassed a broader range of difficulty. This measure also included 2 normed items from the standard mathematics education literature. This third measure was designed with two purposes; to evaluate both the change in the pre and posttest performance of the experimental Grade 6 students as well as to evaluate the performance of normative samples of students, ranging from Grade 4 to adult. For a summary of the methods and design of the three studies please see Table 6.1 below.

Table 6.1
Design of the Three Experimental Studies

	Study 1	Study 2	Study 3
Goals	Test new approach using formal experiment and measure	Replicate experiments with broader range of subjects and more test items	See if approach works with older students who have had traditional introduction to rational number
Subjects	Grade 4 n = 16 Bottom quarter of class excluded	Grade 4 n = 15 All students in class mixed-ability	Grade 6 n = 15 All students in class mixed-ability
Design	pre/post treatment/control	pre/post no treatment control	pre/post no treatment control comparison groups
Teaching hours	18 hrs	18 hrs	13 hrs
Focus on Topics	percent .6 decimals .3 fractions/mixed representation .1	percent .5 decimals .2 fractions/mixed representation .3	percent .3 decimals .1 fractions/mixed representation .6
Measures	41 Items 9 Percents 16 Fractions 16 Decimals	49 Items 16 Percent 17 Fractions 16 Decimals	20 Items Mixed Representations

With all of the changes that were made to the measure, 10 items were retained across the three studies and it is these 10 items that will be analyzed in this chapter. The aim of the present chapter is to assess the robustness of the curriculum across the three studies. In order to accomplish this analysis, I created a new data base using the ten common items. These items are presented below in Table 6.2.

Table 6.2
Ten Items Ordered by Level that Were Retained Across the Three Studies

Level 1	What is 50% of $8.00? Draw a line on this beaker where 1/4 full would be.
Level 2	Can you name any number between .3 and .4? Which is bigger, .20 or .089? Shade 3/4 of this pizza (pizza is divided into 8 sections) How would you write 6% as a decimal? How much is .5 + .38?
Level 3	Another student said that 7 is 3/4 of 10. Is it? Explain. What is 1/8 as a decimal? What is 65% of 160? Explain your answer.

As can be seen these items were taken from each of the three levels of difficulty. Thus, while it is true that these items only represent a portion of each of the longer measure, the range of difficulty that they represent does provide us with the ability to compare the performance of the groups on different types of items.

The analyses in this chapter will be as follows. First I will first present a quantitative analysis that compares the performance of the groups as a whole.

As part of this analysis I will also examine the differences in performance of the high- and low-achievers. Following that, I will present an analysis of the individual items and compare the performance of the three groups on the different items. Finally, using these same items, I will compare the performance of the experimental students to that obtained for the normative samples.

6.2 Results on the Overall Measure

Table 6.3 shows the means and standard deviations of the pretest and posttest scores and the effect sizes for the three experimental groups and the treatment/control group on the 10 items (Zumbo, 1999). (The effect sizes were calculated by subtracting the pretest score from the posttest score and then dividing that difference by the average standard deviation of the pre and post scores). As was to be expected, the Grade 6 class had the highest scores both on the pretest 5.12 (1.81) and posttest 8.61 (1.18) followed by the high-achieving Grade 4's in Study 1, who obtained an overall score at pretest of 3.07 (1.82) and a posttest score of 8.4 (1.60). The mixed Grade 4 class, on the other hand, achieved a score of 4.3 at pretest and 7.2 (1.88) at posttest, with an effect size of 2.55. The students in the control group from Study 1, although achieving a similar score to the experimental students in pretests (scoring 2.85 (1.99), were much less successful on these items at posttest receiving a score of 3.5 (1.78). These results are presented in Table 6.3 below.

Table 6.3

Total Means Scores, Difference Scores and Effect Sizes on the Items Across for All 3 Experimental Groups and the 1 Control Group

Group	n=	Pretest	Posttest	Difference Score	Effect Size
Year 1 Experimental Grade 4	16	3.07 (1.82)	8.4 (1.61)	5.33/1.71 (1.75)	3.32
Year 2 Experimental Grade 4	21	2.43 (1.88)	7.2 (1.88)	4.81/1.88 (1.72)	2.55
Year 3 Experimental Grade 6	15	5.12 (1.81)	8.61 (1.18)	3.49/1.49 (1.40)	2.34
Year 1 Control Grade 4	14	2.85 (1.99)	3.5 (1.78)	.643 /1.89 (1.27)	.34

When a one-way analysis of variance was performed on the pretest score means for the three experimental groups, a significant difference was found between and within the groups, $F, (2, 49) = 11.88$ $p < .0001$. A post hoc Scheffe showed that there was no difference between the two Grade 4 groups at pretest. However, there were significant differences found at pretest between the two Grade 4 groups compared to the Grade 6 students. When a Scheffe was performed on the posttest scores there were no differences found between any of the groups. See figure 6.1. Thus, this analysis based on these ten items suggests that the curriculum seemed to be equally effective at posttest for the three experimental groups.

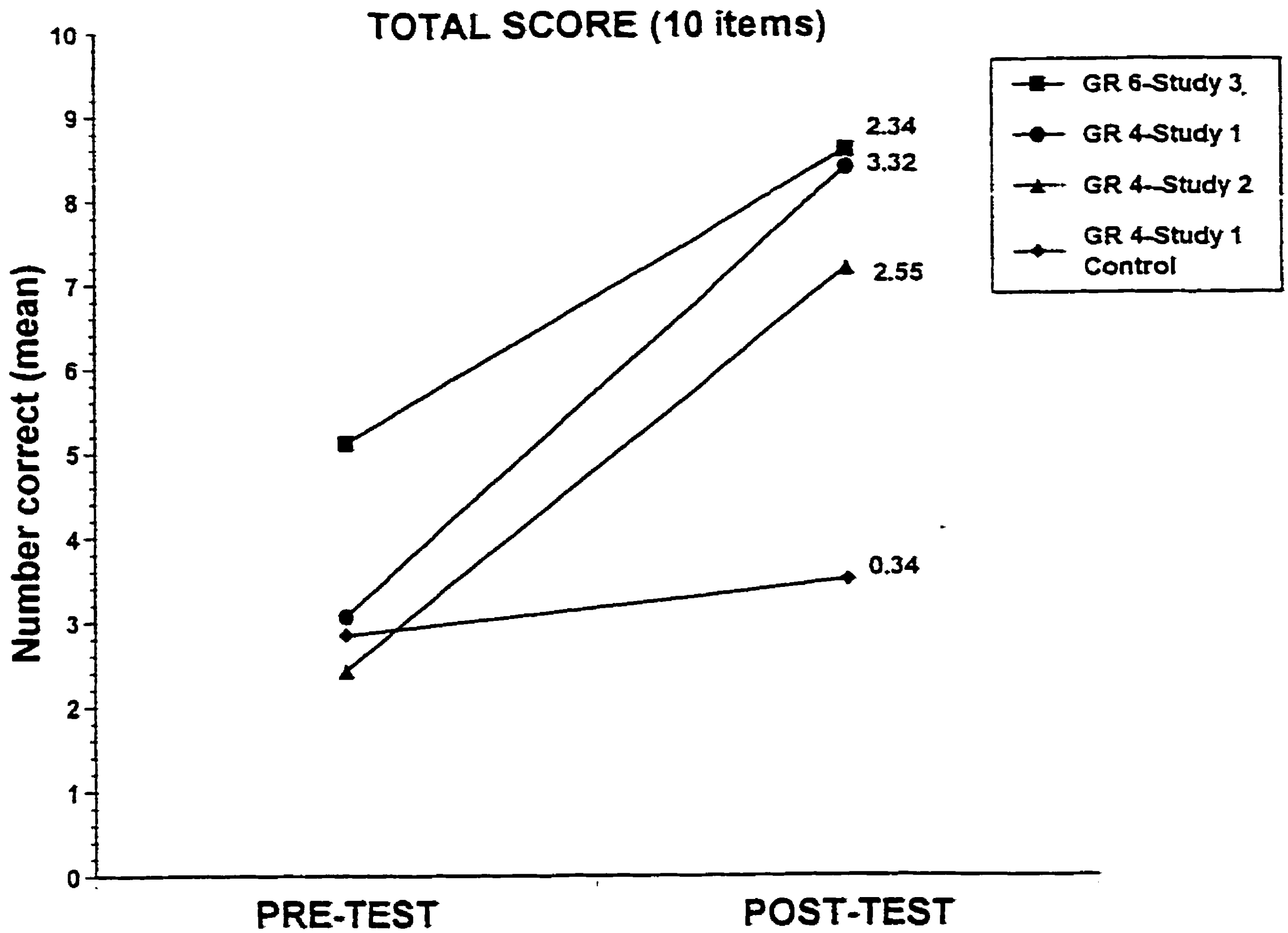


Figure 6.1 Total mean scores and effect sizes for the three experimental groups and the control group from Study 1 on the 10 selected items at pre and posttests

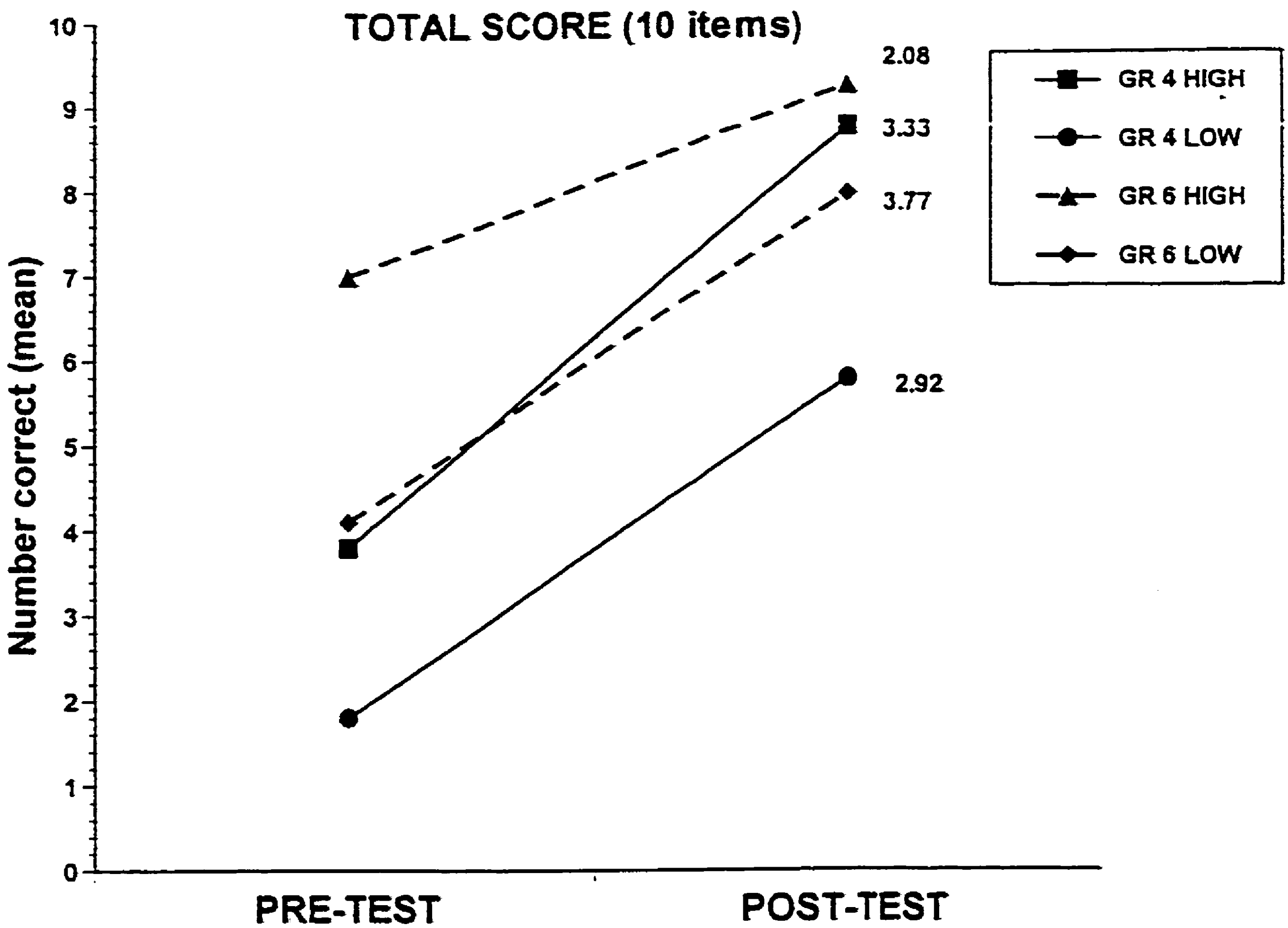
6.3 A Comparison of High- and Low-Achieving Students

Recall the differences that were found between high- and low-achieving students in both Study 2 and Study 3. A surprising result in Study 2 with the mixed Grade 4 class was the significant difference in rate of gain at posttest in favour of the high-achieving students. When similar analyses were conducted following the intervention with the mixed-ability Grade 6 students, however, the results were reversed: the low-achieving students gained significantly more than the students in the upper half of the class. Thus, based on those results I was able to conclude that the rational number curriculum was most effective for high-ability Grade 4 and low- ability Grade 6 students.

As part of the present analyses, I was interested to see if these same differences would still be evident on this new database of ten items and if, in fact, it might be a possible to calculate which of these two groups gained the most. Means and standard deviations of the pre and post scores, and effect sizes for these 4 groups of students on the ten items, are shown in Table 6.4. and Figure 6.2 (To review how students were designated as either high or low achieving, please see chapters 4 and 5). As can be seen, the results of these analyses showed a similar pattern to the results obtained by these groups on the original measures. Again, the most substantial gains were found in the scores of the high-achieving Grade 4 students and the low-achieving Grade 6 students. However, studying the effect sizes that were obtained by these groups, does reveal that although they were very high in both cases, the effect size for the low Grade 6 students (3.76) was even higher than that of the high Grade 4 students (3.57). This result suggests that the experimental curriculum is well suited for both of these groups but perhaps slightly more for the low Grade 6 students. A limitation with this analysis, however, is that there is a possible ceiling effect for the high Grade 6 students. Thus we cannot be sure how they would have done had there been a larger number of intermediate and difficult items.

Table 6.4
Means, Standard Deviations, Difference Scores and Effect Sizes for the Ten Items Across the Three Studies Comparing High- and Low-Achieving Students

Group	n	Pre	Post	Difference Score	Effect Size
Study 2 Grade 4 High-Achieving	10	3.8 (2.20)	8.8 (.69)	5/1.5 (1.90)	3.33
Study 2 Grade 4 Low-Achieving	11	1.80 (1.27)	5.81 (1.47)	4.01/1.37 (1.27)	2.92
Study 3 Grade 6 High-Achieving	7	7 (1.15)	9.29 (1.06)	2.29/1.1 (1.25)	2.08
Study 3 Grade 6 Low-Achieving	8	4.12 (.99)	8 (1.07)	3.88/1.03 (1.13)	3.77


TOTAL SCORE (10 items)
Number correct (mean)
10
9
8
7
6
5
4
3
2
1
0
PRE-TEST
POST-TEST
2.08
3.33
3.77
2.92
GR 4 HIGH
GR 4 LOW
GR 6 HIGH
GR 6 LOW

6.4 Analysis of the Individual Test Items

In order to further compare the groups' responses, I conducted another analysis in which I looked at the difference in the students' responses on the individual items. Table 6.5 presents the percent of correct responses for all of the groups on the level 2 items at pre and posttest. table 6.6 presents similar data for the level items. The items are arranged in descending order according to the scores that were obtained.

Table 6.5
Percentage of the Students from all Three Experimental Groups and Control Group Succeeding on Level 2 Items Before and After Instruction

	Study 1 Control n= 13		Study 1 High G 4 n = 16		Study 2 Mixed G 4 n = 21		Study 3 Mixed G 6 n = 15	
	pre	post	pre	post	pre	post	pre	post
Which bigger, .20 or .089?	50	50	38	81	33	100	67	94
Shade 3/4 pizza	33	33	75	100	38	76	73	100
Number between .3 and .4	43	17	39	100	14	76	60	67
Write 6% as decimal	0	23	0	93	5	81	40	96
what is .5 + .38?	5	33	6	50	5	52	46	80
Average percent correct	5	31	30	85	19	77	57	87

6.4.1 Pretest Results of the Level 2 Items

An examination of the Level 2 pretest items reveals that Grade 6 students were more competent with these questions than were either group of Grade 4 students. Fifty-seven percent of these mixed-ability Grade 6 students succeeded in answering these questions before instruction; by comparison only 5%,of the control students, 30%, of the high-achieving Grade 4's and 19% of the mixed-ability Grade 4 class could correctly answer these questions. Thus, we see that there is a developmental aspect or a least an effect of traditional instruction on these items.

6.4.2 Posttest Results on Level 2 Items

By comparison it can be seen that the posttest results were more unified; the passing rates were as follows 87% for the Grade 6 students and 85% and 77% for the high-achieving and mixed-ability classes respectively. Generally the passing rates were similar across all of the groups on the individual items. There were however 2 exceptions: the first was on the items that asked the students to "Find a number between .3 and .4." On this item both groups of Grade 4 students were successful achieving 100% for the first group and 76% for the second group. By contrast, the Grade 6s only achieved a 67% rate of passing and thus barely made any improvement from pretest. Although the relatively poor showing on this item is difficult to explain, a possible explanation is that the traditional training that the students had received prior to the intervention had created some confusion. Recall that the Grade 4 students had not had any training in decimals outside of the experimental program. It is interesting to note that on this same item the passing rate for the control group (from Study 1) actually dropped from pre- to posttest from 43% at pre- and 17% at posttest.

Thus traditional instruction appeared to have made this item particularly challenging for the students. .

The second anomaly in posttest scores on Level 2 items was, on another item involving decimals "How much is .5 + .38?. On this item, in contrast to the previous one, it was the Grade 6 students who achieved the significantly higher rate of passing (80%) compared to any of the Grade 4 groups. These younger students, while improving considerably from pretest, only had 50% and 52% rate of passing at posttest. Perhaps the fact that this item was more standard in nature meant that the Grade 6 students who had done more standard computation were able to be more successful.

6.4.3 Pretest Results Level 3 Items

Table 6.6 has the pre- and posttest scores that were achieved of for the 3 Level 3 items. As indicated on Table 6.6, the level 3 items were difficult for all of the students in the experimental groups.

Table 6.6
Percentage of students in the Three Experimental Groups and the Control Group Succeeding at Level 3 Items, Before and After Instruction

	Study 1 Control		Study 1 Experimental		Study 2 Grade 4 mixed		Study 3 Grade 6	
	pre	post	pre	post	pre	post	pre	post
Level 3 items								
7 is 3/4 of 10. Is it? Explain	0	6	6	75	24	76	12	73
What is 1/8 as a decimal	0	0	6	75	0	47	14	93
What is 65% of 160? Explain	8	0	0	69	5	52	61	58
Average percentage correct	3	2	4	73	10	58	29	75

6.4.5 Posttest Scores on Level 3 Items

As can be seen from the table, the mixed Grade 4 group experienced difficulty with all of the Level 3 items. However, the item that proved to be the most challenging was "What is 1/8 as a decimal?" On this item, this group achieved a rate of passing of only 47% compared to 75% for the high-achieving Grade 4 students and 93% for the Grade 6 students. This result was surprising. As I have indicated in Chapter 4, these students actually received more direct instruction with mixed-representations of numbers. A speculation is that the explicit instruction in mixed-representations might not be as helpful for the students as repetitious work with percents in measurement.

In summary then these result show that all of the students including the control group were successful in the Level 1 items, and that generally the pretest scores on the Level 2 items were higher for the Grade 6 students.This led to the speculation that traditional training might have been useful in response to these questions. However, the advantage displayed by the Grade 6 students on Level 2 items did not apply to the Level 3 items. On these items there was no difference in passing rates for the Grade 6s and the high-achieving Grade 4s.The performance on the Level 3 items perhaps is the most unified. On these all of the students did well with the exception of the item "What is 1/8 as a decimal?' It appears that even though there were many changes made, the curriculum proved to be helpful to all of the students regardless of age or experience.

6.5 Comparison of the Performance of the Experimental and the Normative Groups

In a final set of analyses I compared the passing rates of the 3 experimental groups and control groups to those of the normative groups on the same 10 items. Table 6.7 presents the passing rates on the level 2 items for the normative students from Grades 4, 6, and 8 and postgraduates. The passing rates for the experimental students are re-presented as well.

Table 6.7
Percentage of Students From the Normative Groups and the Three Experimental Groups After Instruction Succeeding at Level 2 Items

	Normative				Experimental		
Item	G4	G 6	G8	PS	G4-1	G4-2	G 6
Bigger, .20 or .089?	50	70	19	100	81	100	94
Shade 3/4 of pizza	45	84	89	94	100	76	100
Between 0.3 and 0.4.	10	67	80	97	100	76	80
Write 6% as a decimal?	5	42	68	88	93	681	96
How much is .5 + .38?	10	49	47	81	50	52	80
Average percentage correct	24	62	61	92	85	74	90

6.5.1 Level 2 Items

As was to be expected, the Grade 4's found these items very challenging. Although they had reportedly encountered many of these kinds of question in their classroom, nonetheless these questions presented considerable challenge. The performance of the g6 students was much better with 62% of the students able to answer that group of questions successfully, the surprising result however was that obtained by the Grade 8 students. A glance at the table will reveal that the g8 did significantly better on most items than the Grade 6 however there was one item that they were not able to answer at all. "Which is bigger .20 or .089?," the Grade 8 students' performance was markedly below even the Grade 4's (see Table 6.7). In reviewing their explanations offered during the testing, we discovered that they were operating with an erroneous rule, based on their belief that the "longer" the number past the decimal point, the larger its quantity. -Resnick (Resnick, et al. 1988), has described this phenomenon in the development of children's decimal understanding. At the time of testing, the Grade 8 class had just been reintroduced to decimal quantities and operations, allowing me to speculate that they were working from a rule-based system that was imperfectly remembered. It is also interesting to remember that the Grade 6 students were less able to perform similar compare and order question namely what number comes between .3 and .4? Perhaps there is some feature that is conceptually challenging or perhaps this item reinforces an earlier conjecture that students become confused after they have learned rules for this kind of task.

As can be seen, the preservice teachers were able to answer all of the Level 2 items. On these items they were even slightly more successful than the Grade 6s and the high-achieving Grade 4 students from Study 1. Another interesting feature was that the Grade 8 scores were the same as the Grade 4 scores on these items. By contrast, the Grade 6 normative and the Grade 4 normative groups were not as successful on these items. Thus, it seems that our curriculum has brought these Grade 4 students up to the performance of the Grade 8 students.

6.5.2 Level 3 Items

An interesting comparison in looking at the Level 3 results for all of the groups is to discover that the adults while successful at the Level 2 items and as capable or even more than the students in the experimental groups were far less successful than the students on the experimental groups on these items. The items and the average passing rates that the students achieved are presented in the Table 6.8 below.

Table 6.8 *Percentage of Students From the Normative Groups and the Three Experimental Groups After Instruction Succeeding at Level 3 Items*

	Normative				Experimental		
Item	G4	G 6	G8	PS	G4-1	G4-2	G6
Is 7 is 3/4 of 10. explain	20	41	42	59	75	52	73
What is 1/8 as a decimal	0	31	42	50	75	47	93
What is 65% of 160?	0	24	21	47	69	52	58
Average percentage correct	7	32	35	52	73	50	75

As can be seen, on these items, all of the students in the experimental condition had higher passing rates than both the adults and the Grade 8 students. It appears that the curriculum was particularly effective in supporting student learning of these items.

As mentioned previously, the Level 3 items are particularly challenging in that they not only require broad conceptualization of rational number, but that in order to solve them, several strategies need to be integrated. Even given these challenges, however, the adults' results were surprisingly low compared to the results of the students in the experimental groups. In reviewing the protocol from the interviews with the adults students, several features stand out. First, there was general confusion about fraction quantities and operations. This was shown in comments such as "I am not sure how to do fractions" or in response to an ordering item in which decimals and fractions needed to be compared, one student commented that because there were two numbers involved "fractions are always bigger than decimals." And two other students offered equally erroneous suggestions that fractions were always smaller than decimals. Their inability to relate fractions to decimals was evident as well in the low score that they received for the item "What is 1/8 as a decimal?" Four of the students responded that the answer was .8, indicating they had no conceptual understanding of the relationship and the quantities involved. Finally, another problem that was exhibited by the adult students was the difficulty that they had on the two items where they were required to choose an operation ("Is 7, 3/4 of 10?" and "What is 65% of 160?"). As can be seen scores that they achieved for these items were low. During the interview many of these students revealed that they did not "remember" how to do these kinds of problems.

6.5 Conclusion

Although the conclusions that we can draw from these analyses are limited due to the small number of items and the way that they have been selected, still there are some generalizations we can make.

The effect sizes were large for all groups, and well above that of the control groups who received the sort of instruction typically available in good upper middle-class schools, with dedicated math teachers. When the results that were achieved on these ten items were analyzed for differences in high- and low-achieving mathematics students, the high-achieving Grade 4 and the low-achieving Grade 6 students appeared to show the most gains although it is difficult to know how much farther the Grade 6's might go because of ceiling effects. Finally, we saw that the experimental subjects in our program showed a deeper understanding of fractions and decimals than adult teachers in a preservice program, all of whom had at least a B+ average in their undergraduate studies, and most of whom had taken high school and university mathematics .

Chapter 7

Discussion

7.1. Summary of the Results

Overall, the experimental curriculum proved to be effective in enabling students to gain a strong competence with the rational number system and an appreciation of its interconnections. In keeping with the goals of this project, therefore, I was successful in moving children beyond the understanding of any single form of rational number representation toward a deeper understanding of the rational number system as whole. The curriculum, by focusing on the development of benchmark values for moving among the different forms of representation, enabled the students to solve problems in a flexible fashion and to use procedures of their own invention for approaching them. This flexibility was not only evident on the problems requiring direct conversion from one form of representation to another, but also on most of the other types of problems. For example, when students were required to compare and order numbers, whether they were dealing with fractions or decimals or both, they treated the request as one that required them to think in terms of the underlying ratios that were involved rather than on their whole number knowledge. Their solution strategies included a variety of methods to represent these entities, but they never used the sort of simple whole number strategy that has been reported in the literature. Similarly, for standard and nonstandard computation, the students again used a wide variety of strategies. Their responses thus indicated that they had acquired an understanding for rational number and operations, an appreciation of the relationships among the representations, and a disposition to make sense of these quantitative situations. These are the hallmarks of number sense (Charles & Lobato, 1998; NCTM, 1989; Sowder, 1995).

These competencies did not develop equally in all students. High-achieving Grade 4 students made significantly more gains than the students of lesser ability. Conversely, the low-achieving Grade 6 students made more gains than the high-achieving students in their class. Suggesting that the program may be best suited for a particular developmental level. Notwithstanding these differences, however, the results did reveal that all of the students who participated in the experimental curriculum made large and statistically significant gains from pre- to posttest on the rational number measures. Furthermore, these results were not only true for students who were new to rational number. The curriculum also proved to be effective for students who had already received four years of traditional teaching prior to the experiment. This latter finding was contrary to expectations gleaned from studies by Mack and Heibert and Wearne (see Chapter 4).

7.1.1. Comparisons with Traditionally Trained Students

The results also revealed that the performance of the experimental students was significantly better than that of traditionally trained students two or four years their senior. Furthermore, it was discovered that, on the most difficult level of rational number items, many of the students in the experimental programs outperformed adults in a postgraduate teacher training program. These differences between the two groups were found not only in the number of items answered correctly, but also in the quality of the solution methods used. Even when the students in the experimental groups were unable to get the correct answer on a task, protocol analyses revealed that they tended to approach the problem multiplicatively, often finding a strategy that allowed them to at least make a sensible attempt at solving the problem. By contrast, when the traditionally trained students ran into difficulty solving an item, they typically responded in one of two ways. Either they claimed to have "forgotten" how to perform the appropriate calculation, or, they made the classic mistakes that have been reported in the literature, mistakes that involved either a lack of conceptuality or some sort of confusion of the rational numbers with whole numbers. Thus, the responses of the control group were symptomatic of the problem cited in the opening chapters of this thesis: We do succeed in teaching children to manipulate rational numbers with our current instructional methods. However, we fail to help them develop a deep conceptual understanding of these numbers, or to overcome the fundamental misconception with which they start out their learning: namely, that rational numbers are just special kinds of whole numbers.

7.1.2. Robustness: Comparison across the Three Studies

In the preceding chapter I outlined the many significant alterations that were made each time the curriculum was taught—changes in the population of students who participated, changes to the measures and changes in the pacing of the curriculum and the exercises that were presented. Nonetheless, when the results were compared among the three groups, several discoveries were made. First, the effect sizes across the studies were very high: all of the students made significant gains from pre- to posttest achieving effect sizes in the range of 2 1/2 to 3 1/2 standard deviations. Second, in all of the studies the students acquired many similar conceptual strengths. Thus, I can conclude, that despite the variety of age and ability and experience, and despite the alterations that were made to the lessons, the curriculum appeared to be robust across these different situations.

7.2. Multiple Features and the Experimental Rational Number Curriculum

Multiple features were incorporated in the design and implementation of the curriculum any or all of which may have contributed to the outcomes. This is a complex program that produced gains for the students across three studies despite the many changes introduced with each iteration. The purpose of the remainder of this chapter is to present an analysis of the features of the curriculum that may have contributed to the students' learning. I begin this discussion with a brief description of features of the design and implementation of these programs that are consistent with findings in cognition and instruction and mathematics reform. As well, I point out aspects that are common to other research programs in rational number. However, my emphasis will be on the features of the present program that are unique. It is these features that appear to have contributed most significantly to the results of the program. Thus these unique features constitute the essential contribution of the program to the field.

7.2.1. Features of the Curriculum Specific to Reform Curricula in Rational Number Teaching

In the introduction to this thesis, I presented a discussion that highlighted what I see as the limitations of the part/whole subconstruct as the foundational application of rational number. Recall that the part/whole subconstruct is the one that is first introduced to students in their learning of fractions and hence rational number (NCTM, 1997). Briefly, my argument was that this subconstruct, while easily grasped by students, has the disadvantage of being naturally associated with the whole number system and counting. Thus, by featuring a part/whole interpretation there is the danger that students may fail to make the connection to: 1) the proportional nature of rational number, 2) the way that rational numbers are related to the referent whole, and 3) how rational numbers may act as operators. We know that these latter multiplicative interpretations of rational number are conceptually more difficult for students.

Recently, other investigators have voiced similar concerns and have devised instructional programs that have helped students to develop a deeper, more proportionally based understanding of fractions or decimals in the middle school years. For example, to mention only a few, Kieren's (1994) folding exercises enable children to think of fractions such as 1/8 in a multiplicative rather than an additive context, as do Confrey's (1994) exercises on "halving." Streefland's (1992) pizza sharing program for Grade 4 and 5 students, which stresses the equivalence in the portions that are received when, for example, 5 children share two pizzas and 10 children share four pizzas, also is effective in promoting understanding of ratio for students of this age (Case, 1985; Marini & Case, 1994; Noelting, 1980a, 1980b).

The experimental curriculum under discussion shares several important features with the programs designed by these other investigators. These include: (a) a greater emphasis on the proportional nature of rational numbers; (b) greater emphasis on the meaning or semantics of the rational numbers; (c) a greater emphasis on children's natural way of viewing problems, and their spontaneous solution strategies; and (d) the use of an alternative form of visual representation (i.e., an alternative to the standard "pie chart"). As well as these features from rational number research, embedded in the delivery and design of this curriculum are features of a more general nature that owe their inclusion to recent work in the field of cognition and instruction (Bereiter 1990, 1995; Bereiter et al., 1997; Bereiter & Scardamalia, 1993; Bielaczyc & Collins, in press; Bransford, Hasselbring, Narron, Kulewixz, Littlefield, & Goin, 1989; Brown & Palinscar, 1989; Bruer, 1949; CTGV, 1994; Scardamalia & Bereiter, 1991, 1996) as well as in recent developments in mathematics education reform (Ball, 1993; Carpenter & Lehrer, 1999; Fraivillig, Murphy, & Fuson, 1999; Kilpatrick, 1987; Lampert, 1990; Lampert, Rittenhouse, & Crumbaugh, 1997; Maher & Martino, 1996; McClain & Cobb, 1997).

7.2.2. Features from Cognition and Instruction

The organization of all of the classroom activities was based on social constructivist theory. Students regularly worked in collaborative groups with the goal of "knowledge building" (Bereiter, 1997; Bereiter & Scardamalia, 1993, 1997). As well, there was a very strong focus on activities that are known to promote metacognitive thinking and reflection; the students planned lessons for their classmates as well as for younger students (Meichenbaum & Beimiller, 1998), designed assessment tools or tests for other students (Brown, Campione, & Lamon, 1994, 1997), and engaged in planning, designing, and presenting projects for their classmates in the general style of what Brown and Campione have referred to in their Fostering Community of Learners project as "consequential tasks" (Brown & Campione, 1994). These tasks included teaching games and video presentations in which the students were required to explain/teach rational number concepts. Finally, the classroom culture highlighted the importance of inquiry as a basis for learning, promoted the goal of intentional learning (Bereiter & Scardamalia, 1989), and fostered a sense of pride and ownership in the learning process (Schifter & Fosnot, 1993; Wood, Cobb, & Yackel, 1993). Students were encouraged to seek multiple solutions to problems, to discuss the merits of one solution over another (Carpenter et al., 1993; CGTV, 1994; Kamii, 1985, 1994; Lampert, 1990; Maher & Martino, 1996). In keeping with Cobb's orientation to classroom practice, sociomathematical norms were established enabling the class to find ways of discussing personal theories and representing numbers and operations that were common to the class as a whole (Cobb, Gravemeijer, Yackel, McClain, & Whitenack, 1997; Cobb, Jaworski, & Presmeg, 1996; Saxe & Bermudez, 1996; Yackel, 1996; Yackel, Cobb, & Wood, 1993). Finally the mathematical model in these classes was infused with the epistemological bias that mathematics is a human invention and a "science of patterns" (Steen, 1988) rather than a technology of procedures (Schoenfeld, 1989).

While I included these features from cognitive science and mathematics reform in the program, the scope of this work did not permit me to include specific analyses of how these features affected the outcomes of the experiments. I do feel that the inclusion of these features provided the students with optimal conditions for learning, and thus may have contributed to the gains that the students made. However, I also feel that even with these features (which other programs have included as well) the program would not have had the success it did, had not other, more unique features been included as well.

7.3. Unique Features of the Program: Ideas for Broader Applicability

What about the features of the program that are unique? My conjecture is that it is these unique features that most contributed to the success of the program: in particular, to the large gains in performance achieved within a brief intervention. Briefly, these are: 1) the introduction to rational number through the teaching of percent; 2) the use of linear measurement as the central learning context; 3) the unique set of perceptually salient representations that were employed, and the coherent use of these representations throughout the program; and 4) the emphasis on the integration of halving with proportional evaluation and the inclusion of several different representational formats. It was these features that led to the emphasis in this curriculum on promoting an understanding of the number system as a whole and that supported the inclusion of the first 3 above features. Although these features are highly interconnected, I will discuss them separately, beginning with a discussion of the use of percent as the initial representation.

7.4. The Introduction to Rational Number Based on Percent

In the introduction to this thesis I proposed a number of advantages for altering the standard teaching sequence of rational number by introducing this number system through the teaching of percents (rather than fractions and then decimals as is traditionally done). Foremost, I mentioned the benefits of working with the privileged base of 100. Grade 4 students' extensive knowledge of the numbers from 1 - 100 naturally facilitates comparison questions as well as promotes students' ability to translate among the representations of rational number—any percent value can be translated into a fraction or a decimal the converse, however, is not easily done. The ability to invent strategies for calculating is also facilitated by the percent construct; A survey conducted by Lembke and Reys (1994) revealed that students who had not yet had formal training in percents, were able to invent procedures to solve percent tasks and were even better able to solve certain kinds of operations with percents than were older students who had learned percent in school.

7.4.1. Traditional Percent Teaching and Learning Problems

To my knowledge, despite these powerful advantages for the use of percent as an introduction to rational number, no other curriculum for teaching this number system has done this. In fact, percent is usually not introduced until the very end of elementary school or the beginning of middle school and, even then, it is shown to be very difficult for students to master. Fractions and decimals on the other hand are introduced much earlier, with decimal teaching beginning in Grade 3 and fractions introduced as early as Grade 1.

Why this delay in percents teaching? A brief analysis of textbook introductions to this topic and the attendant difficulty that students experience points to the reason. Percents are generally introduced in one of two ways: through "missing term" or "substitution problems," where students are challenged to find one of three possible unknowns in a percents equation; or in "conversion problems" which requires changing between notational systems (Parker & Leinhardt, 1996). Unfortunately, these teaching practices appear to have led to widespread problems: 1) First, students lack consistency in using the percent symbol and either they ignore the percent sign or use it as a label that can be attached or removed, as if it had no operational significance; 2) a second common error, known as the "numerator rule" involves the misconception that any conversion of percents to decimals is achieved by replacing the percents sign at the right of a numeral by a decimal point to the left of the numeral creating misconceptions of the sort that 5% = 0.5 or 110% = .110; 3) another widely noted misconception termed the "random algorithm problem" is exemplified by students asserting that the answer to 4 = _% of 8? is 2, thus indicating that their reasoning is grounded only in the numbers they see, not in the operations that are required. So that when students attempt percent word problems they are not certain if they should divide, multiply, or subtract to find the answer; 4) finally, traditional instruction in percents diminishes the relevance of the base-100 (the very aspect of percents that was so intuitive for the students in the experimental programs).

Parker & Leinhardt (1996) have shown that many of these problems are still evident when students are at the end of high school and early in their college careers. More troubling perhaps, in a study of 70 preservice teachers, it was found that less than half of them scored above 50% on a test of percent exercises that evaluated both conceptual and procedural knowledge. In summary, students do not seem to appreciate the meaning of percents as either operators or as quantities, and these problems persist into adulthood, at least for preservice teachers.

7.4.2. Percent in the Experimental Program

The results of the present studies show a very different learning pattern with regards to percents. Not only did the students' ability to perform percent tasks improve with instruction—this was even true for the Grade 4 students of low mathematical achievement—but it was also discovered that students had substantial intuitions of percent meanings and operations prior to instruction. Recall some of the first day responses that students offered when asked what they knew about percents. These responses pointed to an understanding of 1) magnitude: "1 percent milk is better for diets than 4% milk as it has less fat;" 2) the relative versus absolute nature of percents: "reading 10% of this (short) book is a lot less to read than 10% of that (long) book;" 3) the relationship among the representations "you know, 50% is the same as 1/2;" and 4) the proportional nature of percent "25% of the 80 cm tube (20 cm) is in the same relation as 25% of the 30 cm tube (7 1/2 cm)." These observations revealed an awareness of substantial percent concepts. However, a discussion that occurred on the first day of class with the mixed-ability Grade 4 students in Study 2 points to even more complex understandings.

7.4.3. Classroom Episode: Informal Knowledge and Percents

This discussion began when I asked the students if they thought that there could be a percent greater than 100%—a concept that is known to be difficult to understand (Parker & Leinhardt, 1996). The first student to speak held up a tall tube (80 cm) and made the following claim: "We know that this whole tube is 100%." Next, he picked up a second shorter tube (20 cm) and stood it beside the taller tube and declared that the small tube looked like it was about 25% of the taller tube. To confirm this conjecture he moved the smaller tube along the taller tube and noticed that it fit exactly four times. "O.K., this is definitely 25% of the longer tube. So," he declared, "if you join the two [tubes] together like this" (here he lay both tubes on the ground, and placed the shorter tube end-to-end with the larger), "this new tube is 125% of the first one." Other students used this model with other tubes and made similar assertions. After some time, a boy raised his hand and inquired whether all of the six individual pipes that we had in our classroom had been cut from one very long, single pipe. Given an affirmative response, the student then conjectured that if he joined the two pipes (80 cm and 20 cm) together, they would form a new whole, which would constitute a 100%. "So," he continued, "the two pipes together are either 100% or 125%." In this context, he recognized that the 20 cm tube would no longer be 25% of the whole, but rather, it would now be 20% of the newly formed whole.

The first day observations that the students offered demonstrated that the students had a substantial, principled understanding of percent. The students revealed an awareness of "increase" and "decrease" and the transformative nature of the unit in the rational number system. These are the very same concepts that prove to be so difficult for students and adults who are traditionally trained in percent. Why do we find these differences? Why do these young students appear to have more intuitions for percents than older students? My conjecture is that the bias in traditional percent teaching, rather than promoting the kind of informal knowledge demonstrated by these Grade 4 students, actually inhibits these intuitions and rather, calls on students to reason with percents exclusively as "extensive"(additive) quantities. Support for this conjecture can be seen in Davis' theory of percent applications (1988).

7.4.4. Percent as Number or Operator

In a thought provoking article by Davis (1988) entitled "Is Percent a Number?" he contemplates different interpretations of percent. First of all, he points out that traditional percent instruction is premised on the idea that percent is a number to be used in arithmetic. As an illustration, consider the method that students are taught to solve percent tasks, e.g., to find the answer to 25% of 80, they are taught to reason as follows: Step 1, convert the percent to a decimal (25% = 0.25) and, step 2, multiply the decimal and the number to find the answer (.25 x 80). Here, the percent (25%) is interpreted as a number, 0.25. He asserts that interpreting percents, exclusively as a number, is not only limiting for the students but furthermore, misleading.

Davis' central thesis is that percent is actually understood and used, not as a number, but rather as a relation between two numbers or two variables. Thus, he maintains, that percent is more reasonably interpreted as an operator or a function. Take for example a common usage of percent, namely, percent as tax (

as in the 25% tax). Here Davis maintains that the most appropriate representation would be (1, 0.25), (50, 12.5) (100, 25), (200, 50) ...... i.e., as a linear function. He points out that traditional training in percent (as a number) would have us interpret, for example, 25% of 40 as .25 x 40, i.e., as two numbers involved in a binary operation that accepts two numbers as inputs and outputs a single number (in this case, 10).

If, on the other hand, we take Davis' view that 25% is more intuitively conceived of as a linear function, then, as he suggests, we might also consider writing the 25% relation as _ x 1/4 = X or, we could look at it as a single input/output, such that: one quarter of (something) = output, or: input/4 = output, or: Y = X/4.

Recalling the errors and misconceptions that students reportedly display when calculating with percent (i.e., the lack of meaning of the symbol, the lack of understanding of operations, etc.), it is clear that traditional teaching of this topic is problematic. One hypothesis then, is that the interpretation of percent as a number—certainly the central idea in traditional instruction—promotes these problems. As the qualitative analyses of students' reasoning in the experimental studies point out, the manner in which we introduce percents in our program promotes the meaning of percent as an operator or function. Thus the curriculum has succeeded, at least from Davis' point of view, in grounding the students' learning in the most meaningful interpretation of percent.

7.4.5. Percent as Measure

It is acknowledged however that rational numbers are both operators and numbers and it is through the integration of these two constructs that a full understanding is reached. In my experimental curriculum, while the linear function/operator construct of percent is easily accessed by the students, it is also apparent that they gain an understanding of the magnitude of percents (i.e., the "extensive" property of rational number). The common denominator of 100 clearly provides this sense of magnitude and supports students' understanding of the additive properties of rational number.

Thus, according to the prevailing subconstruct theory (presented in Chapter 2), the students' functional understanding of percents, in the sense that Davis espouses, corresponds to the subconstructs of operator and ratio. And the understanding of magnitude corresponds to the measure and quotient subconstructs. Thus, in the experimental curriculum the students work with percents in such a way as to include all the subconstructs of the rational number system.

7.5. The Use of Linear Measurement as the Main Context for Learning

Not only did I significantly alter the sequence of rational number teaching by introducing percents first, but I also incorporated another unusual feature, namely , the use of linear measurement as the primary context for learning. This decision was consistent with Griffin and Case's whole number programs in which all the exercises were organized around activities that featured the number line (Griffin & Case, 1996, 1997; Griffin, Case. & Siegler, 1994).

7.5.1. Linear Measurement and an Appreciation of Magnitude

Analysis of the data demonstrated that the percent representation with its common denominator of 100 supported an understanding of magnitude. Similarly, the use of linear measurement fostered this understanding as well. We know that "length" expresses magnitude unambiguously and allows students through visual inspection to perform comparisons for both absolute and relative differences. Thus, I included a number of props beside the pipes and tubes that would readily lend themselves to evaluations of relative lengths. Vials and beakers, filled to different degrees, were very useful as these props involved the students in a variety of measurement and calculation situations. Other props that served a similar purpose were cardboard tubes, cut-out dolls, rolled plasticine, and straws, etc. In working with these various objects, the students were regularly comparing percents and fraction magnitudes. The large laminated number lines and the number line games further supported this thinking.

However, as will be shown below, the use of linear measurement did not only provide support for students' understanding of magnitude and the additive properties of this number system, but also supported an understanding of the relational aspects of rational number. For this assertion, I consider the historical roots of rational number as this investigation suggests that the measurement context also reinforces the notion of rational number as operator (Sfard, 1997; Davydov & Tsverkovich, 1991). Just as the language of percent leads the student to focus on the functional/operator interpretation of percent, so too, can the use of measurement.

7.5.2. The History of Fractions as Measurement

Davydov, who has done research in fraction learning, (refs) points out that, historically, fractions evolved in the context of measurement. The Egyptians appeared to be the first to use fractions in response to the growing complexity of their society with its attendant need to obtain more precise measurement. Thus, in Egypt and historically in all other cultures where fractions were used, fractions operated exclusively as a function of the unit whole and served to define a relation between the quantity of the unit and the piece of the unit that was left over: Fractions were primarily descriptions of relationships.

This interpretation of fractions persisted until the mid-19th century. Then, with the advent of modern day mathematics, the definition of fractions changed and fractions became part of the "new algebra." With this change, fractions became a new form of number with properties and axioms that could be mathematized or calculated. As well, this change heralded an identification of rational number with the whole number system. Fractions became defined as a number pair (x/y), that could be operated on as inverted versions of whole numbers. Thus, these relatively new (fraction) numbers became divorced from their origin as functions and relations, and have been turned into abstractions that are suitable for arithmetic calculation.

Thus, by introducing rational number concepts through measurement as we do in our program, we are allowing students to recapitulate in their own learning the historical sequence of rational number development. Hence, we are delaying the learning of fractions as only their most abstract idea—as numbers.

7.5.3. Measurement and Spatial Analogies in Mathematics and Number Sense

However, in this program we did not simply use measurement in a conventional way, for example, as a tool for recording dimension, but our focus was on the use of linear measurement and the way it is intuitively analogous to spatial perception. Case (1998) postulates that there is a "deep commonality in the way in which numerical and spatial knowledge are connected for children." He points first to neurological data which suggests that deficits in mathematics tend to correlate with deficits in spatial cognition; for example, adults with neurological injuries that impair their spatial understanding also often show impaired numerical understanding. He then points out that recent neuro-imaging studies corroborate the connection between spatial and mathematical cognition. Finally it has been noted that mathematicians reason about even the most abstract mathematical concepts in spatial terms (Sfard, 1997, p. 350). Thus a speculation is that there is some kind of "direct connection" between spatial and numerical knowledge.

If there is such a connection between mathematical and spatial knowledge, then it seems possible that we may be able to improve children's mathematical intuitions by creating learning events in which the number system is given some sort of spatial embodiment. Further support for this idea can be found in Greeno's (1992) spatial neighbourhood metaphor which he uses to characterize knowing in a numerical domain. He asserts that just as individuals can find their way around their own neighbourhood, recognize salient landmarks, reason about the relative efficiency of different routes and discern subtle patterns, he asserts that "knowing" in a conceptual mathematical domain means having the ability to find and use, within this environment, the resources needed to understand and reason. Just as each person can move intelligently in their own neighbourhood, so each person is capable of constructing a rich set of interconnections among the"landmarks" in a mathematical domain.

In both Case's previous research programs for developing whole number sense (Griffin et al., 1994; Griffin & Case, 1996, 1997) and in his more recent work in function development (Kalchman & Case, 1999), he and his collaborators have designed highly successful teaching programs in which spatial embodiments of numbers feature strongly.

7.6. The Establishment of a Perceptually Coherent Learning Environment

The rational number program, also includes a variety of different but highly related spatial embodiments for rational numbers—pipes and tubes, beakers of water, number lines, stopwatches, etc. Furthermore when these physical props are no longer available the students spontaneously draw diagrams modeled on these props to help them to conceptualize the work when it become more abstract. What they produce is a rectangular representation of a beaker—or, as it were, an uncalibrated double number line (Klein, Beishuizen, & Treffers, 1998) which I came to term the "percent ribbon." The students labelled this rectangle with 0 to 100 along one edge and from 0 to n along the other edge (see Figure 7.1). This diagram allowed students to visualize relative differences, to use halving schema for calculations,and support understanding of equivalence.

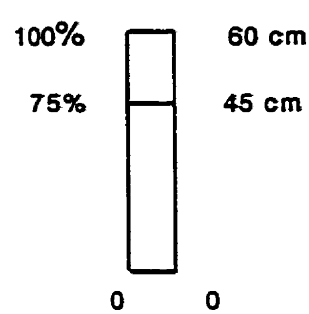


Figure 7.1—The Percent Ribbon

7.6.1. Percent Ribbon for Calculations and Equivalence Evaluations

To illustrate how this percent ribbon was regularly used by the students for calculating, I present a discussion that I had with a Grade 6 student early on in the lesson sequence.

The student was working on a problem that required him to calculate how many millilitres of water he would need to fill a container that holds 240 mls of water 75% full. He answered as follows:

S. "First you need to get 50% of 240 and that equals 120, and then you need to add 25% to 120. So, um, that gives you 120 + 25 = 145.

T. So you think that 145 is 75% of 240?

S. I don't know, it doesn't really seem right. Can I draw it ?

T. Go ahead.

The student drew a long narrow vertical rectangle and wrote the number 240 at the top right corner and 100 on the top left corner. He then estimated the midpoint of the length of the rectangle and wrote 50% on the left side and then put 120 on the corresponding point on the right length of the rectangle. He halved again, and similarly labelled 25% and its equivalent, 60. Then he asserted that 75% must be 180. This strategy, the drawing, and then segmenting by halves of the rectangle as a way to visualize the computation, was used by all of the students.

Another advantage for the students was the way in which the percent ribbon simultaneously represented a magnitude full, and, at the same time, a portion of a quantity and thus afforded students the opportunity to test constructs of order and equivalence of a rational number entity. For example, on an interview question "Draw a diagram to show which number is greater, 2/3 or 3/4?" the students used a diagram of the percent ribbon again to great effect to prove their assertion that 3/4 was greater. The most common strategy that was displayed by students was to draw two adjacent equal sized percent ribbons, perform a halving and quartering operation on one and then a segmenting of thirds on the other. Thus, through visual inspection, they could confirm that the three quarters segment was bigger than the 2/3 portion of the rectangle. Although the majority of the experimental group successfully answered this question (81%) only 38% of the control group were able to do so. Most of the control group asserted that 2/3 and 3/4 were the same size as "they both had one piece taken from them."

As well, I discovered that both the use of the props and the analogous diagrams fostered students' understanding of the density property of rational number. These insights were supported in activities where students for example were challenged to divide units in half using elastic bands on cardboard tubes. By repeating this division action and cutting each successive new length in half they were able to consider that this action could be repeated an infinite number of times. Thus they could begin to understand the notion that a third number can always be inserted between any two other numbers—a central principle of the rational number system.

7.6.2. Vergnaud and Mathematical Modelling

Finally, I argue that the props that were selected and the percent ribbon that was constructed by the students directly modelled the mathematics that were under investigation (Freudenthal, 1983). Thus the curriculum was not only able to provide a learning context that was intuitively salient but also one that was mathematically appropriate. I will elucidate this idea by first reviewing Vergnaud's model for multiplicative structures and then I point out how the core spatial representations of the curriculum supported the central proportional structures in the Multiplicative Conceptual Field.

Vergnaud defines the Multiplicative Conceptual Field as comprising all mathematical topics and situations that consist of simple or multiple proportions. Vergnaud (1996, 1988, 1983) proposes the notion of two measure spaces in his analysis of multiplicative ratio thinking. He asserts that ratios, and hence rational numbers, can be represented either by pairs of elements in the same measure space or elements in two distinct measure spaces (Kieren, 1992). To illustrate this idea, Kieren has suggested the following proportional relationship between pizzas and people. For example, 2 pizzas for 5 people is the same as 6 pizzas for 15 people. In this statement there are two mathematical relationships to consider. The first is the scalar relationship 2 is to 6 (pizzas : pizzas) and 5 is to 15 (people : people) or a/b such that b = 3 x a. Vergnaud calls this a relation between elements in the "same measure space." The second set of relations, he calls the functional relation between elements in "different measure spaces." The functional relationship here is between pizzas and people. Thus we reason that 2 is to 5 (pizza: people) and 6 is to 15 (pizza : people). This relation can be written as a/b such that b = 2 1/2 x a.

Although both the scalar ("same measure space") and the functional ("different measure spaces") relations can be effectively used to solve different kinds of proportion problems, Vergnaud has suggested that it is the first of these relations i.e., the "scalar relation in the same measure space" that is the most fundamental form of ratio reasoning. He calls this scalar relation isomorphism of measure and he notes that is most intuitively salient and most readily accessed by novices. As research with self-taught, unschooled Brazilian children and workers has shown (Carraher, T.N., 1986; Nunes & Bryant, 1996; Saxe, 1988; Schliemann & Nunes, 1990) unschooled workers exclusively use this "isomorphism of measure relationship" in solving the mathematics of the workplace. Furthermore, they even use this "scalar relationship" even when the numbers more readily lend themselves to evaluations of relationships in the second measure space or among the other variables.

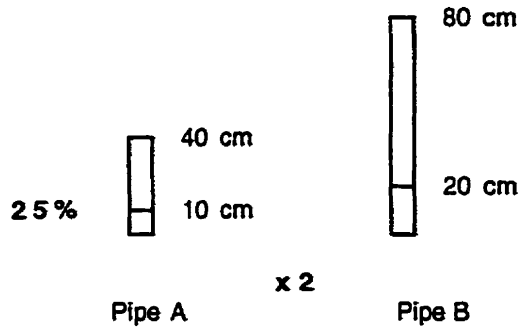


Figure 7.2—"Measure space" model using pipes and tubes from the rational number curriculum.

In the experimental program, the scalar relationship in the same measure space is represented for example, when 10 cm of a 40 cm pipe (Pipe A) is covered with the tubing and the challenge is to cover a second 80 cm pipe (Pipe B) a proportionally equivalent amount. To work out the solution by operating on the variables in the same measure space (scalar relation) the student would reason that since Pipe B is twice the height of Pipe A, therefore the part that should be covered by the tube on the second pipe must also be 2 times that of the covered area on the first pipe, i.e., 2 x 10 cm = 20cm.

While it can be seen that this relationship is modelled by the pipes and tubes, it is not the one that the students in our programs often use to compute the missing term. Rather, the experimental students consider the functional relation across the variables, or in Vergnaud terminology as "the relation in the different measure spaces." Thus, given the same problem situation, they would reason as follows; 10 cm (the covered section) is 25% of the length of Pipe A (40cm). Therefore, to find the covered area on the second 80 cm pipe (Pipe B) the operation they use is to find 25% of 80cm which is 20cm.

Thus not only do the props and the percent ribbon model the mathematics of multiplicative structures, but the context that we have presented to the students with these props generates the more complex of the two types of multiplicative relations.

In this way the spatial embodiments and the core representations permit children to build a rich implicit map of the cognitive elements and relations which are to be learned. Furthermore, as in all of Case's programs, the students are able to utilize this map in a flexible way, to move through the elements of the representation. As well, these core representations are consistently and repeatedly employed so that the students can become flexible and experienced knowers. Thus we might speculate that the use of these various embodiments and the unified way that they are included in the curriculum serve a further purpose. Perhaps it is through their participation in this coherent, spatial program that the students gain the kind of "perceptual attunement" and acquire the degree of automaticity and flexibility that Bereiter has postulated to be at the root of number sense competence. (Bereiter, 1998; Bereiter & Scardamalia, 1997).

7.7 Final Thoughts

We hypothesized a learning trajectory that the students would go through. It was conjectured that the students' learning of rational number would start with basic but separate intuitions for fullness on the one hand and halving and doubling on the other and that these would become merged to form a core organizing schema from which would come future or further stages of knowing; first, of the individual representations with computations that involved numbers that easily lent themselves to halving and doubling and then, to a higher level of rational number knowledge where the students would come to work interchangeably with the entire set of representations: decimals, fractions, percents, and ratios. In effect, the use intuitively understood halving relations became a vehicle for establishing a correspondence across the different forms of representation as well as a vehicle for relating similar representations for different ratio values. Thus the scope of this work was based on the development of a conceptual understanding of the entire number system, as well as a on vision of how that system develops. Because of the hypothesized structure that is generative of an understanding of the domain as a whole, the uniqueness of this program is how it has deliberately fostered from its beginnings, a sense of the entirety of the number system.

In summary, the unique aspects of the program, the introduction based on percents, the linear measurement context for learning, the coherent use of spatial embodiments, and the use of a theoretical framework that focussed on the differentiating and integrating of these various elements of the rational number system, all appear to have made a solid contribution to the students' learning. However, while the students clearly benefited from their participation and gained a strong sense of the rational number and multiplicative reasoning there are questions that remain to be answered and limitations to be addressed.

7.8. Limitations

Halving and Doubling

One of the limitation concerns the widespread use of the halving and doubling schema. On the one hand, we have seen from the data that this schema provided powerful support for the students: this schema formed the basis for students ability to compare and order numbers in different representations, e.g., "3/8 is the same as 37 1/2 % so it is smaller than .40." This halving schema is also at the root of students' ability to solve complex calculations such as "what is 65% of 160? One half of 160 = 80 a half of that is 40, etc." More fundamentally is the salience of halving and doubling for multiplicative reasoning. As Confrey and Kieren have suggested the halving operation is very distinct from the additive notions of counting and thus, it functions to provide the students with a multiplicative basis for their work in rational number.

However, the featuring of halving has its disadvantages. Although students came to know various representations for halves, quarters, eighths, and sixteenths, and could use these quantities with ease, they had less exposure to other single unit fractions such as 1/3, 1/5, etc. and were less able to work with these quantities. As well, many of the students were not successful in working with tenths. All of the students in the experimental group could successfully determine 12 1/2% of a quantity. Many could calculate 6 1/4%. However, this program appeared to accomplish less for the reinforcement of base-10 notions. In fact, when operations with tenths were attempted, students tended to work procedurally rather than conceptually. An important question that remains then is whether we could accomplish more competence with base-10 understandings given a longer teaching sequence. Although this would be desirable, it is not clear that a curriculum that is based on halving can work as successfully with tenths.

Decimals

A second potential and related limitation concerns the depth of decimal learning that the students were able to accomplish. In the rational number program decimals are introduced as another way to represent percent. Although conceptually, this was useful for students when they were working with two place decimals, it was less beneficial for the learning of other kinds of decimals. As well, the lack of attention to tenths and thousandths also made the learning of these other decimals problematic. This question must also be looked at in the scope of a longer intervention.

Discrete Quantity

In this program the students worked almost exclusively with continuous as opposed to discrete quantity. The benefits of using continuous quantity were clear; students could physically manipulate the materials that they were presented to discover relationships, they could more easily understand the density property of rational number, and they rarely resorted to additive strategies. However, the flexible thinking and invention that they could access in working in continuous quantity was not readily available to them when they worked with discrete quantity. This was even true on problems where the same quantities were involved. On the posttest of Study 1 when the students were asked to show which is more 1/2 of 6 pennies or 1/3 of these pennies, many of the students were not able to show that 2 pennies were equal to 1/3 and thus, were not sure how to answer the question. Interestingly, all of the students could successfully compare these quantities in continuous contexts and could make drawings to show that 1/3 was the lesser amount. It is hoped that with a longer intervention there could be a greater focus on working in contexts with discrete quantity.

Advanced Understandings

Finally, questions arise about the potential of this curriculum for students' later learning. Although the results of this study indicate that the short-term gains for the students were very impressive as a result of the implementation of this curriculum, the long-term effects remain to be seen. It is possible that when these students continue their rational number learning using a more traditional approach, they will abandon their highly conceptual approach to problem-solving and come to rely on a more rote and algorithmic method of working. Unfortunately, the scope of my research has not allowed me the necessary follow-up experiments or longitudinal analyses that might provide an empirical answer to this question. What I do propose, however, is to investigate these questions in my future work.

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APPENDIX A

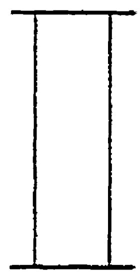
RATIONAL NUMBER TEST

Rational Number Test (Pretest)
Percent Test

1. If I said that we were 90% finished, would you think that we had a long way to go?

2. How much is 50% of eight dollars?

3. Draw a line on this beaker to show what it would look like if the beaker was approximately 25% full.

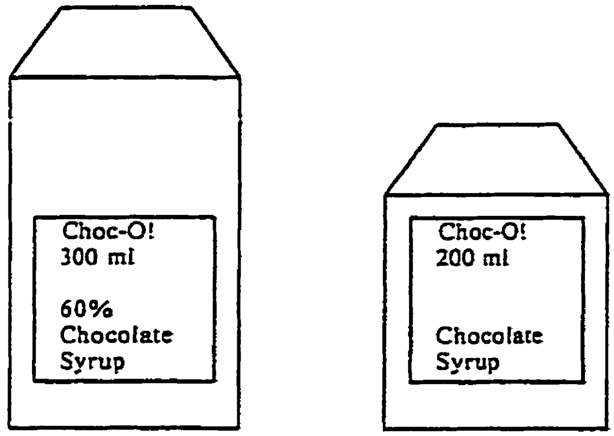


4. If this beaker holds a total of 80 ml. of water, how many mls. of water would there be if you had filled it 75% full?

5. Six blocks spilled out of a bag. This was 25% of the total number of blocks. How many blocks were in the bag to begin with?

6. The school went on a trip to hear Ani De Franco in concert. The total number of students in the school is 815. 70% of the students attended the concert. How many students would that be?

7. Below are two carons of chocolate milk. One carton contains 300ml, the other 200ml. Both cartons of chocolate milk come from the same vat. The milk is a mixture of chocolate syrup and milk. The company forgot to put the percentage of chocolate on the label of the smaller carton. What is the percentage of chocolate syrup in the smaller carton?



300 ml 200 ml

8. How many ml of syrup are in the smaller carton?

9. How much is 10% of ninety cents?

10. How would you write 6% as a decimal?

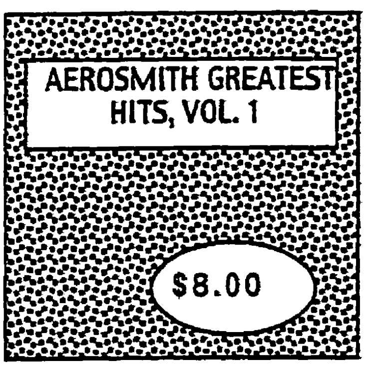
11. As a fraction?

12. What is 1/3 as a percent?

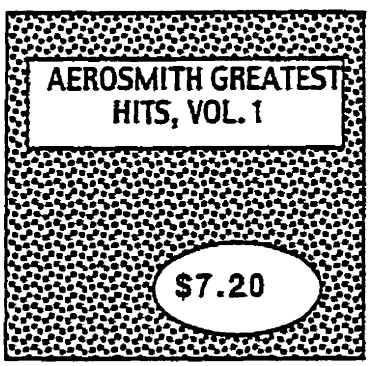
13. Suppose that you got 1/5 of the answers correct on a test, what would that be as a percent?

14. What is 65% of 160? Explain how you got your answer. What is the first thing that you need to do?

15. There was a sale at Sam's. This $8.00 CD (point) was on sale and the new price was $7.20. Sometimes when things go on sale they are say, 25% off the regular price. What do you think the percent discount is for the C.D.?



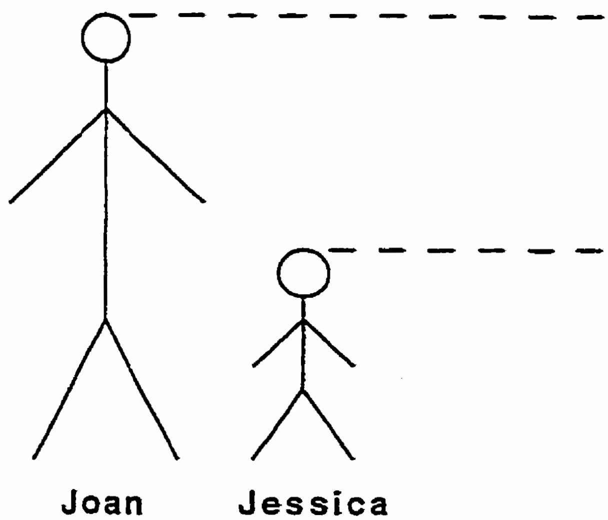
regular price



sale price

16. How much is 1% of four dollars?

17. Joan is 100% taller than Jessica. Jessica's height is _____% of Joan's.



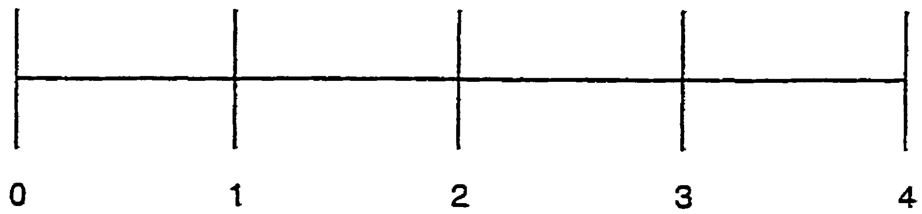
Fraction Test

(Divide 10 blocks into 3 groups of 3, 5 and 2. Shift group of 5 blocks ahead.) Is this half of the blocks?

Here is another group of blocks. *(Show 6 blocks).* Please show me which would be less, 1/3 of these blocks or 1/2 of these blocks. First show me 1/2 of the blocks. Now show me which is less?

Order these three numerals from smallest to largest: 1/2, 1, 1/3.

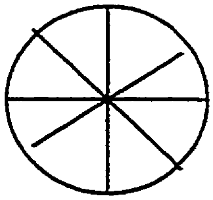
This is a number line. (Point to the whole line.) Where would you put the number 3 1/2?



How about the number 1 1/3?

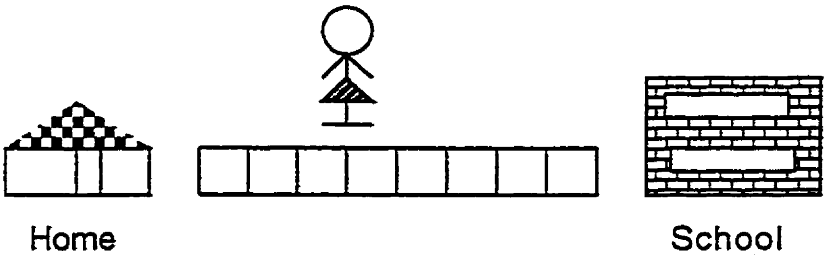
How about the number 1/4?

This is a pizza. Can you shade in 3/4 of this pizza?



Draw a picture to show which is greater, 3/4 or 2/3.

What fraction of the distance has Mary travelled from home to school?



How would you express that as a percent?

Another student told me that 7 is 3/4 of 10. Is it? Explain your answer.

. Can you think of a number taht comes between 1/2 and 1/3?

. What is 2 1/4 + 3 3/8?

. What is 4 3/4 + 6 6/8?

. What is 3 14 - 2 1/2?

. What is 1/2 of 1/8?

. How much is 2/3 of 6/8?

. Can you draw a picture to explain how you got the answer?

. What is 1/2 divided by 1/3? Explain how you got your answer.

Decimals Test

37. A package of blocks contains twenty blocks in all. Ten are yellow blocks and ten are blue blocks. Do you think that the yellow blocks are .5 of all the blocks?

38. Can you tell me a number that comes between .3 and .4?

39. Which is bigger, .20 or .089?

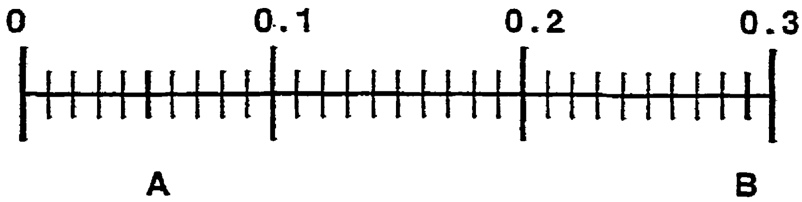
40. Which is bigger, tenths, hundredths, or thousandths?

41. How should you write seventy-five thousandths as a decimal?

42. How much is .5 + .38?

43. Can you construct the number 23.5 with these blocks using the long, 10-unit blocks as ones?

Look at this number line. What number is marked by the letter A?



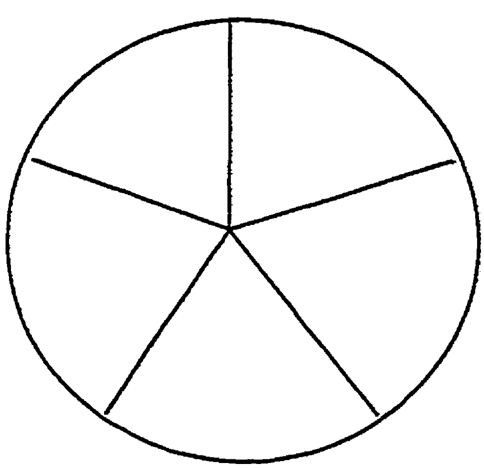
What number is marked by the letter B?

How much is 3.64 - .8?

What is 1/8 as a decimal, do you know? Explain your answer.

If you had 20 candies and you were told to give away .05 of all the candies, how many candies would you have to give away?

Shade in .3 of the circle. How did you know how much to shade?



How much is 3 x .4?

Could these be the same amount, .06 of a tenth and .6 of a hundredth?

Yes ______ or No ______ (Explain)

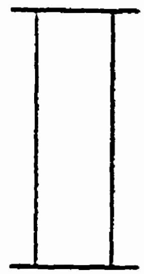
APPENDIX B

RATIONAL NUMBER INTERVIEW

Rational Number Interview

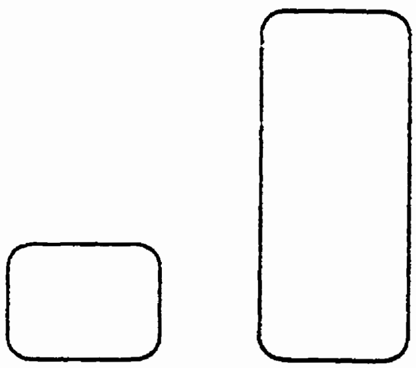
1. What is 50% of $8.00?

2. Draw a line on this beaker to show what it would look like if the beaker was approximately $\frac{1}{2}$ full?



3. Now, draw a line where $\frac{1}{4}$ full would be.

4. The can at the left holds 1 quart of oil which is the same as 2 pints. The can on the right holds 3 quarts of oil. How many pints will it hold?

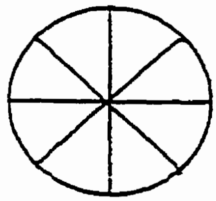


5. If a beaker holds a total of 80 ml. of water, how many mls. of water would there be if you filled it 75% full?

6. Can you tell me a number that comes between .3 and .4.

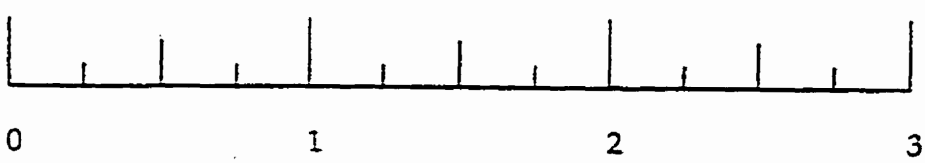
7. Which is bigger, .20 or .089?

8. This is a pizza. Can you shade in $\frac{3}{4}$ of this pizza?



9. What is $\frac{1}{2}$ of 84?

10 Can you place the number $\frac{1}{4}$ on this number line?



11. How much is .5 ÷ .38?

12. How would you write 6% as a decimal?

13 Estimate the answer to $\frac{12}{13} + \frac{7}{8}$

1
2
19
21
I don't know

14. A highschool student said that 7 is $\frac{3}{4}$ of 10. Is it? Explain your answer.

15. Order the following numbers from largest to smallest.

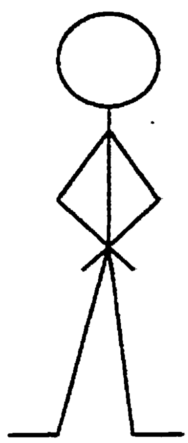
.48, 5/8, 14/13, .99, 1.03

16. Is there a fraction that comes between 1/4 and 2/4?

17. What is 65% of 160? Explain how you got your answer.

18. What is 1/8 as a decimal?

19. Mr. Short's height is 4 buttons or 6 paper clips.
His friend Mr. Tall's height is 6 buttons.
How many paper clips are needed for Mr. Tall's height?



20. Mrs. Cheever is 50% taller than her daughter. Her daughter's height is _____% of Mrs. Cheever's.