Constructive Approximation in Rational Systems

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Abstract

In recent years the following rational system

$$\mathcal{P}_m(a_1, a_2, \dots, a_n) := \left\{ \frac{P(x)}{\prod_{k=1}^{n} |x - a_k|}, \quad P \in \mathcal{P}_m \right\}$$

has been efficiently used in numerical analysis, where $\{a_k\}_{k=1}^{n} \subset \mathbb{C} \backslash [-1, 1]$. This thesis considers constructive approximation problems in the rational system above with prescribed poles $\{a_k\}_{k=1}^{n} \subset \mathbb{C} \backslash [-1, 1]$. Our constructive tools are interpolation, including Lagrange-type interpolation and Hermite-type interpolation, and the Bernstein-type 'polynomials'. We also consider the Bernstein-Markov inequality with respect to this rational system, which plays an important role in Lagrange-type interpolation and Hermite-type interpolation.

Chapter 1 introduces rational systems and related Chebyshev polynomials as well as some notations.

Chapter 2 characterizes the denseness of rational systems $\{\mathcal{P}_{n-1}(a_1, \dots, a_n)\}$ in $C[-1, 1]$. This extends a well-known result of Achiezer.

Chapter 3 is related to inequalities in rational systems. We first give a sharp (to constant) Markov-type inequality for real rational functions in $\mathcal{P}_n(a_1, a_2, \dots, a_n)$. The corresponding Markov-type inequality for high derivatives is established, as well as Nikolskii-type inequalities. A sharp Schur-type inequality is also proved, which plays a key role in the proofs of Markov-type inequalities. Finally, we consider inequalities for rational functions in $\mathcal{P}_n(a_1, a_2, \dots, a_n)$ with constrained conditions such as with curved majorants as well as with restricted real zeros for rational functions in $\mathcal{P}_n(a_1, a_2, \dots, a_n)$, which generalize some well-known results for classical polynomials.

Chapter 4 considers Lagrange-type interpolation in rational systems. The Lagrange-type interpolation is based on the zeros of the Chebyshev polynomial for the rational system

$\mathcal{P}_n(a_1, \ldots, a_n)$ with distinct real poles $\{a_k\}_{k=1}^n \subset \mathbb{R}\backslash[-1,1]$. The corresponding Lebsegue constant is estimated, and is shown to be asymptotically of order $\ln n$ when the poles stay outside an interval which contains $[-1,1]$ in its interior. Moreover, We conclude that the corresponding L^p-convergence $(0 < p < \infty)$ always holds for the continuous functions on $[-1,1]$ when the poles stay outside a circle which contains the unit circle in its interior. This extends the Erdős-Feldheim theorem for classical polynomial interpolation. As an application of the corresponding Lagrange-type interpolation, we also obtain a positive Gaussian-type quadrature formula.

Chapter 5 considers Hermite-type interpolation in rational systems with nonreal elements in $\{a_k\}_{k=1}^n \subset \mathbb{C}\backslash[-1,1]$ paired by complex conjugation. The Hermite-type interpolation is based on the zeros of the Chebyshev polynomial for the rational system $\mathcal{P}_n(a_1, \ldots, a_n)$. This extends some well-known results of Fejér and Grünwald for the classical polynomial case. More precisely, we prove that the corresponding Hermite-Fejér-type interpolation converges uniformly to the continuous function on $[-1,1]$ under the some conditions. Moreover, we characterize the uniform convergence of corresponding Grünwald-type interpolation.

In Chapter 6, we consider Bernstein-type 'polynomials' for the rational space $\{p(x)/\prod_{i=1}^n(1+t_i x),\, p \in \mathcal{P}_n\}$ associated with $t_i > -1$, $i = 1, \ldots, n$ on the interval $[0,1]$. Popoviciu-type theorem and asymptotic formula are established for these Bernstein-type polynomials. Some shape preserving properties of these Bernstein-type polynomials are presented. As an application of these Bernstein-type polynomials, we also consider the approximation problem in $\{p(x)/\prod_{i=1}^n(1+t_i x),\, p \in \mathcal{P}_n\}$ with $p(x)$ having integral coefficients.

Dedication

I would like to dedicate this to my son Michael and my wife Jane .

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Publication List

The material in this thesis has led to the following publications:

- On Denseness of Rational Systems, *J. Approx. Theory*, to appear. Chapter 2.
- Inequalities for Rational Functions with Prescribed Poles, *Canadian J. Mathematics.*, in press. Chapter 3.
- Inequalities for the Derivatives of Rational Functions with Real Zeros, *Acta Math. Hungar.*, to appear. Chapter 3.
- Lagrange Interpolation and Quadrature Formula in Rational Systems, *J. Approx. Theory*, to appear. Chapter 4.
- Lobatto-type Quadrature Formula in Rational Spaces, *J. Comp. and Appl. Math.*, to appear. Chapter 4.
- L^p-Convergence of Lagrange-type Interpolation in Rational Spaces, "Approximation Theory IX" (eds. Charles K. Chui and L. Schumaker), to appear. Chapter 4.
- Hermite-Fejér Interpolation for Rational Systems, *Constructive Approximation*, in press. Chapter 5.
- Bernstein-type Polynomials for Rational Systems with Prescribed Poles, submitted to *J. Approx. Theory*. Chapter 6.

Contents

Chapter 1

Introduction

1.1 Chebyshev System

A *Chebyshev system* $\{u_k\}_{k=0}^n$ on an interval $[a,b]$ is a set of $n+1$ continuous functions on $[a,b]$ such that any element of $H_n := \text{span}\{u_0, u_1, ..., u_n\}$ that has $n+1$ distinct zeros in $[a,b]$ is identically zero. The following simple equivalences hold:

Proposition 1.1.1 (Equivalences) *Let $\{u_k\}_{k=0}^n$ on an interval $[a,b]$ be a set of $n+1$ continuous functions on $[a,b]$. Then the following are equivalent:*

(a) *Every $0 \neq p \in \text{span}\{u_0, u_1, ..., u_n\}$ has at most n distinct zeros in $[a,b]$.*

(b) *If $x_0, \dots, x_n$ are distinct points of $[a,b]$ and $y_0, \dots, y_n$ are real numbers, then there exists a unique $p \in \text{span}\{u_0, u_1, \dots, u_n\}$ such that*

$$p(x_i) = y_i, \quad i = 0, 1, \dots, n.$$

(c) *If $x_0, \dots, x_n$ are distinct points of $[a,b]$, then $D(x_0, \dots, x_n) \neq 0$, where*

$$D(x_0, \dots, x_n) := \begin{vmatrix} u_0(x_0) & \dots & u_n(x_0) \\ u_0(x_1) & \dots & u_n(x_1) \\ \vdots & \ddots & \vdots \\ u_0(x_n) & \dots & u_n(x_n) \end{vmatrix}$$

.

We say that $\{u_0, u_1, ..., u_n\}$ is a *Markov system* on $[a,b]$ if each $u_i \in C[a,b]$ and $\{u_0, u_1, ..., u_m\}$ is a Chebyshev system for each $m = 0, 1, \dots, n$.

For a given Chebyshev system $\{u_k\}_{k=0}^n$, we can define the *generalized Chebyshev polynomial* as (cf. [1] [6] [12] [15] [32] [40] [69])

$$T_n := \sum_{k=0}^{n} \alpha_k u_k$$

for H_n on $[a, b]$ by equi-oscillation properties. More precisely, there exists an alternation set of length $n+1$: $a \le x_0 < x_1 < \cdots < x_n \le b$ for T_n on $[a, b]$, that is

$$T_n(x_k) = (-1)^k \|T_n\|_{[a,b]} = (-1)^k, \quad k = 0, 1, ..., n,$$

In the above formula and throughout this paper, $\|.\|_A$ denotes the supremum norm on $A \subset \mathbb{R}$.

Many extremal problems are solved by the Chebyshev polynomials (cf. [6], [69]) and the denseness of the Markov space is also intimately tied to the location of the zeros of the associated Chebyshev polynomials (cf. [4, Theorem 1]). Chebyshev polynomials are ubiquitous and have many applications, ranging from analysis, statistics, numerical methods, to number theory (cf. [6], [23], [32] and [69]). In this chapter, we introduce rational systems with prescribed poles and related Chebyshev polynomials, which will be often used throughout chapter 2 to chapter 3.

1.2 Rational Systems and Related Chebyshev Polynomials

We let

$$\mathcal{P}_m(a_1, \ldots, a_n) := \left\{ \frac{P(x)}{\prod_{k=1}^n |x - a_k|}, \quad P \in \mathcal{P}_m \right\} \tag{1.2.1}$$

and

$$\mathcal{T}_m(a_1, \ldots, a_n) := \left\{ \frac{P(t)}{\prod_{k=1}^n |\cos t - a_k|}, \quad P \in \mathcal{T}_m \right\}, \tag{1.2.2}$$

where $\{a_k\}_{k=1}^n \subset \mathbb{C} \backslash [-1, 1]$ is a fixed set of poles such that $\prod_{k=1}^n (x - a_k) \in \mathcal{P}_n$. In other words, the nonreal poles form complex conjugate pairs. We define the numbers $\{c_k\}_{k=1}^n$ by

$$a_k := \frac{c_k + c_k^{-1}}{2}, \quad |c_k| < 1,$$

that is,

$$c_k = a_k - \sqrt{a_k^2 - 1}, \quad |c_k| < 1. \tag{1.2.3}$$

Note that $\left(a_k + \sqrt{a_k^2-1}\right)\left(a_k - \sqrt{a_k^2-1}\right) = 1$, throughout this paper, $\sqrt{a_k^2-1}$ will always be defined by (1.2.3).

When all the poles $\{a_k\}_{k=1}^n$ are distinct and real, $\mathcal{P}_n(a_1, a_2, \ldots, a_n)$ and $\mathcal{T}_n(a_1, a_2, \ldots, a_n)$ are simply the real spans of the following two systems

$$\left\{1, \frac{1}{x-a_1}, \frac{1}{x-a_2}, \ldots, \frac{1}{x-a_n}\right\}, \quad x \in [-1, 1], \tag{1.2.4}$$

and

$$\left\{1, \frac{1 \pm \sin t}{\cos t - a_1}, \frac{1 \pm \sin t}{\cos t - a_2}, \ldots, \frac{1 \pm \sin t}{\cos t - a_n}\right\}, \quad t \in [0, 2\pi), \tag{1.2.5}$$

respectively. Moreover, they are Chebyshev systems (cf. [6] [32] [69]).

There are very few situations where Chebyshev polynomials can be explicitly computed. However, the explicit formulae for the Chebyshev polynomials for the systems $\mathcal{P}_n(a_1, a_2, \ldots, a_n)$ and $\mathcal{T}_n(a_1, a_2, \ldots, a_n)$ with distinct real poles outside $[-1, 1]$ are implicitly contained in Achiezer [1]. Recently, Borwein, Erdélyi and Zhang [8] have derived analogue Chebyshev polynomials of the first and second kinds for these systems. Moreover, they allow *repeated poles* and *nonreal poles* in these systems, in which the nonreal poles form complex conjugate pairs. These Chebyshev polynomials are constructed as follows (cf. [8]).

Given $\{a_k\}_{k=1}^n \subset \mathbb{C}\backslash[-1, 1]$ such that its nonreal elements are paired by complex conjugation, therefore $\mathcal{P}_n(a_1, a_2, \ldots, a_n)$ and $\mathcal{T}_n(a_1, a_2, \ldots, a_n)$ are real rational spaces.
Let

$$M_n(z) := \left(\prod_{k=1}^n (z - c_k)(z - \bar{c}_k)\right)^{1/2}, \tag{1.2.6}$$

where the square root is defined so that $M_n^*(z) = z^n M_n(z^{-1})$ is analytic in a neighbourhood of the closed unit disk, and let

$$f_n(z) := \frac{M_n(z)}{z^n M_n(z^{-1})}. \tag{1.2.7}$$

Then the *Chebyshev polynomials of the first kind* for the rational systems $\mathcal{P}_n(a_1, a_2, \ldots, a_n)$ and $\mathcal{T}_n(a_1, a_2, \ldots, a_n)$ are defined by

$$T_n(x) := \frac{f_n(z) + f_n^{-1}(z)}{2}, \quad x = \frac{z + z^{-1}}{2}, \quad |z| = 1, \tag{1.2.8}$$

and

$$\tilde{T}_n(t) = T_n(\cos t), \quad t \in \mathbb{R}, \tag{1.2.9}$$

respectively, and the *Chebyshev polynomials of the second kind* are defined by

$$U_n(x) := \frac{f_n(z) - f_n^{-1}(z)}{z - z^{-1}}, \quad x = \frac{z + z^{-1}}{2}, \quad |z| = 1, \tag{1.2.10}$$

and

$$\tilde{U}_n(t) = U_n(\cos t)\, \sin t, \quad t \in \mathbb{R}. \tag{1.2.11}$$

It is shown in [8] that these Chebyshev polynomials preserve almost all of the elementary properties of the classical Chebyshev polynomials. More precisely, we have

Theorem A (cf. [8, Theorem 1.2, Corollary 4.9]) *Let T_n and U_n be defined by* (1.2.8) *and* (1.2.10) *from $\{a_k\}_{k=1}^n \subset \mathbb{C}\backslash[-1, 1]$ with nonreal elements paired by complex conjugation. Then*

(*a*) $T_n \in \mathcal{P}_n(a_1, a_2, \ldots, a_n)$ *and* $U_n \in \mathcal{P}_{n-1}(a_1, a_2, \ldots, a_n)$.

(*b*) $\|T_n\|_{[-1,1]} = \|\sqrt{1-x^2}\, U_n(x)\|_{[-1,1]} = 1$.

(*c*) *There are* $-1 = y_n < y_{n-1} < \cdots < y_1 < y_0 = 1$ *such that*

$$T_n(y_j) = (-1)^j, \quad j = 0, 1 \ldots, n.$$

(*d*)

$$T_n^2(x) + \left(\sqrt{1-x^2}\, U_n(x)\right)^2 = 1, \quad x \in [-1, 1]$$

(*e*) $T_n(x)$ *has exactly* n *zeros in* $[-1, 1]$:

$$-1 < x_n < \cdots < x_1 < 1. \tag{1.2.12}$$

The conclusion that $T_n(x)$ has exactly n zeros can also be found in [62].

We denote by

$$B_n(x) := \sum_{k=1}^{n} \Re\left(\frac{\sqrt{a_k^2-1}}{a_k-x}\right), \quad \tilde{B}_n(t) := B_n(\cos t), \tag{1.2.13}$$

which are called the *Bernstein factors* and they play important roles in [8]. When the nonreal elements in $\{a_k\}_{k=1}^n$ are paired by complex conjugation, it is easy to check that

$$B_n(x) = \sum_{k=1}^{n} \Re\frac{\sqrt{a_k^2-1}}{a_k-x} = \sum_{k=1}^{n} \frac{\sqrt{a_k^2-1}}{a_k-x} > 0.$$

Theorem B ([8, Theorem 2.1]) *Let $\tilde{T}_n$ and $\tilde{U}_n$ be defined by* (1.2.9) *and* (1.2.11) *from $\{a_k\}_{k=1}^n \subset \mathbb{C}\backslash[-1,1]$ with nonreal elements paired by complex conjugation. Then*

$$\tilde{T}_n'(t) = -\tilde{B}_n(t)\tilde{U}_n(t), \quad \tilde{U}_n'(t) = \tilde{B}_n(t)\tilde{T}_n(t), \quad t \in \mathbb{R} \tag{1.2.14}$$

and

$$\tilde{T}_n'(t)^2 + \tilde{U}_n'(t)^2 = \tilde{B}_n(t)^2, \quad t \in \mathbb{R} \tag{1.2.15}$$

where $\tilde{B}_n(t)$ is defined by (1.2.13).

Theorem C ([8, Lemma 4.4]) *Let T_n be defined by* (1.2.8) *from $\{a_k\}_{k=1}^n \subset \mathbb{R}\backslash[-1,1]$. Then, for $0 \le m \le n$, we have*

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} T_n(x)T_m(x)\,dx = \frac{\pi}{2}(-1)^{n+m}(1+c_1^2\cdots c_m^2)c_{m+1}\cdots c_n, \tag{1.2.16}$$

where $\{c_k\}_{k=1}^n$ is defined by (1.2.3), *and the empty product is understood to be* 1 *for $n=0$ or $m=n$.*

Therefore, $\{T_n(x)\}_{n=0}^\infty$ are *not* orthogonal in general, this property is different from that of the classical Chebyshev polynomials. However, a simple linear combination of T_n and T_{n-1}:

$$R_0 := 1, \quad R_n := T_n + c_n T_{n-1},$$

or

$$R_0^* = \frac{1}{\sqrt{\pi}}, \quad R_n^* = \sqrt{\frac{2}{\pi(1-c_n^2)}}(T_n + c_n T_{n-1}). \tag{1.2.17}$$

is orthogonal with respect to the weight $\frac{1}{\sqrt{1-x^2}}$ from $\{a_k\}_{k=1}^{\infty} \subset \mathbb{R}\backslash[-1,1]$ with distinct numbers. That is,

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} R_n^*(x) R_m^*(x)\, dx = \delta_{m,n}. \tag{1.2.18}$$

Many aspects of orthogonal rational functions and their applications can be found in the literature, for example, see [9] [11] [17] [39] [59] and the references therein.

The next explicit formulae of the Chebyshev polynomials with respect to $\mathcal{P}_n(a_1, \cdots, a_n)$ are found recently by Borwein, Erdélyi and Zhang [8].

Theorem D ([6, Theorem 3.5.4] or [8, Proposition 4.1]) *Let* $\{a_k\}_{k=1}^{n} \subset \mathbb{C}\backslash[-1,1]$ *be distinct such that the nonreal elements are paired by complex conjugation, and let* T_n *be the Chebyshev polynomial of the first kind for* $\mathcal{P}_n(a_1, a_2, \ldots, a_n)$*. Then*

$$T_n(x) = A_0 + \frac{A_1}{x-a_1} + \cdots + \frac{A_n}{x-a_n}, \tag{1.2.19}$$

where

$$A_0 = \frac{(-1)^n}{2}\left((c_1 \cdots c_n)^{-1} + c_1 \cdots c_n\right), \tag{1.2.20}$$

and

$$A_k = -\left(\frac{c_k^{-1} - c_k}{2}\right)^2 \prod_{j=1, j\neq k}^{n} \frac{1-c_k c_j}{c_k - c_j}, \quad k = 1, \ldots, n. \tag{1.2.21}$$

1.3 Why Should We Study the Rational Systems?

It is known that it is not good enough for the approximation of functions to use the classical polynomials in many practical problems. For example, for integrands having poles outside the interval of integration, it would be more natural to design quadrature rules to integrate exactly rational functions (not polynomials), which have the same or almost the same poles, of maximum possible degrees (cf. [25] [78]). Recently, Gautschi (cf. [25], [26]) has successfully used this idea for the computation of the following generalized Fermi-Dirac and Bose-Einstein integrals (also cf. [61] [71]):

$$F_k(\eta, \theta) = \int_0^{\infty} \frac{x^k \sqrt{1+\frac{1}{2}\theta x}}{e^{-\eta+x}+1}\, dx, \quad \theta \geq 0, \eta \in \mathbb{R},$$

$$G_k(\eta,\theta) = \int_0^\infty \frac{x^k\sqrt{1+\frac{1}{2}\theta x}}{e^{-\eta+x}-1}\,dx, \quad \theta \geq 0, \eta \leq 0.$$

The computation of all of these are closely tied to rational interpolation.

Moreover, since orthogonal rational functions play a very important role in Hankel and Toeplitz operators, continued fractions, moment problems, Carathéodory-Fejér interpolation, function algebras, and solving electrical engineering problems, related studies of rational systems and related orthogonal rational systems are now very active (cf. [8] [9] [11] [10] [17] [38] [39] and the references therein).

1.4 Assumption (A)

For the simplicity of the statements of our results, we here introduce an assumption, which plays an important role in the proofs of our main results of this thesis.

Assumption (A) Let the nonreal elements in $\{a_k\}_{k=1}^n \subset \mathbb{C}\backslash[-1,1]$ be paired by complex conjugation. If there exists some constant α such that

$$|a_k| \geq \alpha > 1, \tag{1.4.22}$$

i.e, the poles must stay outside a circle which contains the unit circle in its interior, then we say that $\{a_k\}_{k=1}^n \subset \mathbb{C}\backslash[-1,1]$ satisfy *Assumption* (A).

It is easy to see that Assumption (A) is equivalent to

$$|c_k| \leq \gamma, \quad k = 1, \ldots, n,$$

where $0 \leq \gamma = \alpha - \sqrt{\alpha^2-1} < 1$. If this condition is satisfied, we say that $\{c_k\}_{k=1}^n$ satisfy *Assumption* (C). For convenience, we often use Assumption (C) later, instead of assumption (A)..

Next we conclude that the Assumption (A) is equivalent to the assertion: the orthonormal rational systems $\{R_n^*(x)\}$ defined by (1.2.17) are uniformly bounded on $[-1,1]$.

Theorem 1.4.1 *Let* $\{a_k\}_{k=1}^n \subset \mathbb{R}\backslash[-1,1]$. *Then Assumption (A) is true if and only if* $\{R_n^*(x)\}$ *is uniformly bounded on* $[-1,1]$. *where* $\{R_n^*(x)\}$ *is defined by* (1.2.17).

Proof. Note that Assumption (A) is equivalent to Assumption (C) and $|T_n(x)| \leq 1$ for $x \in [-1,1]$, then we can easily prove the *only if* part. Next we prove the *if* part. We suppose that Assumption (A) doesn't hold, then there exists a subsequence $\{a_{n_k}\}$ such that

$$a_{n_k} \to 1, \quad or \quad a_{n_k} \to -1, \quad k \to \infty.$$

Without loss of the generality, we assume $a_{n_k} \to 1, \quad k \to \infty$. That is,

$$c_{n_k} \to 1, \quad k \to \infty.$$

But, we also have

$$\|R^*_{n_k}\|_{[-1,1]} \geq |R^*_{n_k}(1)| = \sqrt{\frac{2(1+c_{n_k})}{\pi(1-c_{n_k})}} \to \infty, \quad k \to \infty.$$

This contradicts the assumption, and we have completed the proof of the *if* part. □

1.5 Notations

In this section, we give some notations which will be used later.

• Let

$$\mathcal{P}_n := \left\{ p : p(x) = \sum_{k=0}^{n} b_k x^k, \quad b_k \in \mathbb{R} \right\} \tag{1.5.23}$$

be the set of all real algebraic polynomials of degree at most n and

$$\mathcal{T}_n := \left\{ t : t(\theta) = b_0 + \sum_{k=1}^{n} (b_k \cos kt + d_k \sin kt), \quad b_k, d_k \in \mathbb{R} \right\} \tag{1.5.24}$$

be the set of all real trigonometric polynomials of degree at most n.

• Let $\{a_k\}_{k=1}^{l} \subset \mathbb{C} \backslash [-1,1]$. Then we denote

$$\mathcal{P}_m(a_1, a_2, \ldots, a_l) := \left\{ \frac{P(x)}{\prod_{k=1}^{l} |x - a_k|}, \quad P \in \mathcal{P}_m \right\}, \tag{1.5.25}$$

$$\mathcal{T}_m(a_1, a_2, \ldots, a_l) := \left\{ \frac{P(t)}{\prod_{k=1}^{l} |\cos t - a_k|}, \quad P \in \mathcal{T}_m \right\}, \tag{1.5.26}$$

$$\mathcal{R}_m(a_1,\dots,a_l) := \left\{ \frac{P(x)}{\prod_{k=1}^{l}|x-a_k|^2}, \quad P \in \mathcal{P}_m \right\} \tag{1.5.27}$$

$$\mathcal{P}_m^*(a_1,a_2,\dots,a_l) := \left\{ P \in \mathcal{P}_m(a_1,a_2,\dots,a_l) : |P(x)| \le \sqrt{1-x^2},\ x \in [-1,1] \right\}, \tag{1.5.28}$$

and

$$\mathcal{P}_{m-1}^{**}(a_1,a_2,\dots,a_l) := \left\{ P \in \mathcal{P}_{m-1}(a_1,a_2,\dots,a_l) : \sqrt{1-x^2}|P(x)| \le 1,\ x \in [-1,1] \right\} \tag{1.5.29}$$

• **Modulus of Continuity**: Let $f(x)$ be defined on $[-1,1]$. Then

$$\omega(f,\delta) = \sup_{x,y\in[-1,1],|x-y|\le\delta} |f(x)-f(y)| \tag{1.5.30}$$

• **Dini-Lipschitz Condition**: If the condition

$$\lim_{\delta\to 0} \omega(f,\delta)\ln\delta = 0 \tag{1.5.31}$$

holds, then we say that the continuous function f satisfies the *Dini-Lipschitz condition* on $[-1,1]$.

• We use $\|.\|_A$ to denote the supremum norm on $A \subset \mathbb{R}$.

• If $f(x)$ is defined on $[-1,1]$, then we denote

$$E_n(f) := \inf_{p\in\mathbb{P}_n} \|f-p\|_{[-1,1]} \tag{1.5.32}$$

• Let $\{a_k\}_{k=1}^n \subset \mathbb{C}\setminus[-1,1]$. Then

$$E_n^R(f) := \inf_{\beta_k\in\mathbb{R}} \left\| f(x) - \left(\frac{\beta_1}{x-a_1} + \dots + \frac{\beta_n}{x-a_n} \right) \right\|_{[-1,1]}. \tag{1.5.33}$$

and

$$E_{n-1}^Q(f) := \inf_{\beta_k\in\mathbb{R}} \left\| f(x) - \left(\beta_0 + \frac{\beta_1}{x-a_1} + \dots + \frac{\beta_{n-1}}{x-a_{n-1}} \right) \right\|_{[-1,1]}. \tag{1.5.34}$$

• For $0 < p < \infty$, we denote

$$\|f\|_{v,p} := \left(\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} |f(x)|^p\,dx \right)^{1/p} \tag{1.5.35}$$

and

$$\|f\|_p := \left(\int_{-1}^{1} |f(x)|^p\,dx \right)^{1/p}. \tag{1.5.36}$$

• We use $C[-1,1]$ to denote the set of all continuous functions on $[-1,1]$ and we also denote

$$C^*[-1,1] := \{f,\ f \in C[-1,1], f(-1) = f(1) = 0\}.$$

• The symbol "$\sim$" is used as follows: if A and B are two expressions depending on some variables and indices, then

$$A \sim B \Leftrightarrow |AB^{-1}| \le c \quad \text{and} \quad |A^{-1}B| \le c.$$

• We use $d_i(\alpha)$ $(i = 1, 2, \dots)$ to denote some positive constant depending only on α, respectively.

Chapter 2

On the Denseness of Rational Systems

Overview

This chapter characterizes the denseness of the rational system

$$\mathcal{P}_{n-1}(a_1, \ldots, a_n)\,(n = 1, 2\ldots) := \left\{ \frac{P(x)}{\prod_{k=1}^{n}(x - a_k)}, \quad P \in \mathcal{P}_n \right\}\, n = 1, 2\ldots,$$

with the nonreal poles in $\{a_k\}_{k=1}^{\infty} \subset \mathbb{C}\backslash[-1,1]$ paired by complex conjugation. This extends an result of Achiezer.

2.1 Introduction

With respect to the denseness of the system $\text{span}\left\{\frac{1}{x-a_k}\right\}_{k=1}^{\infty}$ in $C[-1,1]$, the following well-known result is due to Achiezer [1, P254, Problem 7]:

Achiezer's Theorem *Let* $\{a_k\}_{k=1}^{\infty} \subset \mathbb{R}\backslash[-1,1]$ *be distinct. Then* $\text{span}\left\{\frac{1}{x-a_k}\right\}_{k=1}^{\infty}$ *is dense in* $C[-1,1]$ *if and only if*

$$\sum_{k=1}^{\infty}(1 - |c_k|) = \infty,$$

where $\{c_k\}_{k=1}^{\infty}$ *are defined in* (1.2.3).

In [5] Borwein and Erdélyi also proved this by using entirely different methods.

Note that $\mathcal{P}_{n-1}(a_1, \dots, a_n)$ is still a real rational system when the nonreal poles of $\{a_k\}_{k=1}^n$ form complex conjugate pairs. So, it is natural to ask: whether we can extend Achiezer's Theorem to the case: the nonreal elements in $\{a_k\}_{k=1}^\infty \subset \mathbb{C}\backslash[-1,1]$ are paired by complex conjugation? We consider the above question and give an affirmative answer.

2.2 Extension of Achiezer's Theorem

Theorem 2.2.1 *Let the nonreal elements in* $\{a_k\}_{k=1}^\infty \subset \mathbb{C}\backslash[-1,1]$ *be paired by complex conjugation. Then* $\{\mathcal{P}_{n-1}(a_1, \dots, a_n)\}$ *is dense in* $C[-1,1]$ *if and only if*

$$\sum_{k=1}^{\infty}(1 - |c_k|) = \infty. \tag{2.2.1}$$

Our proof of Theorem 2.2.1 is mainly based on the Chebyshev polynomials with respect to $\mathcal{P}_n(a_1, \dots, a_n)$. We still use $T_n(x)$ to denote the Chebyshev polynomial of the first kind with respect to $\mathcal{P}_n(a_1, a_2, \dots, a_n)$.

Lemma 2.2.2 *Let the nonreal elements in* $\{a_k\}_{k=1}^n \subset \mathbb{C}\backslash[-1,1]$ *be paired by complex conjugation, and let* $T_n(x)$ *be the Chebyshev polynomial of the first kind with respect to* $\mathcal{P}_n(a_1, a_2, \dots, a_n)$*. Then the best approximation to* 1 *from* $\mathcal{P}_{n-1}(a_1, a_2, \dots, a_n)$ *is*

$$p := 1 - T_n/A_0. \tag{2.2.2}$$

Moreover, we have

$$\|1 - p\|_{[-1,1]} = 1/|A_0|. \tag{2.2.3}$$

where A_0 *is given by* (1.2.20).

Let $a \in \mathbb{R} \setminus [-1,1]$ such that $a \notin \{a_k\}_{k=1}^n$, then we define the constant c by

$$a := \frac{c + c^{-1}}{2}, \quad |c| < 1. \tag{2.2.4}$$

Let T_{n+1} be the Chebyshev polynomial of the first kind with respect to $\mathcal{P}_{n+1}(a_1, \dots, a_n, a)$. Then, Lemma 2.2.3 gives the best approximation to $\frac{1}{x-a}$ from $\mathcal{P}_n(a_1, \dots, a_n)$. That is,

Lemma 2.2.3 *Let the nonreal elements in $\{a_k\}_{k=1}^n \subset \mathbb{C}\backslash[-1,1]$ be paired by complex conjugation. Then, for $a \in \mathbb{R} \setminus [-1,1]$ and $a \notin \{a_k\}_{k=1}^n$, the best approximation to $\frac{1}{x-a}$ from $\mathcal{P}_n(a_1, \dots, a_n)$ is*

$$q := \frac{1}{x-a} - \frac{T_{n+1}(x)}{B_{n+1}} \tag{2.2.5}$$

where

$$B_{n+1} := \left(\frac{c-c^{-1}}{2}\right)^2 \prod_{j=1}^{n} \frac{1-cc_j}{c-c_j}. \tag{2.2.6}$$

Proofs of Lemmas 2.2.2 – 2.2.3. Their proofs are similar and they can be proved by an argument of counting zeros. So we just prove Lemma 2.2.3. Since $a \in \mathbb{R} \setminus [-1,1]$ and $a \notin \{a_k\}_{k=1}^n$, we then can construct the Chebyshev polynomial of the first kind T_{n+1} for $\mathcal{P}_{n+1}(a_1, \dots, a_n, a)$ and it can be expressed as:

$$T_{n+1}(x) := s(x) + \frac{B_{n+1}}{x-a},$$

where $s(x) \in \mathcal{P}_n(a_1, \dots, a_n)$. Since

$$B_{n+1} = \lim_{x \to a} (x-a) T_{n+1}(x),$$

it is easy to show (2.2.6) by a simple calculation. Moreover, $q(x) = -s(x)/B_{n+1}$. Clearly, $q \in \mathcal{P}_n(a_1, \dots, a_n)$. Moreover, note that (cf. [8, Theorem 1.2]) $\|T_{n+1}\|_{[-1,1]} = 1$, we have

$$\left\| \frac{1}{x-a} - q(x) \right\|_{[-1,1]} = \frac{1}{|B_{n+1}|}. \tag{2.2.7}$$

Suppose that there exists some $t \in \mathcal{P}_n(a_1, \dots, a_n)$ such that

$$\left\| \frac{1}{x-a} - t(x) \right\|_{[-1,1]} < \frac{1}{|B_{n+1}|}. \tag{2.2.8}$$

Recall that (cf. [8, Theorem 1.2]) there exist $n+2$ nodes: $-1 = y_{n+1} < y_n < \cdots < y_1 < y_0 = 1$ such that $T_{n+1}(y_j) = (-1)^j$, $j = 0, \dots, n+1$. So,

$$\frac{T_{n+1}}{B_{n+1}} - \left(\frac{1}{x-a} - t(x)\right) = -q + t \in \mathcal{P}_n(a_1, \dots, a_n)$$

changes sign between any two consecutive extrema of T_{n+1}. Furthermore, it has at least $n+1$ zeros in $(-1,1)$, and consequently, it must vanish identically. This contradicts (2.2.8). □

Proof of Theorem 2.2.1. We first prove the *only if* part. Note that $|c_k| < 1\,(k = 1, 2, \dots)$ and by (1.2.20) we then have

$$\prod_{k=1}^{n} |c_k| < \frac{1}{|A_0|} = \frac{2\prod_{k=1}^{n} |c_k|}{1 + \prod_{k=1}^{n} |c_k|^2} \leq 2\prod_{k=1}^{n} |c_k|,$$

that is $1/|A_0| \sim \prod_{k=1}^{n} |c_k|$. If $\{\mathcal{P}_{n-1}(a_1, \dots, a_n)\}_{n=1}^{\infty}$ are dense in $C[-1, 1]$, then by Lemma 2.2.2 we have $1/|A_0| \to 0\,(n \to \infty)$. That is $\prod_{k=1}^{\infty} |c_k| = 0$. This implies (2.2.1).

Next we prove *if* part. By (2.2.6) and (2.2.7) we have

$$\frac{1}{|B_{n+1}|} \to \left(\frac{2}{c - c^{-1}}\right)^2 \prod_{j=1}^{\infty} \left|\frac{c - c_j}{1 - cc_j}\right|, \quad n \to \infty.$$

Note that $\prod_{k=1}^{\infty} \frac{c - c_j}{1 - cc_j}$ is an infinite Blaschke product. Then by [82, Theorem 1, P281] or [70, Theorem 15.23, P311] we conclude that (2.2.1) implies

$$\prod_{j=1}^{\infty} \left|\frac{c - c_j}{1 - cc_j}\right| = 0.$$

Consequently, with (2.2.7), we conclude that $\frac{1}{x-a}$ can be uniformly approximated by $\{\mathcal{P}_n(a_1, \dots, a_n)\}_{n=1}^{\infty}$ on $[-1, 1]$. Also, if (2.2.1) holds, then by the analysis of the proof of the *only if* part and Lemma 2.2.2, we see that any constant can be uniformly approximated by $\{\mathcal{P}_{n-1}(a_1, \dots, a_n)\}_{n=1}^{\infty}$. Thus, we conclude that (2.2.1) implies that $\frac{1}{x-a}$ can be uniformly approximated by $\{\mathcal{P}_{n-1}(a_1, \dots, a_n)\}_{n=1}^{\infty}$ on $[-1, 1]$. Note that $a \in \mathbb{R}\backslash$ is an arbitrary number, so we can take a to be any of a sequence of distinct number satisfying the condition (2.2.1). This means $\frac{1}{x-a}$ can be taken as any of a dense sequence of distinct basis functions formed by $\frac{1}{x-a}$. Therefore, the *if* part follows. □

Chapter 3

Inequalities in Rational Systems

Overview

This chapter considers inequalities for the rational system $\mathcal{P}_n(a_1, a_2, \ldots, a_n)$ with prescribed poles $\{a_k\}_{k=1}^n \subset \mathbb{C}\backslash[-1,1]$. Section 3.2 introduces the Bernstein-Szegő-type inequality and the Bernstein-type inequality in this rational system. These inequalities are developed by Borwein, Erdélyi and Zhang [8]. A sharp Schur-type inequality is proved in section 3.3, and plays a key role in the proof of our Markov-type inequality. Section 3.5 gives a sharp (to constant) Markov-type inequality for real rational functions in $\mathcal{P}_n(a_1, a_2, \ldots, a_n)$. The corresponding Markov-type inequality for higher derivatives is also established in section 3.6. The Nikolskii-type inequalities are established in section 3.7. Section 3.8 considers inequalities for rational functions with some restrictions in $\mathcal{P}_n(a_1, a_2, \ldots, a_n)$. More precisely, some sharp Markov- and Bernstein-type inequalities with curved majorants for rational functions in $\mathcal{P}_n(a_1, a_2, \ldots, a_n)$ are obtained, and Turán-type inequalities are established for the derivatives of rational functions, whose zeros are all real and lie inside $[-1,1]$ but whose poles lie outside $(-1,1)$, in the supremum- and L^2−norms, respectively. Several well-known results for classical polynomials are generalized.

3.1 Introduction

The following two inequalities are fundamental to the proofs of many inverse theorems in polynomial approximation theory and of course have their own intrinsic interest, see, for example, Borwein and Erdélyi [6, Chapter 5], Cheney [12], Lorentz [40], Milovanović, Mitrinović and Rassias [45, Chapter 6], Natanson [54], Rivlin [69].

Markov Inequality. The inequality

$$\|P_n'\|_{[-1,1]} \le n^2 \|P_n\|_{[-1,1]}$$

holds for $P_n \in \mathcal{P}_n$.

Bernstein Inequality. The inequality

$$|P_n'(x)| \le \frac{n}{\sqrt{1-x^2}} \|P_n\|_{[-1,1]}, \quad x \in (-1,1)$$

holds for $P_n \in \mathcal{P}_n$.

There are many results on Bernstein's and Markov's inequalities and their generalization. For the interested readers, see, for example, Borwein and Erdélyi [6], Milovanović, Mitrinović and Rassias [45, Chapter 6] and Rahman and Schmeisser [68] and references therein.

On the other hand, the Bernstein-Markov type inequality does not exist for the arbitrary rational function. For example, if $r(x) = -\frac{\delta^2}{x^2+\delta^2}$, then $\|r\|_{[-1,1]} \le 1$ but $r'(\delta) = \frac{1}{2\delta}$ (cf. Lorentz [40]).

However, we can develop Bernstein-Markov type inequalities for rational functions with restricted denominators (cf. Borwein [4]). Recently, Borwein and Erdélyi and Zhang [8] considered the inequalities of rational functions with prescribed poles. For more information about inequalities of rational functions with prescribed poles on the unit disk or on the whole real axis, see, for example, Borwein and Erdélyi [6, Section 7.1] [7], Li [37], Jones Li, Mohapatra and Rodriguez [31] [38] and Petrushev and Popov [60]. This is an area of current research activity. For the application in the numerical analysis and related historical remarks concerning this kind of inequalities, see [36] [72] and [83].

3.2 Bernstein-Szegő-type Inequality

Borwein, Erdélyi and Zhang (cf. [8, Theorem 3.1]) obtained a remarkable extension of the well-known Bernstein-Szegő inequality for system $\mathcal{T}_n(a_1, a_2, \cdots, a_n)$, that is,

Theorem E (Bernstein-Szegő-type Inequality) *Let* $\{a_k\}_{k=1}^n \subset \mathbb{C}\backslash[-1,1]$ *with its nonreal elements being complex conjugation, and* $\tilde{B}_n(t)$ *be defined by* (1.2.13). *Then*

$$P'(t)^2 + \tilde{B}_n^2(t)P^2(t) \leq \tilde{B}_n^2(t) \max_{\tau\in\mathbb{R}} |P(\tau)|^2, \quad t \in \mathbb{R} \tag{3.2.1}$$

hold for every P *in* $\mathcal{T}_n(a_1, a_2, \ldots, a_n)$, *and equality holds in* (3.2.1) *if and only if* t *is a maximum point of* $|P|$, *or* P *is a linear combination of* $\tilde{T}_n$ *and* $\tilde{U}_n$, *where* $\tilde{T}_n$ *and* $\tilde{U}_n$ *are defined by* (1.2.9) *and* (1.2.11) *respectively.*

Borwein, Erdélyi and Zhang [8] also got a Bernstein-type inequality (cf. [8, Corollary 3.4]):

Theorem F (Bernstein-type Inequality) *Let the nonreal elements in* $\{a_k\}_{k=1}^n \subset \mathbb{C}\backslash[-1,1]$ *be paired by complex conjugation, and* $B_n(x)$ *be defined by* (1.2.13). *Then*

$$|P'(x)| \leq \frac{B_n(x)}{\sqrt{1-x^2}} \|P\|_{[-1,1]}, \quad x \in (-1,1) \tag{3.2.2}$$

holds for every $P \in \mathcal{P}_n(a_1, a_2, \ldots, a_n)$, *and equality holds in* (3.2.2) *if and only if* P *is a constant multiple of* T_n *and* x *is one of zeros of* T_n, *where* T_n *is defined by* (1.2.8).

3.3 Schur-type Inequality

In this section, we establish a sharp Schur-type inequality which plays a key role in the proof of our Markov-type inequality.

Theorem 3.3.1 (Schur-type Inequality) *Let the nonreal elements in* $\{a_k\}_{k=1}^n \subset \mathbb{C}\backslash[-1,1]$ *be paired by complex conjugation, and* $B_n(x)$ *be defined by* (1.2.13). *Then*

$$\|P\|_{[-1,1]} \leq \|B_n\|_{[-1,1]} \, \|\sqrt{1-x^2}P(x)\|_{[-1,1]} \tag{3.3.3}$$

holds for every $P \in \mathcal{P}_{n-1}(a_1, a_2, \ldots, a_n)$.

Proof. We may assume that $\sqrt{1-x^2}|P(x)| \leq 1$, and we must prove $\|P\|_{[-1,1]} \leq \|B_n\|_{[-1,1]}$.

It is easy to see that our hypothesis implies that $\sin t\, P(\cos t) \in \mathcal{T}_n(a_1, a_2, \ldots, a_n)$ and $|\sin t\, P(\cos t)| \leq 1$. Applying the Bernstein-Szegő inequality (3.2.1) for $\sin t\, P(\cos t)$, we then have

$$\tilde{B}_n^2(t) \sin^2 t\, P^2(\cos t) + \left(\cos t P(\cos t) + \sin t \left\{\frac{d}{dt} P(\cos t)\right\}\right)^2 \leq \tilde{B}_n^2(t). \tag{3.3.4}$$

Let t_0 be a maximum point of $|P(\cos t)|$, that is, $|P(\cos t_0)| = \|P(\cos t)\|$. Then we have that $\frac{d}{dt}\{P(\cos t)\}|_{t=t_0} = 0$. Therefore,

$$\tilde{B}_n^2(t_0)\sin^2 t_0\, P^2(\cos t_0) + \cos^2 t_0\, P^2(\cos t_0) \le \tilde{B}_n^2(t_0), \tag{3.3.5}$$

or

$$\left(\tilde{B}_n^2(t_0) - 1\right)\sin^2 t_0\, P^2(\cos t_0) + P^2(\cos t_0) \le \tilde{B}_n^2(t_0). \tag{3.3.6}$$

We distinguish two cases: (i) $\tilde{B}_n(t_0) \ge 1$ and (ii) $\tilde{B}_n(t_0) < 1$. In the first case, (3.3.6) implies that $|P(\cos t_0)| \le \tilde{B}_n(t_0) \le \|B_n\|_{[-1,1]}$.

In the second case, (3.3.5) implies that

$$P^2(\cos t_0) + \left(\frac{1}{\tilde{B}_n^2(t_0)} - 1\right)\cos^2 t_0\, P^2(\cos t_0) < 1,$$

hence, $|P(\cos t_0)| < 1$ follows. Also, from the definition of $B_n(x)$ we have $\|B_n\|_{[-1,1]} \ge 1$. Thus, we still have $|P(\cos t_0)| \le \|B_n\|_{[-1,1]}$. On combining cases (i) and (ii), we complete the proof of Theroem 3.3.1. □

Remark. For the real poles case $\{a_k\}_{k=1}^n \subset \mathbb{R}\backslash[-1,1]$, [6, E.8, p. 337] also showed (3.3.3) using an entirely different method.

3.4 How Large Are the Bernstein Factors?

We now make an observation about the Bernstein factors $B_n(x)$ as defined by (1.2.13). We will show that $B_n(x)$ is a convex function on $[-1,1]$ when the poles $\{a_k\}_{k=1}^n$ are real. Therefore, we can easily calculate its norm which is usually dependent on $\{a_k\}_{k=1}^n$. Moreover, we show that its L^1-norm with Chebyshev weight is independent of $\{a_k\}_{k=1}^n$. More precisely, we have

Lemma 3.4.1. *Let $\{a_k\}_{k=1}^n \subset \mathbb{R}\backslash[-1,1]$ and $B_n(x)$ be defined by* (1.2.13)*. Then $B_n(x)$ is a convex function on $[-1,1]$ and its maximum on $[-1,1]$ is always attained at ± 1:*

$$\begin{aligned}\|B_n\|_{[-1,1]} &= \max\{B_n(-1), B_n(1)\} \\ &= \max\left\{\sum_{k=1}^n \frac{1-c_k}{1+c_k}, \sum_{k=1}^n \frac{1+c_k}{1-c_k}\right\} \ge n.\end{aligned} \tag{3.4.7}$$

Moreover, we have

$$\|B_n\|_{1,v} = \pi n.$$

Proof. Note that

$$B_n(x) = \sum_{a_k>0} \frac{\sqrt{a_k^2-1}}{a_k-x} + \sum_{a_k<0} \frac{\sqrt{a_k^2-1}}{-a_k+x},$$

where $\sqrt{a_k^2-1}$ denotes the principal square root of a_k^2-1. We can quickly show that $B_n''(x) \geq 0$ on $[-1,1]$. This implies that $B_n(x)$ is a convex function on $[-1,1]$. Note that $B_n(x) > 0$, so the first equality of (3.4.7) follows. By a slightly longer calculation we may show the second equality in (3.4.7). Note that

$$n^2 = \left(\sum_{k=1}^{n} \sqrt{d_k}\frac{1}{\sqrt{d_k}}\right)^2 \leq \sum_{k=1}^{n} d_k \sum_{k=1}^{n} \frac{1}{d_k}$$

for any $d_k > 0$. Hence, we may also prove the last inequality in (3.4.7).

In fact, $B_n(x)$ can be expressed as

$$B_n(x) = \sum_{k=1}^{n} sgn(a_k)\frac{\sqrt{a_k^2-1}}{a_k-x},$$

where $\sqrt{a_k^2-1}$ denotes the principal square root of a_k^2-1. By a simple calculation, we have

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\frac{1}{a-x}\,dx = \frac{sgn(a)\pi}{\sqrt{a^2-1}}.$$

Thus, it follows,

$$\int_{-1}^{1} \frac{B_n(x)}{\sqrt{1-x^2}}\,dx = \pi n.$$

This completes the proof. □

Remark. In general, Lemma 3.4.1 does not hold for $\{a_k\}_{k=1}^n \subset \mathbb{C}\backslash[-1,1]$. For example, take $a_1 = i, a_2 = -i$. It is not hard to show that

$$B_2(x) = \frac{2\sqrt{2}}{x^2+1},$$

which is not a convex function and $\|B_2\|_{[-1,1]} = B_2(0)$.

Note that $\|B_n\|_{[-1,1]} \geq n$ and equality holds when $c_k = 0\,(k = 1, \dots, n)$, that is classical polynomial case. Lemma 3.4.2 gives a sufficient condition which guarantees $B_n(x)$ to be asymptotic to n.

Lemma 3.4.2 *Let* $\{a_k\}_{k=1}^n \subset \mathbb{C}\backslash[-1,1]$ *satisfy Assumption (A). Then*

$$\frac{1-\gamma}{1+\gamma}n \leq B_n(x) \leq \left(\frac{1+\gamma}{1-\gamma}\right)^2 n, \quad x \in [-1,1], \tag{3.4.8}$$

and

$$|B_n'(x)| \leq d(\alpha)n, \quad x \in [-1,1]. \tag{3.4.9}$$

Proof. By a simple calculation we can show that

$$B_n(x) = \sum_{k=1}^n \frac{1-|c_k|^2}{|c_k - z|^2}.$$

Therefore, we have

$$\frac{1-\gamma}{1+\gamma}n \leq \sum_{k=1}^n \frac{1-|c_k|}{1+|c_k|} \leq B_n(x) \leq \sum_{k=1}^n \frac{1+|c_k|}{1-|c_k|} \leq \frac{1+\gamma}{1-\gamma}n,$$

and

$$|B_n'(x)| \leq \sum_{k=1}^n \frac{\sqrt{a_k^2-1}}{(|a_k|-1)^2} \leq 2\sum_{k=1}^n \frac{|c_k|}{1-|c_k|^2} \leq d(\alpha)n.$$

So, we have completed the proof. □

Example. We let

$$a_k := \frac{1}{\cos\frac{2k-1}{2n}\pi}, \quad k = 1, \ldots, n,$$

where $\cos\frac{2k-1}{2n}\pi$ $(k = 1, \ldots, n)$ are the zeros of the classical n-th Chebyshev polynomial of the first kind. Then, for $k = 1, \ldots, n$, we have

$$c_k = \sec\theta_k - \tan\theta_k,$$

where $\theta_k = \frac{2k-1}{2n}\pi$, $k = 1, \ldots, n$.

Note that the zeros of the classical Chebyshev polynomial are symmetric about the y-axis, and one then can check that $B_n(x)$ is an even function on $[-1,1]$. Furthermore, we have $\|B_n\|_{[-1,1]} = B_n(1)$ by Lemma 3.4.1. Therefore, by a simple calculation, we have

$$\|B_n\|_{[-1,1]} = \sum_{k=1}^n \frac{1+\sec\theta_k - \tan\theta_k}{1-\sec\theta_k + \tan\theta_k} = \sum_{k=1}^n \frac{1+\cos\theta_k - \sin\theta_k}{-1+\cos\theta_k+\sin\theta_k}.$$

Note that

$$\int_0^\pi \left(\frac{1+\cos x - \sin x}{-1+\cos x + \sin x} - \frac{2}{x}\right) dx = 2\ln\frac{2}{\pi},$$

and we conclude that

$$\frac{1}{n}\sum_{k=1}^{n}\left(\frac{1+\cos\theta_k-\sin\theta_k}{-1+\cos\theta_k+\sin\theta_k}-\frac{2}{\theta_k}\right)\sim 1.$$

Recall that

$$\frac{1}{n}\sum_{k=1}^{n}1/\theta_k\sim\ln n,$$

and we thus have

$$\|B_n\|_{[-1,1]}\sim n\ln n.$$

3.5 Markov-type Inequality

Now we state our main result.

Theorem 3.5.1 (Markov-type Inequality) *Let the nonreal elements in $\{a_k\}_{k=1}^n\subset\mathbb{C}\backslash[-1,1]$ be paired by complex conjugation, and $B_n(x)$ be defined by* (1.2.13). *Then*

$$\|P'\|_{[-1,1]}\le 2\|B_n\|_{[-1,1]}^2\|P\|_{[-1,1]} \tag{3.5.10}$$

holds for every $P\in\mathcal{P}_n(a_1,a_2,\cdots,a_n)$.

Corollary 3.5.2 *if $\{a_k\}_{k=1}^n\subset\mathbb{R}\backslash[-1,1]$ are real poles, then*

$$\|B_n\|_{[-1,1]}^2\le\sup_{0\ne P\in\mathcal{P}_n(a_1,a_2,\cdots,a_n)}\frac{\|P'\|_{[-1,1]}}{\|P\|_{[-1,1]}}\le 2\|B_n\|_{[-1,1]}^2. \tag{3.5.11}$$

Corollary 3.5.3 *Let the nonreal elements in $\{a_k\}_{k=1}^n\subset\mathbb{C}\backslash[-1,1]$ be paired by complex conjugation, and $B_n(x)$ be defined by* (1.2.13). *Then*

$$\|P'\|_{[-1,1]}\le\|B_n\|_{[-1,1]}^2\left(\max_{-1\le x\le 1}P(x)-\min_{-1\le x\le 1}P(x)\right) \tag{3.5.12}$$

for $p\in\mathcal{P}_n(a_1,a_2,\cdots,a_n)$. Particularly, if $0\le P(x)\le 1$ for $-1\le x\le 1$, we have

$$\|P'\|_{[-1,1]}\le\|B_n\|_{[-1,1]}^2, \tag{3.5.13}$$

and these inequalities are sharp up to a constant for $\{a_k\}_{k=1}^n \subset \mathbb{R}\backslash[-1,1]$.

Proofs of Theorem 3.5.1 *and Corollary* 3.5.2. Note that we allow repeated poles in $\mathcal{P}_n(a_1, a_2, \dots, a_n)$, and we denote $a_{n+1} := a_1, \dots, a_{2n} := a_n$. We can consider

$$P' \in \mathcal{P}_{2n-1}(a_1, a_2, \dots, a_{2n}).$$

Thus, by Theorem 3.3.1 we have

$$\|P'\|_{[-1,1]} \le \|B_{2n}\|_{[-1,1]} \|P'(x)\sqrt{1-x^2}\|_{[-1,1]}. \tag{3.5.14}$$

But, in this case, it is easy to see that

$$B_{2n}(x) = 2B_n(x).$$

Hence, combining (3.5.13) and (3.2.2) we conclude that

$$\|P'\|_{[-1,1]} \le 2\|B_n\|_{[-1,1]}^2 \|P\|_{[-1,1]}.$$

Next we will show the left-side inequality in (3.5.11). From Theorem A(d) we have

$$U_n^2(x) = \frac{1 - T_n^2(x)}{1 - x^2}.$$

Thus one can easily get that

$$U_n^2(\pm 1) = |T_n'(\pm 1)| = |B_n(\pm 1) U_n(\pm 1)|.$$

This implies

$$T_n'(1) = (B_n(1))^2, \quad |T_n'(-1)| = (B_n(-1))^2. \tag{3.5.15}$$

Hence, by taking $P := T_n \in \mathcal{P}_n(a_1, a_2, \dots, a_n)$ and using Lemma 3.4.1, one can show the left-side inequality in (3.5.11). □

Remark. If $\{a_k\}_{k=1}^n \subset \mathbb{R}\backslash[-1,1]$, then (3.5.11) can also be expressed as

$$\max\{|T_n'(-1)|, |T_n'(1)|\} \le \sup_{0 \ne P} \frac{\|P'\|_{[-1,1]}}{\|P\|_{[-1,1]}} \le 2 \max\{|T_n'(-1)|, |T_n'(1)|\}, \tag{3.5.16}$$

where the supremum is taken for $P \in \mathcal{P}_n(a_1, a_2, \dots, a_n)$, and $T_n(x)$ is defined by (1.2.8).

From the above theorems, the estimate of $\|P'\|_{[-1,1]}$ (Markov-type inequality) and the pointwise estimate of $|P'(x)|$ (Bernstein-type inequality) are dependent on the given poles $\{a_k\}_{k=1}^n$ for $P \in \mathcal{P}_n(a_1, a_2, \dots, a_n)$. However, Borwein, Erdélyi and Zhang [8] observed the following result:

$$|P'(0)| \le n\|P\|_{[-1,1]} \tag{3.5.17}$$

for $P \in \mathcal{P}_n(a_1, a_2, \dots, a_n)$ and real poles $\{a_k\}_{k=1}^n \subset \mathbb{R}\backslash[-1,1]$. Therefore, by a linear transformation , they obtained (cf. [8, Corollary 3.7])

$$|P'(x)| \le \frac{n}{1-|x|}\|P\|_{[-1,1]}, \quad x \in (-1,1) \tag{3.5.18}$$

for every $P \in \mathcal{P}_n(a_1, a_2, \dots, a_n)$ and real poles $\{a_k\}_{k=1}^n \subset \mathbb{R}\backslash[-1,1]$.

It may be reasonable to replace the factor $1-|x|$ by $1-x^2$ in (3.5.18). Indeed, we have

Lemma 3.5.4 *Let $\{a_k\}_{k=1}^n \subset \mathbb{R}\backslash[-1,1]$. Then*

$$|P'(x)| \le \frac{n}{1-x^2}\|P\|_{[-1,1]} \tag{3.5.19}$$

for $P \in \mathcal{P}_n(a_1, a_2, \dots, a_n)$ and $x \in (-1,1)$.

Proof. Let $Q(y) := P\left(\frac{x+y}{1+xy}\right)$ for given $x \in (-1,1)$, then it is easy to see that $Q \in \mathcal{P}_n(b_1(x), b_2(x), \dots, b_n(x))$ and $\|Q\|_{[-1,1]} = \|P\|_{[-1,1]}$, where $b_k(x) = (a_k - x)/(1 - xa_k) \in \mathbb{R}\backslash[-1,1]$. Moreover, by (3.5.17) we have

$$(1-x^2)|P'(x)| = |Q'(0)| \le n\|Q\|_{[-1,1]} = n\|P\|_{[-1,1]},$$

which is nothing but (3.5.19). □

Hence, it is possible to obtain the following Markov-type inequality by exactly the same way as the proof of [8, Theorem 3.5]:

Theorem 3.5.5 *Let $\{a_k\}_{k=1}^n \subset \mathbb{R}\backslash[-1,1]$ and $\{c_k\}_{k=1}^n$ be defined by (1.2.3). Then*

$$\begin{aligned} \|P'\|_{[-1,1]} &\le \frac{2\sqrt{n}}{\sqrt{n}+\sqrt{n-1}}\left(\sum_{k=1}^n \frac{1+|c_k|}{1-|c_k|}\right)^2 \|P\|_{[-1,1]} \\ &\le \sqrt{\frac{n}{n-1}}\left(\sum_{k=1}^n \frac{1+|c_k|}{1-|c_k|}\right)^2 \|P\|_{[-1,1]} \end{aligned} \tag{3.5.20}$$

hold for every $P \in \mathcal{P}_n(a_1, a_2, \dots, a_n)$, $n = 1, 2, \dots$.

Remark. [8, Theorem 3.5] got similar estimates to (3.5.20), but instead of $\sqrt{\frac{n}{n-1}}$ by $\frac{n}{n-1}$.

3.6 Markov-type Inequality for High Derivatives

We may establish a corresponding Markov-type inequality for high derivatives.

Theorem 3.6.1 (Markov-type Inequality for High Derivatives) *Let the nonreal elements in* $\{a_k\}_{k=1}^n \subset \mathbb{C}\backslash[-1,1]$ *be paired by complex conjugation, and* $B_n(x)$ *be defined by* (1.2.13). *Then*

$$\|P^{(m)}\|_{[-1,1]} \le m!\,(m+1)!\|B_n\|_{[-1,1]}^{2m}\|P\|_{[-1,1]} \tag{3.6.21}$$

holds for every $P \in \mathcal{P}_n(a_1, a_2, \dots, a_n)$ *and* $m = 1, 2, \dots$.

Proof. We prove this by induction on m. The case of $m = 1$ is from Theorem 3.5.1. Suppose that (3.6.21) is true for $m = k$, that is

$$\|P^{(k)}\|_{[-1,1]} \le k!(k+1)!\|B_n\|_{[-1,1]}^{2k}\|P\|_{[-1,1]} \tag{3.6.22}$$

for every $P \in \mathcal{P}_n(a_1, a_2, \dots, a_n)$.

Let $a_{in+1} = a_1, \dots, a_{(i+1)n} = a_n$, $i = 1, \dots, k+2$, then we can consider

$$P^{(k+1)} \in \mathcal{P}_{(k+2)n-1}(a_1, a_2, \dots, a_{(k+2)n})$$

as in the proof of Theorem 3.5.1. Similarly, we have

$$B_{(k+2)n}(x) = (k+2)B_n(x),$$

where $B_{(k+2)n}(x)$ is the corresponding Bernstein factor with respect to $\mathcal{P}_{(k+2)n}(a_1, a_2, \dots, a_{(k+2)n})$. Now using (3.6.22) and applying the Schur-type inequality (3.3.3) and the Bernstein-type inequality (3.2.2) for $P^{(k+1)}$, we have

$$\begin{aligned}\|P^{(k+1)}\|_{[-1,1]} &\le \|B_{(k+2)n}\|_{[-1,1]}\|\sqrt{1-x^2}P^{(k+1)}(x)\|_{[-1,1]} \\ &\le (k+2)\|B_n\|_{[-1,1]}\|B_{(k+1)n}\|_{[-1,1]}\|P^{(k)}\|_{[-1,1]} \\ &\le (k+1)!(k+2)!\left(\|B_n\|_{[-1,1]}\right)^{2(k+1)}\|P\|_{[-1,1]}.\end{aligned}$$

Hence (3.6.21) holds for $m = k+1$ and we complete the proof. □

3.7 Nikolskii-type Inequality

Theorem 3.7.1 (Nikolskii-type Inequality) *Let the nonreal elements in* $\{a_k\}_{k=1}^n \subset \mathbb{C}\backslash[-1,1]$ *be paired by complex conjugation, and* $B_n(x)$ *be defined by* (1.2.13)*, then*

$$\|P\|_p \leq 2\left\{2\|B_n\|_{[-1,1]}\right\}^{2(1/q-1/p)}\|P\|_q \tag{3.7.23}$$

holds for every $P \in \mathcal{P}_n(a_1, a_2, \dots, a_n)$*, where* $\|P\|_p := \left(\int_{-1}^1 |P(x)|^p\,dx\right)^{1/p}$ *and* $0 < q < p \leq \infty$*.*

Proof. First we prove this for $p = \infty$. For given $P \in \mathcal{P}_n(a_1, a_2, \dots, a_n)$, we may suppose that $|P(y)| = \|P\|_{[-1,1]}$, where $y \in [-1,1]$. Also we denote $\lambda_n := 2\|B_n\|_{[-1,1]}^2$. Then by Theorem 3.5.1 and the Mean Value Theorem we get that

$$|P(x)| > \frac{1}{2}P(y) = \frac{1}{2}\|P\|_{[-1,1]}$$

for every $x \in I := \left\{t : |t-y| \leq \frac{1}{2\lambda_n}, t \in [-1,1]\right\}$. Thus

$$\|P\|_q^q \geq \int_I |P(t)|^q\,dt \geq \frac{1}{2^q}\|P\|_{[-1,1]}^q \frac{1}{2\lambda_n},$$

it follows that

$$\|P\|_{[-1,1]} \leq 2\left\{2\lambda_n\right\}^{1/q}\|P\|_q.$$

Therefore, for $0 < q < p < \infty$, we conclude that

$$\begin{aligned}\|P\|_p^p = \int_{-1}^1 |P(t)|^{p-q+q}\,dt &\leq \|P\|_{[-1,1]}^{p-q}\|P\|_q^q \\ &\leq \{2\lambda_n\}^{p-q/q}\,\|P\|_q^{p-q}\|P\|_q^q.\end{aligned}$$

This yields (3.7.23). □

In a certain weighted L^2-norm, we can get an exact Nikolskii-type inequality which has a smaller Nikolskii constant under some conditions. Precisely, we have

Theorem 3.7.2 *Let* $\{a_k\}_{k=1}^n \subset \mathbb{R}\backslash[-1,1]$ *be distinct and* $\{c_k\}_{k=1}^n$ *be defined by* (1.2.3)*. Then*

$$\|P\|_{[-1,1]} \leq \left(\frac{1}{\pi} + \frac{2}{\pi}\sum_{k=1}^n \frac{1+|c_k|}{1-|c_k|}\right)^{1/2}\|P\|_{2,v} \tag{3.7.24}$$

for $P \in \mathcal{P}_n(a_1, a_2, \cdots, a_n)$. *Moreover, if* $\{a_k\}_{k=1}^n$ *keep constant sign, then* (3.7.24) *is exact. Here* $\|P\|_{2,v} := \left(\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}|P(x)|^2\,dx\right)^{1/2}$.

Proof. Let $\{R_k^*\}_{k=0}^{\infty}$ be the orthonormal system with respect to the rational system $\mathcal{P}_n(a_1, a_2, \cdots, a_n)$, which is defined by (1.2.17). We may denote $P := \sum_{k=0}^n \alpha_k R_k^*$ and assume that $\|P\|_{2,v} = 1$, which implies $\sum_{k=0}^n \alpha_k^2 = 1$. Moreover, note that $\|T_n\|_{[-1,1]} = 1$ and by Cauchy's inequality we then have

$$\begin{aligned} P^2 \le \sum_{k=0}^n \alpha_k^2 \sum_{k=0}^n (R_k^*)^2 &\le \frac{1}{\pi} + \sum_{k=1}^n \left(\sqrt{\frac{2}{\pi(1-c_k^2)}}(1+|c_k|)\right)^2 \\ &= \frac{1}{\pi} + \frac{2}{\pi}\sum_{k=1}^n \frac{1+|c_k|}{1-|c_k|}. \end{aligned}$$

Hence, conclusion (3.7.24) follows. Taking

$$P = \frac{1}{\left(\frac{1}{\pi} + \frac{2}{\pi}\sum_{k=1}^n \frac{1+\mathrm{sgn}(c_k)c_k}{1-\mathrm{sgn}(c_k)c_k}\right)^{1/2}} \left\{\frac{1}{\sqrt{\pi}}R_0^* + \sum_{k=1}^n \left(\frac{2}{\pi}\frac{1+\mathrm{sgn}(c_k)c_k}{1-\mathrm{sgn}(c_k)c_k}\right)^{1/2} R_k^*\right\},$$

we can easily show that (3.7.24) is best possible under the hypotheses. □

3.8 Inequalities for Rational Functions with Restricted Conditions

Since the well-known Bernstein-Markov inequalities for the derivatives of polynomials were established, a series of papers has been devoted to sharpen and generalize these inequalities to polynomials with some curved majorants, with restricted zeros, and for other (weighted) norms. In these cases, the corresponding Bernstein-Markov inequalities can be improved.

In 1970, at a conference on "Constructive Function Theory" held in Varna, Bulgaria, P. Turán raised the following problem:

Problem. Determine $\max_{-1\le x\le 1}|P_n'(x)|$ for all polynomials $P_n(x)$ of degree at most n satisfying the restriction that

$$\sup_{-1<x<1} \frac{|P_n(x)|}{\sqrt{1-x^2}} = 1. \tag{3.8.25}$$

For real-valued polynomials, the hypothesis says that the graph of $P_n(x)$ on the interval $-1 < x < 1$ is contained in the closed unit disk.

Rahman [66, Theorem 1] completely solved the above problem by

Theorem G *Let $P_n \in \mathcal{P}_n$ satisfy $|P_n(x)| \le \sqrt{1-x^2}$ for $x \in [-1,1]$. Then*

$$\|P_n'\| \le 2(n-1), \tag{3.8.26}$$

and (3.8.26) *is sharp by $P_n(x) = (1-x^2)U_{n-2}(x)$, where $U_{n-2}(x)$ is the classical Chebyshev polynomial of the second kind.*

For the case of the restriction

$$|P_n(x)| \le (1-x^2)^{-1/2}, \tag{3.8.27}$$

Lachance [35] obtained the following Bernstein- and Markov- type inequalities

Theorem H *Let $P_n \in \mathcal{P}_n$ satisfy* (3.8.27). *Then*

$$|P_n'(x)| \le 2(n+1)(1-x^2)^{-1}, \quad -1 < x < 1, \tag{3.8.28}$$

and

$$\|P_n\|_{[-1,1]} \le n(n+1)^2, \tag{3.8.29}$$

and these inequalities are sharp to constant, respectively.

Rahman and his associates have extensively investigated these kinds of inequalities for classical polynomials. For more details, see, for example, [45, Section 6.1.4] and the references therein.

For Bernstein-Markov inequalities of polynomials with restricted zeros, The starting point of these generalizations is the following well-known result of Erdős [18]:

Theorem I (Erdős) *Let $P_n \in \mathcal{P}_n$ having all its zeros in $\mathbb{R}\backslash(-1,1)$. Then*

$$\|P_n'\|_{[-1,1]} \le \frac{en}{2}\|P_n\|_{[-1,1]} \tag{3.8.30}$$

and

$$|P_n'(x)| \le \frac{4\sqrt{n}}{(1-x^2)^2}\|P_n\|_{[-1,1]}, \quad x \in (-1,1). \tag{3.8.31}$$

Another fundamental result is the so-called Turán inequality which establishes a converse Markov inequality for polynomials with restricted real zeros (cf. Turán [76]), more precisely,

Theorem J (Turán) *Let $P_n \in \mathcal{P}_n$ having all its zeros in* $[-1,1]$. *Then*

$$\|P_n'\|_{[-1,1]} > \frac{\sqrt{n}}{6}\|P_n\|_{[-1,1]}. \tag{3.8.32}$$

Since then, a lot of extensions of Erdős' and Turán's results have been made, see, for example, Borwein and Erdélyi [6, Appendix A5] and Milovanović, Mitrinović and Rassias [45, Section 6.2] and the references therein.

It is natural to ask if we can extend the above results to the rational system $\mathcal{R}_n = \{p/q, p, q \in \mathcal{P}_n\}$ with restricted zeros and poles?

In the sixties Rahman [65] and Malik [44] established an analogue of (3.8.31) for rational functions which have neither zeros nor poles inside the unit circle. Moreover, in 1991 Rahman [67] sharpened it and found the corresponding best constant. But as far as I know, the question of how to establish the analogue of (3.8.30) for the rational functions is still open.

3.8.1 Inequalities for the Rational Functions with Some Curved Majorants

Here we try to generalize the above results for polynomials with curved majorants to rational systems. More precisely, we have

Theorem 3.8.1 *Let the nonreal elements in $\{a_k\}_{k=1}^n \subset \mathbb{C}\backslash[-1,1]$ be paired by complex conjugation, and $B_n(x)$ be defined by* (1.2.13). *Then, for $P \in \mathcal{P}_n^*(a_1, a_2, \cdots, a_n)$, we have*

$$\|P'\|_{[-1,1]} \le 2\|B_n\|_{[-1,1]} \tag{3.8.33}$$

and

$$|P'(x)| \le \left(x^2(1-x^2)^{-1} + \left(\|B_n\|_{[-1,1]} - 1\right)^2\right)^{1/2} \tag{3.8.34}$$

for $-1 < x < 1$.

Furthermore, if $\{a_k\}_{k=1}^n$ *satisfy Assumption (A), then*

$$2\left(\frac{1-\gamma}{1+\gamma}\right)^5 (n-2) \le \sup_{P\in\mathcal{P}_n^*(a_1,a_2,\dots,a_n)} \|P'\|_{[-1,1]} \le 2\frac{1+\gamma}{1-\gamma}n, \tag{3.8.35}$$

for $\{a_k\}_{k=1}^n \subset \mathbb{R}\backslash[-1,1]$, $n = 2, 3 \dots$., *where* $\mathcal{P}_n^*(a_1, a_2, \dots, a_n)$ *are defined by* (1.5.28).

Theorem 3.8.2 *Let the nonreal elements in* $\{a_k\}_{k=1}^n \subset \mathbb{C}\backslash[-1,1]$ *be paired by complex conjugation, and* $B_n(x)$ *be defined by* (1.2.13). *Then, for* $P \in \mathcal{P}_{n-1}^{**}(a_1, a_2, \dots, a_n)$, *we have*

$$(1-x^2)|P'(x)| \le 2\|B_n\|_{[-1,1]}, \quad x \in [-1,1] \tag{3.8.36}$$

and

$$\|P'\|_{[-1,1]} \le 2\|B_n\|_{[-1,1]}^3. \tag{3.8.37}$$

Furthermore, if $\{a_k\}_{k=1}^n$ *satisfy Assumption (A), then* (3.8.36) *is sharp up to a constant and*

$$\frac{1}{3}\left(\left(\frac{1-\gamma}{1+\gamma}\right)^3 n^3 - \frac{(1+\gamma)^2+2\gamma}{1-\gamma^2}n\right) \le \sup_P \|P'\|_{[-1,1]} \le 2\left(\frac{1+\gamma}{1-\gamma}\right)^3 n^3, \tag{3.8.38}$$

holds, where the supremum is taken for $P \in \mathcal{P}_{n-1}^{**}(a_1, a_2, \dots, a_n)$ *defined by* (1.5.29), $\{a_k\}_{k=1}^n \subset \mathbb{R}\backslash[-1,1]$.

Proof of Theorem 3.8.1. Our hypothesis implies that

$$P(x) = \frac{(1-x^2)P_{n-2}(x)}{\prod_{k=1}^n (x-a_k)} = \sqrt{1-x^2}\frac{\sqrt{1-x^2}P_{n-2}(x)}{\prod_{k=1}^n (x-a_k)} := \sqrt{1-x^2}Q(x), \tag{3.8.39}$$

and $|Q(x)| \le 1$.

Since

$$\begin{aligned} |P'(x)| &= \left|-x(1-x^2)^{-1/2}Q(x) + \sqrt{1-x^2}Q'(x)\right| \\ &\le |x|(1-x^2)^{-1/2}|Q(x)| + \left|\sqrt{1-x^2}Q'(x)\right| \end{aligned} \tag{3.8.40}$$

and $Q(\cos t) \in \mathcal{T}_n(a_1, a_2, \dots, a_n)$, by the Bernstein-Szegő-type inequality (3.2.1) for $\mathcal{T}_n(a_1, a_2, \dots, a_n)$ we have

$$\left|\frac{d}{dt}\{Q(\cos t)\}\right| \le \tilde{B}_n(t).$$

That is,

$$\left|\sqrt{1-x^2}Q'(x)\right| \le B_n(x). \tag{3.8.41}$$

Note that $(1-x^2)^{-1/2}Q(x) \in \mathcal{P}_{n-1}(a_1, a_2, \cdots, a_n)$. Moreover,

$$\left|(1-x^2)^{1/2}(1-x^2)^{-1/2}Q(x)\right| = |Q(x)| \le 1, \quad -1 \le x \le 1.$$

Thus, Theorem 3.3.1 yields

$$\left|(1-x^2)^{-1/2}Q(x)\right| \le \|B_n\|_{[-1,1]} \tag{3.8.42}$$

for $-1 < x < 1$. Combining (3.8.40), (3.8.41) and (3.2.2), we obtain (3.8.33).

Next we prove (3.8.35). Obviously, the right-side inequality in (3.8.35) follows from (3.8.33) and (3.8.39).

Let

$$P(x) := \frac{(1-x^2)U_{n-2}(x)}{(x-a_{n-1})(x-a_n)}(\mathrm{sgn}(a_{n-1}) - a_{n-1})(\mathrm{sgn}(a_n) - a_n)$$

where $U_n(x)$ is defined by (1.2.11). Since (cf. [8]) $\sqrt{1-x^2}|U_{n-2}(x)| \le 1$ for $-1 \le x \le 1$, thus, $P \in \mathcal{P}_n(a_1, a_2, \cdots, a_n)$ and $|P(x)| \le \sqrt{1-x^2}$. By Theorem B, it is easy to see that

$$T_n'(x) = B_n(x)U_n(x), \tag{3.8.43}$$

and

$$U_n'(x) = \frac{xU_n(x) - B_n(x)T_n(x)}{1-x^2}. \tag{3.8.44}$$

Furthermore, in this case, we easily get from (2.8.44) that

$$\begin{aligned}
|P'(1)| &\ge |U_{n-2}(1) + B_{n-2}(1)|\frac{(|a_{n-1}|-1)(|a_n|-1)}{(|a_{n-1}|+1)(|a_n|+1)} \\
&= 2B_{n-2}(1)\left(\frac{1-|c_{n-1}|}{1+|c_{n-1}|}\right)^2\left(\frac{1-|c_n|}{1+|c_n|}\right)^2 \\
&\ge 2\left(\frac{1-\gamma}{1+\gamma}\right)^4 B_{n-2}(1) \ge 2\left(\frac{1-\gamma}{1+\gamma}\right)^5 (n-2).
\end{aligned} \tag{3.8.45}$$

Similarly, we have

$$|P'(-1)| \ge 2\left(\frac{1-\gamma}{1+\gamma}\right)^5 (n-2). \tag{3.8.46}$$

Hence, we have shown the left-side inequality in (3.8.35).

Moreover, since $Q(\cos t) \in \mathcal{P}_n(a_1, a_2, \dots, a_n)$, on using Rahman's argument (cf.[66]), the Bernstein-Szegő inequality (3.2.1) and (3.8.40) we have

$$\begin{aligned}
|P'(x)| &\le |x|(1-x^2)^{-1/2}|Q(x)| + \left|\sqrt{1-x^2}Q'(x)\right| \\
&\le |x|(1-x^2)^{-1/2}|Q(x)| + \left(\|B_n\|_{[-1,1]} - 1\right)(1-|Q(x)|^2)^{1/2} \\
&\le \max_{-\le y \le 1} \left\{|x|(1-x^2)^{-1/2}y + \left(\|B_n\|_{[-1,1]} - 1\right)(1-y^2)^{1/2}\right\} \\
&\le \left(x^2(1-x^2)^{-1} + \left(\|B_n\|_{[-1,1]} - 1\right)^2\right)^{1/2}.
\end{aligned}$$

This implies (3.8.34). □

Proof of Theorem 3.8.2. From our hypothesis, we know that $\sin tP(\cos t) \in \mathcal{T}_n(a_1, a_2, \dots, a_n)$ and $|\sin tP(\cos t)| \le 1$. Then, applying the Bernstein-Szegő inequality (3.2.1) to $\sin tP(\cos t)$, we have

$$\left|\cos tP(\cos t) - \sin^2 tP'(\cos t)\right| \le \tilde{B}_n(t),$$

and combining Theorem 3.3.1 we get

$$(1-x^2)|P'(x)| = |\sin^2 tP'(\cos t)| \le \tilde{B}_n(t) + \|P\|_{[-1,1]} \le 2\|B_n\|.$$

Next we show that (3.8.36) is sharp up to the constant under the hypothesis. Let $P(x) := U_n(x)$, the Chebyshev polynomial of the second kind defined by (1.2.11). Taking $x = x_k$, as the zeros $U_n(x)$ in Theorem A(d), we have from (3.8.44)

$$(1-x_k^2)|U_n'(x_k)| = |B_n(x_k)T_n(x_k)| = B_n(x_k).$$

Hence (3.8.36) is sharp to constant by Lemma 3.4.2.

Combining the Markov-type inequality (3.5.10) and the Schur-type inequality (3.3.3), one can easily see that

$$\|P'\|_{[-1,1]} \le 2\|B_n\|^2_{[-1,1]}\|P\|_{[-1,1]} \le 2\|B_n\|^3_{[-1,1]},$$

and (3.8.37) follows.

On the other hand, by (3.8.44) we can easily get

$$U_n'(\pm 1) = -\frac{U_n(\pm 1) + U_n'(\pm 1) - B_n'(\pm 1) - B_n^2(\pm 1)U_n(\pm 1)}{2}, \tag{3.8.47}$$

this implies

$$|U_n'(1)| = \frac{1}{3}\left|B_n^3(1) + B_n'(1) - B_n(1)\right| > \frac{1}{3}\left(B_n^3(1) - |B_n'(1)| - B_n(1)\right).$$

Similarly, we have

$$|U_n'(-1)| > \frac{1}{3}\left(B_n^3(-1) - |B_n'(-1)| - B_n(-1)\right).$$

Hence combining Lemma 3.4.2 and (3.8.47) we can show that

$$\|U_n'\|_{[-1,1]} \geq \max\{|U_n'(1)|, |U_n'(-1)|\} \geq \frac{1}{3}\left(\left(\frac{1-\gamma}{1+\gamma}\right)^3 n^3 - \frac{(1+\gamma)^2 + 2\gamma}{1-\gamma^2} n\right), \quad (3.8.48)$$

but $U_n \in \mathcal{P}_{n-1}^{**}(a_1, a_2, \ldots, a_n)$, so the left-side inequality in (3.8.38) follows. The right-side inequality in (3.8.38) follows from (3.8.37). □

3.8.2 Inequalities for the Derivatives of Rational Functions with Real Zeros

In this section, we shall try to generalize the Turán inequality to rational functions whose zeros are all real and lie inside $[-1, 1]$ but whose poles lie outside $(-1, 1)$.

For convenience in stating our results, we give the following definition:

Definition 3.8.3 Let $\{\mu_k\}$ be a complex sequence, if there exists some $\rho > 0$ such that $|\mu_k| - 1 > \rho$, then we say that the $\{\mu_k\}$ are *away from the unit circle centered at the origin by* ρ. In particular, we call $\{\mu_k\}$ *away from the interval* $[-1, 1]$ *by* ρ if $\{\mu_k\}$ is a real sequence.

Theorem 3.8.4 (Turán Type Inequality in the Supremum-norm) *Let the nonreal elements in* $\{a_k\}_{k=1}^n \subset \mathbb{C}\backslash[-1, 1]$ *be paired by complex conjugation, and let* $\{a_k\}_{k=1}^n$ *be away from the unit circle centered at the origin by* ρ, $\rho > 2$. *Then, for* $P \in \mathcal{P}_n(a_1, a_2, \ldots, a_n)$ *with all its zeros in* $[-1, 1]$, *we have*

$$\|P'\|_{[-1,1]} > \frac{\sqrt{\rho^2 - 4}}{6\rho}\sqrt{n}\|P\|_{[-1,1]} \quad (3.8.49)$$

for $n \geq \max\left\{\frac{\rho+2}{9(\rho-2)}, \frac{4\rho^2}{\rho^2-4}\right\}$.

Note that the Bernstein-Markov inequality does not exist for an arbitrary rational function (cf. [40]), therefore, it is reasonable to restrict the poles of rational functions. However, whether the restriction of $\rho > 2$ can be removed is still open. Anyway, Turán's result (3.8.32) is the limiting case of the above result on letting all the poles go to $\pm\infty$.

Theorem 3.8.5 (Turán Type Inequality in the L^2-norm) *Let the nonreal elements in* $\{a_k\}_{k=1}^n \subset \mathbb{C}\backslash[-1,1]$ *be paired by complex conjugation. Then, for* $P \in \mathcal{P}_n(a_1, a_2, \ldots, a_n)$ *with all its zeros in the interval* $[-1,1]$*, we have*

$$\int_{-1}^{1} (1-x^2)(P'(x))^2\,dx \geq \frac{1}{2}\int_{-1}^{1} \mathcal{B}_n(x)P^2(x)\,dx \tag{3.8.50}$$

and the equality holds if and only if $P(x) = e_n(1-x)^m(1+x)^\ell / \prod_{k=1}^n (x-a_k)$*, where* $m+\ell = n$*,* $m, \ell \in \mathbb{N}$ *and* $e_n \in \mathbb{R}$*. In particular, we have*

$$\int_{-1}^{1} (P'(x))^2\,dx \geq \frac{1}{2}\int_{-1}^{1} \mathcal{B}_n(x)P^2(x)\,dx, \tag{3.8.51}$$

where

$$\mathcal{B}_n(x) := \sum_{k=1}^{n} \frac{a_k^2-1}{(x-a_k)^2}\,(>0) \tag{3.8.52}$$

is a convex function on $[-1,1]$ *for* $\{a_k\}_{k=1}^n \subset \mathbb{R}\backslash[-1,1]$.

Corollary 3.8.6 *Let* $\{a_k\}_{k=1}^n \subset \mathbb{R}\backslash[-1,1]$*. Then, for* $P \in \mathcal{P}_n(a_1, a_2, \ldots, a_n)$ *with all its zeros in* $[-1,1]$*, we have*

$$\int_{-1}^{1} (1-x^2)(P'(x))^2\,dx \geq \frac{1}{2}\sum_{k=1}^{n} \frac{|a_k|-1}{|a_k|+1}\int_{-1}^{1} P^2(x)\,dx. \tag{3.8.53}$$

In particular, if $\{a_k\}_{k=1}^n$ *are away from* $[-1,1]$ *by* ρ *for some* $\rho > 0$*, then,*

$$\int_{-1}^{1} (P'(x))^2\,dx \geq \frac{\rho n}{2(2+\rho)}\int_{-1}^{1} P^2(x)\,dx. \tag{3.8.54}$$

Obviously, Varma's result (cf. [79, Theorem 1] or [80]) is the limiting case of the above result (3.8.54) on letting all the poles go to $\pm\infty$.

Proof of Theorem 3.8.4. We modify Turán's argument to prove this. Let $\{x_k\}_{k=1}^n$ be the zeros of $P(x)$, that is, $P(x) := e_n \prod_{k=1}^n (x - x_k/x - a_k)$, where $e_n \in \mathbb{R}$. One can show that

$$\frac{P'(x)}{P(x)} = \sum_{k=1}^{n} \frac{1}{x-x_k} - \sum_{k=1}^{n} \frac{1}{x-a_k} \tag{3.8.55}$$

and

$$(P'(x))^2 - P(x)P''(x) = P^2(x)\left(\sum_{k=1}^{n}\frac{1}{(x-x_k)^2} - \sum_{k=1}^{n}\frac{1}{(a_k-x)^2}\right). \qquad (3.8.56)$$

Assume that $\|P\|_{[-1,1]} = 1$ and some $a \in [-1,1]$ such that $|P(a)| = 1$ (for example, without loss of generality, $P(a) = 1$). Now we distinguish two cases.

Case 1. If $a = \pm 1$, for example, $a = 1$. Then from (3.8.55) we have

$$\begin{aligned}|P'(a)| = \left|\frac{P'(a)}{P(a)}\right| &= \left|\sum_{k=1}^{n}\frac{1}{1-x_k} - \sum_{k=1}^{n}\frac{1}{1-a_k}\right| \\ &\geq \frac{n}{2} - \sum_{k=1}^{n}\frac{1}{|a_k|-1} \geq \left(\frac{1}{2}-\frac{1}{\rho}\right)n \\ &> \frac{\sqrt{\rho^2-4}}{6\rho}\sqrt{n}\end{aligned}$$

for $n \geq \frac{\rho+2}{9(\rho-2)}$.

Case 2. If $a \in (-1,1)$, then $P'(a) = 0$. Without loss of generality, we assume $a \in [-1,0]$. Let $I := [a, a + 2\rho/\sqrt{\rho^2-4}\sqrt{n}] \subset [-1,1]$ for $n \geq \frac{4\rho^2}{\rho^2-4}$.

(i). If $|P'(x)| \leq \frac{\sqrt{\rho^2-4}}{6\rho}\sqrt{n}$ on I, then $P(x) \geq 2/3$ on I, otherwise, by the Mean Value Theorem we see if there exists $\xi_1 \in I$ such that $P(\xi_1) < 2/3$. Then

$$\frac{\sqrt{\rho^2-4}}{6\rho}\sqrt{n} < \left|\frac{P(a)-P(\xi_1)}{\xi_1 - a}\right| = |P'(\xi_2)|,$$

where $\xi_2 \in I$.

(ii). We can also assume that $|P''(\xi)| \leq \frac{\rho^2-4}{12\rho^2}n$ for the some $\xi \in (a, a+2\rho/\sqrt{\rho^2-4}\sqrt{n})$, otherwise, we have

$$\left|P'(a+2\rho/\sqrt{\rho^2-4}\sqrt{n})\right| = \left|\int_a^{a+\frac{2\rho}{\sqrt{\rho^2-4}}\frac{1}{\sqrt{n}}} P''(t)\,dt\right| > \frac{\sqrt{\rho^2-4}}{6\rho}\sqrt{n}.$$

In short, we can assume that $P(x) \geq 2/3$ and that there exists some $\xi \in I$ such that $|P''(\xi)| \leq \frac{\rho^2-4}{12\rho^2}n$. Then, under the hypotheses, we have from (3.8.56)

$$(P'(\xi))^2 - P(\xi)P''(\xi) \geq \frac{4}{9}\left(\frac{n}{4} - \sum_{k=1}^{n}\frac{1}{(|a_k|-1)^2}\right) \geq \frac{4}{9}\left(\frac{n}{4}-\frac{n}{\rho^2}\right),$$

and (3.8.49) now follows by a simple calculation. □

Proofs of theorem 3.8.5 *and Corollary* 3.8.6. We still denote $P(x)$ by $P(x) := e_n \prod_{k=1}^{n}(x - x_k/x - a_k)$, where $e_n \in \mathbb{R}$. From (3.8.55) and (3.8.56) and by a slightly longer calculation, we have

$$\begin{aligned} &2(1-x^2)(P'(x))^2 - \frac{d}{dx}\left\{(1-x^2)P(x)P'(x)\right\} \\ &= P^2(x)\left(\sum_{k=1}^{n}\frac{1-x_k^2}{(x-x_k)^2} + \sum_{k=1}^{n}\frac{a_k^2-1}{(x-a_k)^2}\right). \end{aligned} \tag{3.8.57}$$

Hence, under the hypotheses, (3.8.57), we have

$$2(1-x^2)(P'(x))^2 - \frac{d}{dx}\left\{(1-x^2)P(x)P'(x)\right\} \geq P^2(x)\mathcal{B}_n(x)$$

and the equality holds if and only if all zeros of $P(x)$ are ± 1. Therefore, integrating both sides of the above inequality from -1 to 1, (3.8.50) follows.

Next we claim that $\mathcal{B}_n(x) > 0$ for $x \in [-1, 1]$. Since the nonreal elements in $\{a_k\}_{k=1}^{n}$ are paired by complex conjugation, we have

$$\mathcal{B}_n(x) = \sum_{k=1}^{n} \Re\frac{a_k^2-1}{(x-a_k)^2}.$$

Hence, we need only to show that $\Re\frac{a_k^2-1}{(x-a_k)^2} > 0$ for $x \in [-1, 1]$.

Let $a_k := \alpha_k + i\beta_k, \gamma_k := \sqrt{\alpha_k^2 + \beta_k^2}(> 1)$, then,

$$\Re\frac{a_k^2-1}{(x-a_k)^2} = \frac{(\alpha_k^2-\beta_k^2-1)x^2 - 2\alpha_k(\gamma_k^2-1)x + \gamma_k^4 - (\alpha_k^2-\beta_k^2)}{\left((x-\alpha_k)^2-\beta_k^2\right)^2 + (2\beta_k(x-\alpha_k))^2} := \frac{C_k(x)}{D_k(x)}.$$

Note that $D_k(x) > 0$ for $x \in [-1, 1]$ and

$$\begin{aligned} C_k(x) &= (\gamma_k^2-1)x^2 + 2\beta_k^2(1-x^2) - 2\alpha_k(\gamma_k^2-1)x + \gamma_k^2(\gamma_k^2-1) \\ &\geq (\gamma_k^2-1)x^2 - 2\alpha_k(\gamma_k^2-1)x + \gamma_k^2(\gamma_k^2-1) \\ &= (\gamma_k^2-1)(x^2 - 2\alpha_k x + \gamma_k^2) > 0, \quad x \in [-1, 1], \end{aligned}$$

hence we see that $\mathcal{B}_n(x) > 0$ for $x \in [-1, 1]$.

On the other hand, since $\mathcal{B}_n''(x) > 0$ on $[-1, 1]$ for $\{a_k\}_{k=1}^{n} \subset \mathbb{R}\backslash[-1, 1]$, it is strictly a convex function on $[-1, 1]$, which implies that $\|\mathcal{B}_n\|_{[-1,1]} = \max\{\mathcal{B}_n(1), \mathcal{B}_n(-1)\}$. Thus we conclude that

$$\sum_{k=1}^{n}\frac{|a_k|-1}{|a_k|+1} \leq \mathcal{B}_n(x) \leq \|\mathcal{B}_n\|_{[-1,1]} = \max\left\{\sum_{k=1}^{n}\frac{a_k+1}{a_k-1}, \sum_{k=1}^{n}\frac{a_k-1}{a_k+1}\right\}, \tag{3.8.58}$$

for $\{a_k\}_{k=1}^n \subset \mathbb{R}\backslash[-1,1]$, now (3.8.53) follows. ☐

Actually, from (3.8.57) we can obtain an analogue of Erdős' result (3.8.31) for the rational system $\mathcal{P}_n(a_1, a_2, \dots, a_n)$ in L^2–norm:

Theorem 3.8.7 *Let the nonreal elements in $\{a_k\}_{k=1}^n \subset \mathbb{C}\backslash[-1,1]$ be paired by complex conjugation, and $\mathcal{B}_n(x)$ be defined by (3.8.52). Then, for $P \in \mathcal{P}_n(a_1, a_2, \dots, a_n)$ with all its zeros in $\mathbb{R}\backslash(-1,1)$, we have*

$$\int_{-1}^{1} (1-x^2)(P'(x))^2\,dx \le \frac{1}{2}\int_{-1}^{1} \mathcal{B}_n(x)P^2(x)\,dx \tag{3.8.59}$$

and the equality holds if and only if $P(x) = e_n(1-x)^m(1+x)^\ell / \prod_{k=1}^n (x-a_k)$, where $m+\ell = n$, $m, \ell \in \mathbb{N}$ and $e_n \in \mathbb{R}$. In particular, we have

$$\int_{-1}^{1} (1-x^2)(P'(x))^2\,dx \le \frac{1}{2}\|\mathcal{B}_n\|_{[-1,1]} \int_{-1}^{1} P^2(x)\,dx \tag{3.8.60}$$

for $\{a_k\}_{k=1}^n \subset \mathbb{R}\backslash[-1,1]$, where $\|\mathcal{B}_n\|_{[-1,1]}$ is given by (3.8.58).

3.9 Problems

Problem 3.10.1. The best constant in Markov-type inequality (3.5.10) is still an open problem.

Problem 3.10.2. It would be interesting to establish an L_p version of the Markov-type inequality for rational functions with prescribed poles.

Chapter 4

Lagrange-type Interpolation in Rational Systems

Overview

This chapter considers Lagrange-type interpolation in the rational system $\mathcal{P}_{n-1}(a_1, \cdots, n)$ with distinct $\{a_k\}_{k=1}^n \subset \mathbb{R}\backslash[-1, 1]$. This Lagrange-type interpolation is based on the zeros of the Chebyshev polynomial of the first kind for the rational system $\mathcal{P}_n(a_1, \cdots, n)$. The corresponding Lebesegue constant is estimated, and is shown to be asymptotically of order $\ln n$ when the poles stay outside an interval which contains $[-1, 1]$ in its interior. The mean convergence of this Lagrange-type interpolation is also established. As an application of this interpolation, we construct a quadrature formula, it is a positive quadrature formula; moreover, it is exact for any element in $\mathcal{R}_{2n-1}(a_1, \cdots, n)$. Some well-known results of classical Lagrange polynomial interpolations are extended.

4.1 Introduction

Interpolation by polynomials is probably the oldest profession in Approximation. Turán [77] wrote "Newton, who wanted to draw conclusions from the observed location of comets at equidistant times as to their location at arbitrary times arrived at problem of determining a 'geometric' curve passing through arbitrarily many given points, solved this problem by the interpolation polynomial bearing his name. How highly he esteemed his result is revealed by his letter to Oldenburg of 1676, in which he wrote that this was one of the most beautiful results he had ever achieved. Newton uses his formula to give the exact value of $\int_a^b f(x)\,dx$ in terms of the values of $f(x_k)$ when $f(x)$ is a polynomial of degree n, and $x_k = a + ((b-a)/n)k, k = 0, \ldots, n$. His student Cotes called this quadrature formula 'pulcherrima et utilissima regula' and calculated its coefficients for $n \leq 10$. This work, based on Newton's interpolation formula, must have been quite awkward. Application of Lagrange's interpolation formula would have simplified it, but that was published only in 1795. Gauss' quadrature formula was also motivated by astronomy, namely by the investigation of the orbit of the planet Pallas. How important this formula was for Gauss is shown by the fact that unlike many other results, this one was not only worked out in his diary but also published, even prepublished. He used the zeros of n-th Legendre polynomial instead of equidistant points of observation. His treatment was later greatly simplified by Jacobi.

Thus we see that interpolation and the theory of mechanical quadrature are just two aspects of the study of functions given by a finite number of observations."

Interpolation theory up to now serves as an important tool of numerical analysis and computer science.

Let us briefly describe it. For a given function $f(x)$ on $[-1,1]$, let

$$-1 \leq x_n < \cdots < x_1 \leq 1, \tag{4.1.1}$$

then the corresponding *Lagrange interpolatory polynomial* of degree$\leq n-1$ is defined by

$$L_n(f,x) := \sum_{k=1}^{n} f(x_k) l_k(x), \quad n = 1, 2, \ldots, \tag{4.1.2}$$

where $\omega_n(x) := \prod_{k=1}^{n}(x - x_k)$, and

$$l_k(x) = \frac{\omega_n(x)}{\omega_n'(x_k)(x - x_k)}, \quad k = 1, \ldots, n; n = 1, 2, \ldots, \tag{4.1.3}$$

are the *fundamental polynomials* of Lagrange interpolation.

It's well-known that $L_n(f)$ is not guaranteed to converge to a continuous function $f(x)$ uniformly on $[-1,1]$. This result, which can be considered as the starting point of the divergence theory for Lagrange interpolation, is due to Faber [21].

Faber Theorem *For any nodes system* $\{x_k\}_{k=1}^{n}$ *there exists an* $f_1 \in C[-1,1]$ *with*

$$\overline{\lim}_{n\to\infty} \|L_n(f_1)\|_{[-1,1]} = \infty. \tag{4.1.4}$$

Note that this result does not exclude a pointwise convergence result at least at a single point. This question was negatively answered by Bernstein (cf. [2] [73, Chapter 4] or [54, Chapter 2 Vol. III]).

Bernstein Theorem *For any nodes system* $\{x_k\}_{k=1}^n$ *there exists a point* $x_0 \in [-1,1]$ *and an* $f_2 \in C[-1,1]$ *such that*

$$\overline{\lim}_{n\to\infty}|L_n(f_2,x_0)| = \infty. \tag{4.1.5}$$

Later, Bernstein showed (cf. [54, Chapter 2, Vol. III] or [73, Chapter 4]) that Lagrange interpolation diverges everywhere for the function $|x|$ on $[-1,1]$ except $x = 0$ when the system of nodes is taken as the "bad" equidistant matrix $E := \{-1+2(k-1)/(n-1)\}_{k=1}^n$. Moreover, Grünwald [28] [29] and Marcinkiewicz [46] showed that even if the system of nodes is taken as the "good" nodes: the zeros of the Chebyshev polynomial of the first kind, there exists some function $f_3 \in C[-1,1]$ such that

$$\overline{\lim}_{n\to\infty}|L_n(f_3,x)| = \infty \quad for\, all\, x \in [-1,1].$$

A stronger result was obtained by Erdős and Vértesi [20] in 1980. That is,

Erdős and Vértesi Theorem *For any nodes system, there exist* $f_4 \in C[-1,1]$ *such that*

$$\overline{\lim}_{n\to\infty}|L_n(f_4,x)| = \infty \quad a.e.\, in\, [-1,1]. \tag{4.1.6}$$

Moreover, the divergence set is of second category in $[-1,1]$.

4.2 Lagrange-type Interpolation in $\mathcal{P}_{n-1}(a_1,\dots,a_n)$

Let f be a function defined on $[-1,1]$. We construct the Lagrange interpolation based on the zeros $\{x_k\}_{k=1}^n$ of Chebyshev polynomial of the first kind $T_n(x)$ with the rational system $\mathcal{P}_n(a_1,\dots,a_n)$ as follows:

$$L_n(f,x) := \sum_{k=1}^{n} f(x_k)l_k(x), \tag{4.2.7}$$

where $T_n(x)$ is defined by (1.2.8) and $\{l_k(x)\}_{k=1}^n$ are the Lagrange fundamental functions:

$$l_k(x) := \frac{T_n(x)}{T_n'(x_k)(x-x_k)}, \quad k = 1,...,n. \tag{4.2.8}$$

It is easy to check that

$$L_n(f, x_k) = f(x_k), \quad k = 1, ..., n,$$

Lemma 4.2.1 *Let f be defined on $[-1,1]$ and $\{a_k\}_{k=1}^n \subset \mathbb{C}\backslash[-1,1]$. Then*

$$L_n(f) \in \mathcal{P}_{n-1}(a_1, \cdots, a_n).$$

Proof. Since we may suppose that

$$T_n(x) := \frac{Q_n(x)}{R_n(x)}, \tag{4.2.9}$$

where $Q_n(x) := e_n(x - x_1)\cdots(x - x_n)$, $R_n(x) := (x - a_1)\cdots(x - a_n)$, and e_n depends on both n and a_k. Then we have

$$l_k(x) = \frac{R_n(x_k)}{R_n(x)} q_k(x), \tag{4.2.10}$$

where

$$q_k(x) := \frac{Q_n(x)}{Q_n'(x_k)(x - x_k)}, \quad k = 1, \dots, n.$$

Therefore, it is easy to see that $L_n(f) \in \mathcal{P}_{n-1}(a_1, \cdots, a_n)$. □

4.2.1 Lebesgue Constant and Uniform Approximation

It is well-known that the Lebesgue constant of classical polynomial interpolation plays an important role in uniform polynomial approximations (cf. [54] [69] [73]). For given $n \in \mathbb{N}$, we also define the associated *Lebesgue function*:

$$L_n(x) := \sum_{k=1}^{n} |l_k(x)|$$

and the *Lebesgue constant*:

$$L_n := \|L_n(x)\|_{[-1,1]},$$

where $l_k(x)$ is defined by (4.2.8). Clearly, L_n depends on $\{a_k\}_{k=1}^n$.

Theorem 4.2.2 *Let $\{a_k\}_{k=1}^n \subset \mathbb{R}\backslash[-1,1]$ be distinct and satisfy Assumption (A). Then*

$$L_n \sim \ln n. \tag{4.2.11}$$

Therefore, with respect to the uniform approximation, we have

Corollary 4.2.3 *Let $f \in C[-1,1]$ and $\{a_k\}_{k=1}^n \subset \mathbb{R}\backslash[-1,1]$ be distinct and satisfy Assumption (A). Then*

$$\|L_n(f) - f\|_{[-1,1]} \le d_1(\alpha) \ln n E_n^R(f), \tag{4.2.12}$$

where $E_n^R(f)$ is defined by (1.5.33). Furthermore, if $f(x)$ satisfies the Dini-Lipschitz condition, then

$$\lim_{n\to\infty} L_n(f,x) = f(x)$$

uniformly on $[-1,1]$.

In order to prove the above results, we first prove several auxiliary results which will be used later.

Lemma 4.2.4 *Let $f \in C[-1,1]$, $\{a_k\}_{k=1}^n \subset \mathbb{R}\backslash[-1,1]$ be distinct and $\{c_k\}_{k=1}^n$ be defined by (1.2.3). Then*

$$E_n^R(f) \le E_n(f) + \left(\|f\|_{[-1,1]} + E_n(f)\right) \prod_{k=1}^n |c_k|, \tag{4.2.13}$$

where $E_n(f)$ is defined by (1.5.32) .

Proof. Let $p_n(x) := \gamma_n x^n + \cdots + \gamma_0$ ($\gamma_n \neq 0$) be the best polynomial approximation of degree n to $f(x)$ on $[-1,1]$, that is

$$\|p_n - f\|_{[-1,1]} = E_n(f).$$

Using [1, Problem 7, p. 254], we know that there exist μ_k $(k = 1, ..., n)$ such that

$$\left\|\frac{p_n(x)}{\gamma_n} - \frac{1}{\gamma_n}\sum_{k=1}^n \frac{\mu_k}{x-a_k}\right\|_{[-1,1]} \le \frac{1}{2^{n-1}}\prod_{k=1}^n |c_k|.$$

Thus,

$$\begin{aligned}\left\|f(x) - \sum_{k=1}^n \frac{\mu_k}{x-a_k}\right\|_{[-1,1]} &\le \|f - p_n\|_{[-1,1]} + \left\|p_n(x) - \sum_{k=1}^n \frac{\mu_k}{x-a_k}\right\|_{[-1,1]} \\ &\le E_n(f) + \frac{|\gamma_n|}{2^{n-1}}\prod_{k=1}^n |c_k|.\end{aligned}$$

Moreover, by the Chebyshev inequality (cf. [54, Corollary 2, Vol. I, p. 56]) we have

$$|\gamma_n| \le 2^{n-1}\|p_n\|_{[-1,1]} \le 2^{n-1}\left(\|f\|_{[-1,1]} + E_n(f)\right).$$

Then (4.2.13) follows. □

By the classical Jackson Theorem (cf. [54, Vol. 1]) and Lemma 4.2.4 we can prove the following corollary in the usual way.

Corollary 4.2.5 *Let* $f \in C^1[-1,1]$ *and* $\{a_k\}_{k=1}^n \subset \mathbb{R}\backslash[-1,1]$ *be distinct. If* $\prod_{k=1}^n |c_k| = O(1/n)$*, then*

$$E_n^R(f) = O(1)\,(\omega(f, 1/n) + 1/n)\,. \tag{4.2.14}$$

Lemma 4.2.6 gives an explicit formula for the Lagrange fundamental interpolatory functions.

Lemma 4.2.6 *Let* $\{a_k\}_{k=1}^n \subset \mathbb{C}\backslash[-1,1]$*. Then, for* $k = 1, 2, \dots, n$*,*

$$l_k(x) = \epsilon \frac{\sqrt{1-x_k^2}T_n(x)}{B_n(x_k)(x-x_k)}, \tag{4.2.15}$$

where $\epsilon = 1$ *or* -1*.*

Proof. By Theorem A(d) and (4.2.8) we can deduce Lemma 4.2.6. □ .

Lemma 4.2.7 *Let* $\{l_k(x)\}_{k=1}^n$ *be defined by* (4.2.8)*. If* $\{a_k\}_{k=1}^n \subset \mathbb{C}\backslash[-1,1]$ *satisfy Assumption (A), then we have*

$$\sum_{k=1}^n |l_k(x)|^m \le d_2(m, \alpha), \quad m = 2, 3..., \tag{4.2.16}$$

(4.2.16) *implies*

$$|l_k(x)| \le \sqrt{d_2(2,\alpha)}, \quad k = 1, ..., n. \tag{4.2.17}$$

Proof. We need to prove this only for $m = 2$. Since T_n can be expressed as (4.2.9), then we have

$$\sum_{k=1}^n \frac{1}{x-x_k} = \frac{T_n'(x)}{T_n(x)} + \frac{R_n'(x)}{R_n(x)} = \frac{T_n'(x)}{T_n(x)} + \sum_{k=1}^n \frac{1}{x-a_k},$$

and

$$\sum_{k=1}^{n} \frac{1}{(x-x_k)^2} = \frac{(T_n'(x))^2 - T_n(x)T_n''(x)}{T_n^2(x)} + \sum_{k=1}^{n} \frac{1}{(x-a_k)^2}. \tag{4.2.18}$$

Under the given assumption, it is easy to see that

$$\frac{1}{(x-a_k)^2} \le \frac{1}{(|a_k|-1)^2} \le \frac{1}{\alpha^2-1}.$$

From (3.8.44) we conclude that

$$T_n''(x) = B_n'(x)U_n(x) + B_n(x)U_n'(x), \tag{4.2.19}$$

and on combining Theorem A(b), (3.8.44), (3.8.45), Lemma 3.4.2 and Theorem 3.3.1 we have

$$\frac{T_n^2(x)(1-x^2)}{n^2} \sum_{k=1}^{n} \frac{1}{(x-x_k)^2} \le d_3(\alpha). \tag{4.2.20}$$

Similarly, we can prove that

$$\frac{T_n^2(x)}{n^2} \left| \sum_{k=1}^{n} \frac{1}{x-x_k} \right| \le d_4(\alpha). \tag{4.2.21}$$

Furthermore, from Lemma 3.4.2 and Lemma 4.2.6 we conclude that

$$\sum_{k=1}^{n} l_k^2(x) \le \left(\frac{1+\gamma}{1-\gamma}\right)^2 \frac{T_n^2(x)}{n^2} \sum_{k=1}^{n} \frac{1-x_k^2}{(x-x_k)^2}.$$

Therefore, on combining (4.2.19) (4.2.20) and by some simple calculations, we have

$$\begin{aligned}
&\sum_{k=1}^{n} l_k^2(x) \\
&\le \left(\frac{1+\gamma}{1-\gamma}\right)^2 \frac{T_n^2(x)}{n^2} \sum_{k=1}^{n} \frac{1-x^2+x^2-x_k^2}{(x-x_k)^2} \\
&\le \left(\frac{1+\gamma}{1-\gamma}\right)^2 \left\{ \frac{T_n^2(x)(1-x^2)}{n^2} \sum_{k=1}^{n} \frac{1}{(x-x_k)^2} + \frac{T_n^2(x)}{n^2} \left(\sum_{k=1}^{n} \frac{2x}{x-x_k} - n \right) \right\} \\
&\le d_2(\alpha),
\end{aligned}$$

and this lemma follows. □

We let $\{u_k\}_{k=0}^n$ be a Markov system on $[a,b]$ and let

$$E(f) := \sup_{f \in C^1[-1,1]} E_n(f),$$

where $E_n(f)$ is the best approximation of f from span$\{u_0, u_1, ..., u_n\}$. Then we have

Lemma 4.2.8 (cf. [34]) *Let $\{u_k\}_{k=0}^n$ be a Markov system on $[a, b]$. Then*

$$M_n := \max_{1 \le k \le n+1} |x_k - x_{k+1}| \le 32E(f), \tag{4.2.22}$$

where $\{x_k\}_{k=1}^n$ are the zeros of the corresponding Chebyshev polynomial $T_n(x)$ with respect to span$\{u_0, u_1, ..., u_n\}$, *$x_0 = a, x_{n+1} = b$.*

Lemma 4.2.9 gives the estimate of the distance between two consecutive zeros of Chebyshev polynomials of the first kind with respect to the rational system $\mathcal{P}_n(a_1, \dots, a_n)$. This will be used in the proof of Theorem 4.2.2.

Lemma 4.2.9 (i) *Assume $\{a_k\}_{k=1}^n \subset \mathbb{R} \backslash [-1, 1]$ are distinct and $\{c_k\}_{k=1}^n$ are defined by (1.2.3). If $\prod_{k=1}^n |c_k| = O(1/n)$, then the largest distance between two consecutive zeros of Chebyshev polynomial of the first kind satisfies*

$$M_n = O(1/n), \tag{4.2.23}$$

where $x_0 := 1, x_{n+1} := -1$. Moreover, if $\{a_k\}_{k=1}^n$ satisfy Assumption (A), then

$$|x_k - x_{k+1}| \ge d_5(\alpha) \frac{1}{n^2}. \tag{4.2.24}$$

(ii) *Let $x_k = \cos\theta_k$, and $\{a_k\}_{k=1}^n \subset \mathbb{R} \backslash [-1, 1]$ be distinct and satisfy assumption (A). Then*

$$|\theta_{k+1} - \theta_k| \sim \frac{1}{n}. \tag{4.2.25}$$

Proof. By Corollary 4.2.4 we know that if $f \in C^1[-1, 1]$, then

$$E_n^R(f) = O(1) \left(\frac{1}{n}\right),$$

so, it is easy to obtain (4.2.23) by Lemma 4.2.8.

On the other hand, since

$$1 = \left| \frac{l_k(x_k) - l_k(x_{k+1})}{x_k - x_{k+1}} (x_k - x_{k+1}) \right| = |l_k'(\eta)| |x_k - x_{k+1}|,$$

by Theorem 3.5.1, Lemma 3.4.2 and (4.2.16), (4.2.24) follows.

Now we let $\tilde{l}_k(\theta) := l_k(\cos\theta)$, $k = 1, ..., n$. Note that (cf. the proof of Theorem 4.2.2) assumption (A) implies that $\prod_{k=1}^{n} |c_k| = O(1/n)$. Thus, by (4.2.23) and the Bernstein-Szegő-type inequality (3.2.1), it is easy to show (4.2.25) by the same method as the estimates in (4.2.23) and (4.2.24). □

Proof of Theorem 4.2.2. First we claim that Assumption (A) implies that $\prod_{k=1}^{n} |c_k| = O(1/n)$. By the inequality

$$\frac{1-x}{1+x} \le e^{-2x}, \quad x \ge 0,$$

and with $x = \frac{1-|c_k|}{1+|c_k|}$ in the above inequality, we have

$$|c_k| \le e^{-2\frac{1-|c_k|}{1+|c_k|}}.$$

Hence Assumption (A) implies that

$$\prod_{k=1}^{n} |c_k| \le e^{-2\sum_{k=1}^{n}\frac{1-|c_k|}{1+|c_k|}} \le e^{-2\frac{1-\gamma}{1+\gamma}n}.$$

Let $x = \cos\theta, x_k = \cos\theta_k$ and $x_j = \cos\theta_j$ be the nearest point to x and let $i = |k-j|$. Then by (4.2.25) we have

$$d_6(\alpha)\frac{i}{n} \le d_7(\alpha)|\theta_k - \theta_j| \le |\theta - \theta_k| \le d_8(\alpha)|\theta_k - \theta_j| \le d_9(\alpha)\frac{i}{n}. \tag{4.2.26}$$

Since

$$L_n(x) = |T_n(x)| \sum_{k=1}^{n} \frac{\sqrt{1-x_k^2}}{B_n(x_k)|x - x_k|},$$

$$\sin\theta_k \le 2\sin\frac{\theta+\theta_k}{2}, k = 1, ..., n,$$

$$|\cos\theta - \cos\theta_k| = 2\left|\sin\frac{\theta+\theta_k}{2}\sin\frac{\theta-\theta_k}{2}\right|,$$

and

$$0 \le \frac{\theta}{\sin\theta/2} \le \pi, \quad |\theta| \le \pi,$$

so, by Lemma 3.4.2 one can show that

$$\begin{aligned} L_n(x) &\le |l_j(x)| + \sum_{k\ne j}|l_k(x)| \le \sqrt{d_2(\alpha)} + \frac{1+\gamma}{1-\gamma}\frac{1}{n}\sum_{k\ne j}\frac{\sin\theta_k}{|\cos\theta - \cos\theta_k|} \\ &\le \sqrt{d_2(\alpha)} + \frac{1+\gamma}{1-\gamma}\frac{1}{n}\sum_{k\ne j}\frac{1}{|\sin(\theta-\theta_k)/2|} \\ &\le \sqrt{d_2(\alpha)} + \frac{1+\gamma}{1-\gamma}\frac{1}{n}\frac{\pi}{d_6(\alpha)}\sum_{k\ne j}\frac{n}{i}, \quad x \in [-1,1]. \end{aligned}$$

Hence

$$L_n = O(\ln n).$$

On the other hand, by Lemma 3.4.2 we have

$$L_n \geq L_n(1) \geq \frac{1-\gamma}{1+\gamma}\frac{1}{n}\sum_{k=1}^{n}\sqrt{\frac{1+x_k}{1-x_k}} = \frac{1-\gamma}{1+\gamma}\frac{1}{n}\sum_{k=1}^{n}\cot\frac{\theta_k}{2}.$$

Note that

$$\int_0^{\frac{\pi}{2}}\left(\cot x - \frac{1}{x}\right)\,dx = \ln\frac{2}{\pi},$$

and (4.2.26) implies $\theta_k \sim \frac{k}{n}$, $k = 1, \dots, n$. We now conclude that

$$\frac{1}{n}\sum_{k=1}^{n}(\cot\theta_k/2 \, - 1/\theta_k) \sim 1$$

and

$$\frac{1}{n}\sum_{k=1}^{n}1/\theta_k \sim \ln n.$$

But

$$L_n \geq \frac{1-\gamma}{1+\gamma}\frac{1}{n}\left(\sum_{k=1}^{n}(\cot\theta_k/2 \, - 1/\theta_k) + \sum_{k=1}^{n}1/\theta_k\right),$$

and hence, we can prove that

$$L_n \geq d_{10}(\alpha)\ln n.$$

Thus, Theorem 4.2.2 follows. □

Lemma 4.2.10 *Let the nonreal elements in* $\{a_k\}_{k=1}^{n} \subset \mathbb{C}\backslash[-1,1]$ *be paired by complex conjugation. Then*

$$L_n\,(p,x) \equiv p(x) \tag{4.2.27}$$

for $p \in \mathcal{P}_{n-1}(a_1,\dots,n)$.

Proof. The proof is similar to that of Lemma 5.2.2, and we omit it. □

Proof of Corollary 4.2.3. Let $p(x)$ be the best approximation for f from $\mathcal{P}_{n-1}(a_1,\dots,n)$ on $[-1,1]$, then

$$\|p - f\|_{[-1,1]} \leq E_n^R(f). \tag{4.2.28}$$

Lemma 4.2.10 yields

$$L_n(f,x) - f(x) = L_n(f-p,x) + (p(x) - f(x)), \tag{4.2.29}$$

hence, it is easy to obtain (4.2.12) in the usual way.

Since Assumption (A) implies that $\prod_{k=1}^{n}|c_k| = O(1/n)$, Corollary 4.2.5 implies that $\lim_{n\to\infty} L_n(f,x) = f(x)$ uniformly on $[-1,1]$. □

4.2.2 L^p-Convergence

Theorem 4.2.2 has indicated that the Lagrange-type interpolation does not converge uniformly for any continuous function on $[-1,1]$. However, in this section, we show that its L^p-convergence always holds under the Assumption (A). More precisely, we have

Theorem 4.2.11 *Let $\{a_k\}_{k=1}^{n} \subset \mathbb{C}\backslash[-1,1]$ satisfy Assumption (A). Then for any $0 < p < \infty$, we have*

$$\|L_n(f) - f\|_{v,p} \le c(p) E_n^R(f) \tag{4.2.30}$$

for $f \in C[-1,1]$.

Therefore, with Theorem 2.2.1, we have

Corollary 4.2.12 *Let the nonreal elements in $\{a_k\}_{k=1}^{\infty} \subset \mathbb{C}\backslash[-1,1]$ be paired by complex conjugation, and let $\{a_k\}_{k=1}^{\infty} \subset \mathbb{C}\backslash[-1,1]$ satisfy Assumption (A). Then*

$$\|L_n(f) - f\|_{v,p} \to 0, \quad (n\to\infty)$$

for $f \in C[-1,1]$.

Remark. When all $a_k \to \infty$ $(k = 1, 2\dots)$, Theorem 4.2.11 degenerates the case of classical polynomial interpolation, which was considered by Erdős and Feldheim [19].

First we need to prove a Lemma which generalizes a classical result of Erdős and Feldheim (cf. [69]).

Lemma 4.2.13 *Let the nonreal elements in $\{a_k\}_{k=1}^n \subset \mathbb{C}\backslash[-1,1]$ be paired by complex conjugation, and let $\rho_1, \ldots, \rho_k$ be distinct integers between 1 and n. Then, for an even number k, we have*

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} l_{\rho_1}(x) \cdots l_{\rho_k}(x)\, dx = 0. \tag{4.2.31}$$

Proof. Note that

$$\prod_{i=1}^{s} \frac{1}{x-x_i} = \sum_{i=1}^{s} \frac{A_i}{x-x_i},$$

where

$$A_i = \prod_{j=1, j\neq i}^{s} \frac{1}{x_i - x_j}.$$

Hence, it follows from Lemma 4.2.6 that

$$\begin{aligned} l_{\rho_1}(x) \cdots l_{\rho_k}(x) &= \varepsilon^k T_n^k(x) \prod_{j=1}^{k} \frac{(-1)^{\rho_j+1}\sqrt{1-x_{\rho_j}^2}}{B_n(x_{\rho_j})} \frac{1}{x - x_{\rho_j}} \\ &= \varepsilon^k T_n^k(x) \prod_{j=1}^{k} \frac{(-1)^{\rho_j+1}\sqrt{1-x_{\rho_j}^2}}{B_n(x_{\rho_j})} \left(\sum_{i=1}^{k} \frac{A_{\rho_i}}{x - x_{\rho_i}} \right) \\ &= \varepsilon^k \sum_{i=1}^{k} A_{\rho_i} \prod_{j=1, j\neq i}^{k} \frac{(-1)^{\rho_j+1}\sqrt{1-x_{\rho_j}^2}}{B_n(x_{\rho_j})} l_{\rho_i}(x) T_n^{k-1}(x). \end{aligned}$$

Therefore, to prove Lemma 4.2.13, it is sufficient to prove that

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} l_j(x) T_n^{k-1}(x)\, dx = 0, \quad j = 1, \ldots, n. \tag{4.2.32}$$

First we show that

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \frac{T_n^{k-1}(x)}{(x-a_m)^\rho}\, dx = 0, \quad \rho = 1, \ldots, r \tag{4.2.33}$$

if $a_k \in \mathbb{R}$ with the multiplicity r in $\{a_k\}_{k=1}^n$. Using the transformation $x = (z+z^{-1})/2$, we have

$$\begin{aligned} \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \frac{T_n^{k-1}(x)}{(x-a_m)^\rho}\, dx &= \frac{1}{2^{k-\rho}} \int_{C^+} \left(f_n(z) + f_n^{-1}(z) \right)^{k-1} \frac{1}{(-c_m - c_m^{-1} + z + z^{-1})^\rho} \frac{dz}{iz} \\ &= \frac{1}{2^{k-1-\rho_i}} \sum_{j=0}^{k-1} \binom{k-1}{j} \int_{C^+} f_n^{k-1-2j}(z) \frac{z^{s-1} dz}{((c_m - z)(c_m^{-1} - z))^\rho}, \end{aligned}$$

where C^+ is the upper half circle.

Note that $k-1$ is an odd number, so applying the transformation $w = 1/z$ to convert the terms $f_n^{-1}(z), \dots, f_n^{-(k-1)}(z)$ to the lower half circle C^-, we conclude that

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \frac{T_n^{k-1}(x)}{(x-a_m)^\rho} dx$$
$$= \frac{1}{2^{k-1-\rho} i} \sum_{j=0}^{[\frac{k}{2}]-1} \binom{k-1}{j} \int_C f_n^{k-1-2j}(z) \frac{z^{\rho-1} dz}{((c_m - z)(c_m^{-1} - z))^\rho}.$$

Recall that the function

$$f_n^{k-1-2j}(z) \frac{z^{\rho-1}}{((c_m - z)(c_m^{-1} - z))^\rho}$$

is analytic in the unit disk for $\rho = 1, \dots, r$ and $j = 0, \dots, [\frac{k}{2}] - 1$, and (4.2.33) immediately follows.

For the case of $\Im a_i \neq 0$ with multiplicity q in $\{a_k\}_{k=1}^n$, and $\gamma = 1, \dots, q$, by the same fashion, we conclude that

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \frac{T_n^{k-1}(x)(ax+b)}{((x-a_i)(x-\overline{a}_i))^\gamma} dx$$
$$= \frac{1}{2^{k-1-2\gamma} i} \sum_{j=0}^{[\frac{k}{2}]-1} \binom{k-1}{j} \int_C f_n^{k-1-2j}(z) \frac{z^{2\gamma-2}\left(\frac{a}{2}(z^2+1)+bz\right)}{((z-c_i)(z-c_i^{-1})(z-\overline{c}_i)(z-\overline{c}_i^{-1}))^\gamma} dz \tag{4.2.34}$$
$$= 0$$

since the integrand is analytic in the unit disk.

Note that $l_j(x) \in \mathcal{P}_{n-1}(a_1, \dots, a_n)$, by the partial fraction decomposition combined with (4.2.33) and (4.2.34), (4.2.32) follows. This finishes the proof of Lemma 4.2.13. □

Proof of Theorem 4.2.11. With Lemma 4.2.7 and Lemma 4.2.13, we can prove this in the same fashion as in [19] or [53]. Obviously, for any given $0 < p < \infty$, there exists $r \in \mathbb{N}$ such that $2(r-1) \leq p \leq 2r$. Then, by the Hölder inequality we have

$$\left(\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} |L_n(f) - f|^p \, dx\right)^{1/p}$$
$$\leq \left(\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} |L_n(f) - f|^{p\frac{2r}{p}} \, dx\right)^{1/2r} \left(\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \, dx\right)^{\frac{2r-p}{2r}\frac{1}{p}}$$
$$\leq c(p) \left(\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} |L_n(f) - f|^{2r} \, dx\right)^{1/2r}.$$

We suppose that $p_n \in \mathcal{P}_{n-1}(a_1, \dots, a_n)$ is the best approximation to $f(x)$ on $[-1, 1]$. By Lemma 4.2.10 we have $L_n(p_n, x) \equiv p_n$, and the Minkowski's' inequality yields

$$\|L_n(f) - f\|_{v,2r} \le \|L_n(f - p_n)\|_{v,2r} + \|p_n - f\|_{v,2r}.$$

Hence, we need only to prove

$$\|L_n(f - p_n)\|_{v,2r} \le c(r) E_n^R(f). \tag{4.2.35}$$

We denote $\Delta(x) := f(x) - p_n(x)$ and

$$I_n := \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} |L_n(\Delta(x)|^{2r}\, dx.$$

Recall that

$$I_n = \sum\nolimits_{r_1+\dots+r_s=2r} \frac{(2r)!}{r_1! \cdots r_s!} \Delta^{r_1}(x_{i_1}) \cdots \Delta^{r_s}(x_{i_s}) \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} l_{i_1}^{r_1}(x) \cdots l_{i_s}^{r_s}(x)\, dx, \tag{4.2.36}$$

where $\sum$ denotes multiple sum and $0 \le r_i \le 2r$, $i = 1, \dots, s$. Note that the first term (single sum) in (4.2.36) is

$$I_{n_1} := \sum_{k=1}^{n} \Delta^{2r}(x_k) \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} l_k^{2r}(x)\, dx.$$

Lemma 4.2.7 yields

$$I_{n_1} \le c(r) (E_n^R(f))^{2r}. \tag{4.2.37}$$

Also, the second term (double sum) in (4.2.36) is

$$\begin{aligned} I_{n_2} : &= \sum_{m=1}^{2r-1} \sum_{k \ne j} \sum \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \left(\Delta(x_k) l_k(x)\right)^m \left(\Delta(x_j) l_j(x)\right)^{2r-m} dx \\ &:= \sum_{m=1}^{2r-1} I_{n_2(m)}. \end{aligned} \tag{4.2.38}$$

For $m = 1$, we have

$$I_{n_2(1)} = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \sum_{k=1}^{n} \Delta(x_k) l_k(x) \sum_{j=1}^{n} (\Delta(x_j) l_j(x))^{2r-1}\, dx - I_{n_1}. \tag{4.2.39}$$

Using the Cauchy's inequality and recalling Lemma 4.2.13, we conclude that

$$\begin{aligned} \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \left| \sum_{k=1}^{n} \Delta(x_k) l_k(x) \right| dx &\le \sqrt{\pi} \left(\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \left[\sum_{k=1}^{n} \Delta(x_k) l_k(x) \right]^2 dx \right)^{1/2} \\ &\le c E_n^R(f). \end{aligned} \tag{4.2.40}$$

Combining (4.2.38) - (4.2.40) we conclude that

$$I_{n_2(1)} \leq c(r)(E_n^R(f))^{2r}. \tag{4.2.41}$$

In the same fashion, we can prove that

$$I_{n_2(m)} \leq c(r)(E_n^R(f))^{2r}, \quad 2 \leq m \leq 2r-1. \tag{4.2.42}$$

From (4.2.41) and (4.2.42) we conclude that

$$I_{n_2} \leq c(r)E_n^{2r}(f). \tag{4.2.43}$$

Similarly, we can prove that

$$I_{n_i} \leq c(r)E_n^{2r}(f), \quad 3 \leq i \leq 2r-1, \tag{4.2.44}$$

where I_{n_i} is the i-th term ($i-$fold sum). For the $2r-$th term in I_n, by Lemma 4.2.13 we have

$$\begin{aligned} I_{n_{2m}} &= \overbrace{\sum \sum_{k\neq j\neq\cdots\neq i} \cdots \sum}^{2r} \Delta(x_k)\Delta(x_j)\cdots\Delta(x_i) \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} l_k(x)l_j(x)\cdots l_i(x)\,dx \\ &= 0. \end{aligned} \tag{4.2.45}$$

On combining (4.2.37) and (4.2.43) - (4.2.45), (4.2.35) follows. This completes the proof of Theorem 4.2.11. □

Next we show that the L^2-convergence also holds even if the Assumption (A) is dropped. That is

Theorem 4.2.14 *Let $f \in C[-1,1]$, and let the nonreal elements in $\{a_k\}_{k=1}^n \subset \mathbb{C}\backslash[-1,1]$ be paired by complex conjugation. Then*

$$\|L_n(f,x) - f(x)\|_{v,2} \leq 2\sqrt{\pi}E_n^R(f). \tag{4.2.46}$$

In particular, if $\sum_{k=1}^{\infty}(1-|c_k|) = \infty$, then

$$\|L_n(f,x) - f(x)\|_{v,2} \to 0, \quad n \to \infty, \tag{4.2.47}$$

We now let

$$\lambda_k := \int_{-1}^{1} \frac{l_k(x)}{\sqrt{1-x^2}}\, dx, \quad k = 1, ..., n. \tag{4.2.48}$$

Next we prove that $\{\lambda_k\}_{k=1}^n$ are positive. This extends a classical result for polynomial interpolation. That is,

Lemma 4.2.15 *Let the nonreal elements in* $\{a_k\}_{k=1}^n \subset \mathbb{C}\backslash[-1,1]$ *be paired by complex conjugation. Then*

$$\lambda_k = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\, l_k^2(x)\, dx > 0, \quad k = 1, ..., n, \tag{4.2.49}$$

and

$$\sum_{k=1}^{n} \lambda_k = \pi - \frac{2\pi}{1 + (c_1 \cdots c_n)^{-2}}. \tag{4.2.50}$$

Remark. Note that $l_k(x) \in \mathcal{P}_{n-1}(a_1, \ldots, a_n)$ $(k = 1, \ldots, n)$ implies that $\sum_{k=1}^{n} l_k(x) \not\equiv 1$, which differs from the classical Lagrange polynomial interpolation. Hence, we cannot use the standard method to prove (4.2.50) (cf. [54, Vol. III]). Moreover, (4.2.50) implies that

$$\sum_{k=1}^{n} \lambda_k < \pi.$$

Proof. By (4.2.32), we conclude that

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\, (x - x_k) l_k^2(x)\, dx = 0, \quad k = 1, ..., n. \tag{4.2.51}$$

Since

$$H_n(l_k(x), x) = l_k(x), \quad k = 1, ..., n, \tag{4.2.52}$$

where $H_n(f, x)$ is the so called Hermite-type interpolation defined by (5.2.6). From (5.2.6)-(4.2.8) and (4.2.51) we obtain

$$\begin{aligned} \lambda_k &= \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\, l_k(x)\, dx = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\, H_n(l_k(x), x)\, dx \\ &= \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\, l_k^2(x)\, dx > 0, \quad k = 1, ..., n. \end{aligned}$$

and then (4.2.49) follows.

Recall that Lemma 2.2.2 or Theorem D implies that $T_n(x) - A_0 \in \mathcal{P}_{n-1}(a_1, \dots, a_n)$. On applying Lemma 4.2.10 we have

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} (T_n(x) - A_0)\, dx = \sum_{k=1}^{n} \lambda_k (T_n(x_k) - A_0) = -A_0 \sum_{k=1}^{n} \lambda_k. \tag{4.2.53}$$

It is easy to show that (cf. [8, Corollary 4.6 (4.13)]),

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} T_n(x)\, dx = (-1)^n \pi c_1 \cdots c_n, \tag{4.2.54}$$

and on combining (4.2.53) and (4.2.54), (4.2.50) follows. □

Proof of Theorem 4.2.14. Let $P(x)$ be the best approximation of $f(x)$ in $\mathcal{P}_{n-1}(a_1, \dots, a_n)$. Then, by (4.2.29) and Lemma 4.2.16 we have

$$\begin{aligned}
&\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} |L_n(f,x) - f(x)|^2\, dx \\
&\le 2\left(\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} |L_n(f-p,x)|^2\, dx + \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} |p(x) - f(x)|^2\, dx\right) \\
&\le 2\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \sum_{k=1}^{n} (f(x_k) - p(x_k))^2 l_k^2(x)\, dx + 2\pi (E_n^R(f))^2 \\
&\le 2\left(E_n^R(f)\right)^2 \left(\sum_{k=1}^{n} \lambda_k + \pi\right) \le 4\pi \left(E_n^R(f)\right)^2,
\end{aligned}$$

so, (4.2.46) follows.

Moreover, $\sum_{k=1}^{\infty}(1 - |c_k|) = \infty$ implies that $\{\mathcal{P}_{n-1}(a_1, \dots, a_n)\}$ is dense in $C[-1,1]$ (cf. Theorem 2.2.1). Hence we obtain (4.2.47). □

4.3 Quadrature Formula

By using Lagrange-type interpolation (4.2.7) we obtain a *quadrature formula:*

$$\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}\, dx \approx Q_n(f) := \sum_{k=1}^{n} f(x_k)\lambda_k, \tag{4.3.55}$$

where λ_k $(k = 1, \dots, n)$ are defined by (4.2.48). We denote its error by

$$E_n^0(f) := \left| \int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}\, dx - Q_n(f) \right|. \tag{4.3.56}$$

With respect to this quadrature formula, based on the above results about mean convergence of Lagrange-type interpolation, it is easy to show that

Theorem 4.3.1 *Let $f \in C[-1,1]$, and let the nonreal elements in $\{a_k\}_{k=1}^{n} \subset \mathbb{C}\backslash[-1,1]$ be paired by complex conjugation. Then*

(a) *$Q_n(f)$ is a positive quadrature formula, that is $\lambda_k > 0$, $k = 1, ..., n$.*

(b) *For any $f \in \mathcal{R}_{2n-1}(a_1, \dots, a_n)$, (4.3.59) is exact.*

(c)

$$E_n^0(f) = O(1)E_n^R(f). \tag{4.3.57}$$

The quadrature formulas based on the rational interpolation were recently considered by Van Assche and Vanherwegen [78] and Gautschi [25]. Gautschi (cf. [25], [26]) has successfully used this idea for the computation of generalized Fermi-Dirac and Bose-Einstein integrals (also cf. [61] [71]).

Theorem 4.3.2 characterizes the convergence of quadrature formula (4.3.55).

Theorem 4.3.2 *Let the nonreal elements in $\{a_k\}_{k=1}^{\infty} \subset \mathbb{C}\backslash[-1,1]$ be paired by complex conjugation. Then*

$$\lim_{n\to\infty} Q_n(f) = \int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}\,dx, \quad \forall f \in C[-1,1] \quad \Leftrightarrow \quad \sum_{k=1}^{\infty}(1-|c_k|) = \infty. \tag{4.3.58}$$

Proof. By Theorem 2.2.1, we may prove the sufficient condition. On the other hand, since

$$Q_n(f) \to \int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}\,dx, \quad n \to \infty$$

for every continuous function, by Steklov's Theorem (cf. [54, Theorem 4, Vol. III, p. 124]) we know that the system of interpolating nodes is dense on $[-1,1]$, that is $M_n \to 0, (n \to \infty)$, where M_n is defined by (4.2.22). Thus, using Borwein's Theorem (cf. [4, Theorem 1]) we have $E_n^R(f) \to 0, (n \to \infty)$. Therefore, we complete the proof of the necessary condition by using Theorem 2.2.1 again. □

Remark. It's well-known that all of the weight coefficients $\{\lambda_k\}_{k=1}^{n}$ are equal in the classical case based on the zeros of the classical Chebyshev polynonial of the first kind. That is,

$\lambda_k = \pi/n\,(k = 1, \dots, n)$. But, in general, the weight coefficients λ_k in (4.2.48) are not equal. Here we give an example to illustrate this.

Example. Let $n = 2$, $a_1 = 10$ and $a_2 = 11$, then we denote the corresponding Chebyshev polynomial of the first kind by $T_2(x)$. By some simple calculations, we can show that

$$\lambda_1 \approx 1.466875541,$$

and

$$\lambda_2 \approx 1.674128336.$$

4.4 Remarks and Problems

1. For the case of $\{a_k\}_{k=1}^n \subset \mathbb{R}\backslash[-1,1]$ distinct, as we have mentioned before, $l_k(x) \in \text{span}\left\{\frac{1}{x-a_1}, \dots, \frac{1}{x-a_n}\right\}$ implies

$$\sum_{k=1}^{n} l_k(x) \not\equiv 1, \tag{4.4.59}$$

where $\{l_k(x)\}$ are defined by (4.2.8).

Therefore, in order to keep the property $\sum_{k=1}^n l_k(x) \equiv 1$ we must construct Lagrange interpolation in another rational system $\mathcal{P}_{n-1}(a_1, \dots, a_{n-1})$, that is

$$span\left\{1, \frac{1}{x-a_1}, \frac{1}{x-a_2}, \dots, \frac{1}{x-a_{n-1}}\right\}. \tag{4.4.60}$$

It is easy to see that

$$\mathcal{L}_n(f, x) := \sum_{k=1}^{n} f(x_k)\ell_k(x) \tag{4.4.61}$$

satisfies

$$\mathcal{L}_n(f, x_k) = f(x_k), \quad k = 1, \dots, n,$$

and

$$\mathcal{L}_n(f) \in \mathcal{P}_{n-1}(a_1, \dots, a_{n-1}),$$

where

$$\ell_k(x) := \frac{x - a_n}{x_k - a_n} l_k(x) \tag{4.4.62}$$

is the corresponding Lagrange fundamental function with respect to the rational system $\mathcal{P}_{n-1}(a_1, \dots, a_{n-1})$.

One can check that

$$\mathcal{L}_n(1, x) \equiv 1, \tag{4.4.63}$$

and

$$\mathcal{L}_n\left(\frac{1}{x - a_i}, x\right) \equiv \frac{1}{x - a_i}, \quad i = 1, \dots, n-1, \tag{4.4.64}$$

and it follows that

$$\sum_{k=1}^{n} \ell_k(x) \equiv 1. \tag{4.4.65}$$

We can also prove the following approximation theorem with respect to uniform approximation:

Theorem 4.4.1 *Let $f \in C[-1,1]$ and $\{a_k\}_{k=1}^{\infty} \subset \mathbb{R} \backslash [-1,1]$ satisfy assumption (A). Then*

$$\|\mathcal{L}_n(f) - f\|_{[-1,1]} \le d_{11}(\alpha) \ln n E_{n-1}^{Q}(f). \tag{4.4.66}$$

Furthermore, if $f(x)$ satisfies the Dini-Lipschitz condition, then

$$\lim_{n \to \infty} \mathcal{L}_n(f, x) = f(x)$$

uniformly on $[-1,1]$

The proof is the same as that of Theorem 4.2.2, and we omit it here.

2. On the other hand, by some simple calculations we may get

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \ell_k(x) \ell_j(x)\, dx$$
$$= \frac{\sqrt{1-x_k^2}\sqrt{1-x_j^2}}{B_n(x_k) B_n(x_j)(x_k - a_k)(x_j - a_j)} \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} T_n^2(x)\, dx.$$

Hence we conclude that

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \ell_k(x)\ell_j(x)\, dx \neq 0. \tag{4.4.67}$$

Therefore, in order to keep the property $\sum_{k=1}^{n} \ell_k(x) \equiv 1$, the beautiful orthogonality (4.2.31) is destroyed.

We may also get a similar theorem with respect to mean convergence. We omit the details here.

3. When $\{a_k\}_{k=1}^{n} \subset \mathbb{R}\backslash[-1,1]$ satisfy Assumption (A), we have shown that the corresponding Lebesgue constant for the rational system is asymptotically of order $\ln n$. This is the same as the case of classical polynomial interpolation based on the zeros of the classical Chebyshev polynomial of the first kind (cf. [54, Theorem 2, Vol. III, p. 48], [69] or [73]). But, whether (4.2.11) implies Assumption (A) is still open.

4. We also construct a Lobatto-type quadrature formula for the rational space

$$\mathcal{R}_{2n-1}(a_1, \dots, a_n) := \left\{ \frac{P(x)}{\prod_{k=1}^{n} |x-a_k|^2}, \quad P \in \mathcal{P}_{2n-1} \right\}$$

with the nonreal elements in $\{a_k\}_{k=1}^{n} \subset \mathbb{C}\backslash[-1,1]$ paired by complex conjugation. This Lobatto-type quadrature formula is based on the extreme points of the Chebyshev polynomial of the first kind with the rational space $\mathcal{P}_n(a_1, \dots, a_n)$. Moreover, it is exact for any element in $\mathcal{R}_{2n-1}(a_1, \dots, a_n)$. For more details, see [50].

Chapter 5

Hermite-type Interpolation in Rational Systems

Overview

This chapter considers Hermite-type interpolation in rational systems. The corresponding uniform approximation theorem of Hermite-Fejér-type interpolation is established. The characterization theorem of corresponding Grünwald-type interpolation is also given. Furthermore, we prove that L^p-convergence of Grünwald-type interpolation always holds for every continuous function on $[-1,1]$.

5.1 Introduction

It is known that the Lagrange interpolation can not be guaranteed to converge uniformly to a continuous function on $[-1,1]$. On the other hand, Fejér [22] discovered, on considering a special nodes system and interpolatory process, that we can get positive results. This interpolatory process now called *Hermite-Fejér interpolation* is as follows (where $y'_k = 0$):

$$H_n(f,x) := \sum_{k=1}^{n} f(x_k)h_k(x) + \sum_{k=1}^{n} y'_k\sigma_k(x), \tag{5.1.1}$$

where

$$h_k(x) := (1 - 2l'_k(x_k)(x - x_k))\, l_k^2(x), \quad k = 1, \dots, n, \tag{5.1.2}$$

and

$$\sigma_k(x) := (x - x_k) l_k^2(x), \quad k = 1, \ldots, n. \tag{5.1.3}$$

One can verify that

$$H_n(f, x_k) = f(x_k), \quad k = 1, \ldots, n,$$

and

$$H_n'(f, x_k) = y_k', \quad k = 1, \ldots, n.$$

Fejér [22] gave a constructive proof of Weierstrass' Theorem using the above interpolation. More precisely, we have

Fejér Theorem *Let $f \in C[-1,1]$, $\{x_k\}_{k=1}^n$ be the zeros of the classical n-th Chebyshev polynomial of the first kind and $y_k' = 0\,(k = 1, \ldots, n)$ in* (5.1.1). *Then*

$$\lim_{n\to\infty} \|H_n(f) - f\|_{[-1,1]} = 0. \tag{5.1.4}$$

Fejér's result was extended by G. Szegő [75]. For a systematic investigation of interpolation for the classical case, readers may consult [73] [74] [75] [55] and [57]. In this chapter, we will consider Hermite(-Fejér) interpolation in the rational space $\mathcal{R}_{2n-1}(a_1, \ldots, a_n)$:

$$\mathcal{R}_{2n-1}(a_1, \ldots, a_n) := \left\{ \frac{P(x)}{\prod_{k=1}^n |x - a_k|^2}, \quad P \in \mathcal{P}_{2n-1} \right\} \tag{5.1.5}$$

When the nonreal elements in $\{a_k\}_{k=1}^n$ are paired by complex conjugation, $\mathcal{R}_{2n-1}(a_1, a_2, \ldots, a_n)$ is a real rational space. In particular, when $\{a_k\}_{k=1}^n$ are real and distinct, $\mathcal{R}_{2n-1}(a_1, a_2, \ldots, a_n)$ is simply the real span of

$$\left\{ \frac{1}{x - a_1}, \frac{1}{(x - a_1)^2}, \ldots, \frac{1}{x - a_n}, \frac{1}{(x - a_n)^2} \right\}.$$

5.2 Hermite-type Interpolation In $\mathcal{R}_{2n-1}(a_1, \ldots, a_n)$

Let f be a function defined on $[-1,1]$ and $\{y_k'\}_{k=1}^n \in \mathbb{R}$. We know (cf. [8, Corollary 4.9] or [62, Theorem 1.1]) that the Chebyshev polynomial of the first kind $T_n(x)$ for the rational system $\mathcal{P}_n(a_1, \ldots, a_n)$ has exactly n distinct zeros on $[-1, 1]$: $-1 < x_n < \cdots < x_1 < 1$. We construct the Hermite interpolation based on the zeros $\{x_k\}_{k=1}^n$ of $T_n(x)$ as follows.

$$H_n(f, x) := \sum_{k=1}^{n} f(x_k)h_k(x) + \sum_{k=1}^{n} y_k' \mu_k(x), \tag{5.2.6}$$

where

$$h_k(x) := (1 - 2l_k'(x_k)(x - x_k))\, l_k^2(x), \quad k = 1, \ldots, n, \tag{5.2.7}$$

and

$$\mu_k(x) := (x - x_k) l_k^2(x), \quad k = 1, \ldots, n, \tag{5.2.8}$$

where $l_k(x)$ $(k = 1, \ldots, n)$ are defined by (4.2.8).

One can verify that

$$H_n(f, x_k) = f(x_k), \quad k = 1, \ldots, n,$$

and

$$H_n'(f, x_k) = y_k', \quad k = 1, \ldots, n.$$

Lemma 5.2.1 *Let the nonreal elements in $\{a_k\}_{k=1}^n \subset \mathbb{C}\backslash[-1, 1]$ be paired by complex conjugation. Then*

$$H_n(f) \in \mathcal{R}_{2n-1}(a_1, \ldots, a_n). \tag{5.2.9}$$

Proof. Since $\{l_k(x)\} \in \mathcal{P}_{n-1}(a_1, \ldots, a_n)$ (cf. Lemma 4.2.1) and $T_n \in \mathcal{P}_n(a_1, \ldots, a_n)$, then, (5.2.9) follows from (5.2.6)-(5.2.8). □

Lemma 5.2.2 gives the fixed elements of the generalized Hermite-type interpolation $H_n(f, x)$.

Lemma 5.2.2 *Let the nonreal elements in $\{a_k\}_{k=1}^\infty \subset \mathbb{C}\backslash[-1, 1]$ be paired by complex conjugation. Then for any $p \in R_{2n-1}(a_1, \ldots, a_n)$, we have*

$$H_n(p, x) \equiv p. \tag{5.2.10}$$

Proof. If $\mathcal{P}_n(a_1, \ldots, a_n)$ has a pole of order m at the point $a_i \in \mathbb{R}$, then we first claim that

$$H_n\left(\frac{1}{(x - a_i)^r}, x\right) \equiv \frac{1}{(x - a_i)^r}, \quad r = 1, \ldots, m. \tag{5.2.11}$$

Here we just prove this for $r = m$ since the other cases can be proved in the same fashion. By a simple calculation we conclude that

$$l_k'(x_k) = q_k'(x_k) - \sum_{j=1}^{n} \frac{1}{x_k - a_j}.$$

Then by (5.2.6)-(5.2.8) we have

$$\begin{aligned} H_n\left(\frac{1}{(x-a_i)^m}, x\right) =& \frac{1}{R_n^2(x)} \sum_{k=1}^{n} \frac{R_n^2(x_k)}{(x_k - a_i)^m}(1 - 2q_k'(x_k)(x - x_k))q_k^2(x) \\ &+ \frac{1}{R_n^2(x)} \sum_{k=1}^{n} \left(-\frac{m}{x_k - a_i} + \sum_{j=1}^{n} \frac{2}{x_k - a_j} \right) \frac{R_n^2(x_k)}{(x_k - a_i)^m}(x - x_k)q_k^2(x). \end{aligned}$$

We now denote

$$t_n(x) := \frac{R_n^2(x)}{(x - a_i)^m} \ (\in \mathcal{P}_{2n-m}),$$

then

$$t_n'(x) = \left(-\frac{m}{x_k - a_i} + \sum_{j=1}^{n} \frac{2}{x_k - a_j} \right) \frac{R_n^2(x_k)}{(x_k - a_i)^m}.$$

Therefore, by the classical Hermite interpolation we have

$$\begin{aligned} H_n\left(\frac{1}{(x-a_i)^m}, x\right) &= \frac{1}{R_n^2(x)} \left\{ \sum_{k=1}^{n} t_n(x_k)\left(1 - 2q_k'(x_k)(x - x_k)\right) q_k^2(x) + \sum_{k=1}^{n} t_n'(x_k)(x - x_k)q_k^2(x) \right\} \\ &= \frac{1}{R_n^2(x)} \frac{R_n^2(x)}{(x - a_i)^m} \equiv \frac{1}{(x - a_i)^m}, \end{aligned}$$

which is nothing but (5.2.11).

Similarly, we can prove

$$H_n\left(\frac{1}{(x-a_i)^{2r}}, x\right) \equiv \frac{1}{(x - a_i)^{2r}}, \quad r = 1, \ldots, m, \tag{5.2.12}$$

and, If $\mathcal{P}_n(a_1, \ldots, a_n)$ has a pole of order l in the point a_j and $\Im a_j \neq 0$, then, we also have

$$H_n\left(\frac{1}{((x-a_j)(x-\overline{a}_j))^s}, x\right) \equiv \frac{1}{((x-a_j)(x-\overline{a}_j))^s}, \quad s = 1, \ldots, l, \tag{5.2.13}$$

$$H_n\left(\frac{x}{((x-a_j)(x-\overline{a}_j))^s}, x\right) \equiv \frac{x}{((x-a_j)(x-\overline{a}_j))^s}, \quad s = 1, \ldots, l, \tag{5.2.14}$$

$$H_n\left(\frac{1}{((x-a_j)(x-\overline{a}_j))^{2s}}, x\right) \equiv \frac{1}{((x-a_j)(x-\overline{a}_j))^{2l}}, \quad s=1,\dots,l \tag{5.2.15}$$

and

$$H_n\left(\frac{x}{((x-a_j)(x-\overline{a}_j))^{2s}}, x\right) \equiv \frac{x}{((x-a_j)(x-\overline{a}_j))^{2l}}, \quad s=1,\dots,l. \tag{5.2.16}$$

Therefore, on combining (5.2.11) - (5.2.16) and using the partial fraction decomposition we have completed the proof of Lemma 5.2.2. □

We now state our main result. Theorem 5.2.3 asserts that Hermite-Fejér-type interpolation converges uniformly to the continuous function on $[-1,1]$ under the Assumption (A).

Theorem 5.2.3 *Let $f \in C[-1,1]$, $y_k' = 0\,(k=1,\dots,n)$, and let $\{a_k\}_{k=1}^{\infty} \subset \mathbb{C}\backslash[-1,1]$ satisfy the Assumption (A). Then*

$$\lim_{n\to\infty} H_n(f,x) = f(x) \tag{5.2.17}$$

uniformly on $[-1,1]$.

The proof is based on several lemmas given below. Lemma 5.2.4 shows that the corresponding Chebyshev polynomials for $\mathcal{P}_n(a_1,a_2,\dots,a_n)$ still satisfy a certain differential equation.

Lemma 5.2.4 *Let $\{a_k\}_{k=1}^{n} \subset \mathbb{C}\backslash[-1,1]$, and let T_n and U_n be the Chebyshev polynomials of the first and second kinds for $\mathcal{P}_n(a_1,a_2,\dots,a_n)$, respectively. Then*

$$(1-x^2)T_n''(x) - xT_n'(x) - B_n^2(x)T_n(x) = (1-x^2)B_n'(x). \tag{5.2.18}$$

Proof. Let $\tilde{T}_n(t) := T_n(\cos t)$ and $\tilde{U}_n(t) := U_n(\cos t)\sin t$. We then have (cf. [8, Theorem 2.1])

$$\tilde{T}_n'(t) = -\tilde{B}_n(t)\tilde{U}_n(t), \quad \tilde{U}_n'(t) = \tilde{B}_n(t)\tilde{T}_n(t), \quad t \in \mathbb{R}$$

where $\tilde{B}_n(t) := B_n(\cos t)$. Hence,

$$T_n'(x) = B_n(x)U_n(x), \quad U_n'(x) = \frac{xU_n(x) - B_n(x)T_n(x)}{1-x^2}. \tag{5.2.19}$$

So, by (5.2.19) it is easy to check (5.2.18). □

Remark. When $a_k \to \infty$, then we have $B_n(x) \equiv n$ for $x \in [-1,1]$, consequently, $B_n'(x) \equiv 0$ on $[-1,1]$. Then (5.2.18) degenerates to the case of the classical Chebyshev polynomial of the first kind.

Lemma 5.2.5 gives an explicit formula for the Hermite fundamental interpolatory functions.

Lemma 5.2.5 *Let $\{a_k\}_{k=1}^n \subset \mathbb{C}\backslash[-1,1]$. Then, for $k = 1, 2, \dots, n$,*

$$h_k(x) = \left(1 - xx_k - (1-x_k^2)\frac{B_n'(x_k)}{B_n(x_k)}(x - x_k)\right)\left(\frac{T_n(x)}{B_n(x_k)(x-x_k)}\right)^2, \tag{5.2.20}$$

and

$$\mu_k(x) = \frac{(1-x_k^2)T_n^2(x)}{B_n^2(x_k)(x-x_k)}. \tag{5.2.21}$$

Proof. From (5.2.18) and (5.2.19) we have

$$\begin{aligned}\frac{T_n''(x_k)}{T_n'(x_k)} &= \frac{B_n'(x_k)U_n(x_k) + B_n(x_k)U_n'(x_k)}{B_n(x_k)U_n(x_k)} \\ &= \frac{x_k}{1-x_k^2} + \frac{B_n'(x_k)}{B_n(x_k)}.\end{aligned} \tag{5.2.22}$$

Also, by L'Hospital rule we can show that

$$h_k(x) = \left(1 - \frac{T_n''(x_k)}{T_n'(x_k)}(x - x_k)\right) l_k^2(x).$$

Therefore (5.2.20) follows, and (5.2.21) follows immediately from Lemma 4.2.6. ☐

Lemma 5.2.6 *Let $\{a_k\}_{k=1}^n \subset \mathbb{C}\backslash[-1,1]$ satisfy the Assumption (A). Then*

$$\sum_{k=1}^n |h_k(x)| = O(1), \quad x \in [-1,1], \tag{5.2.23}$$

$$\sum_{k=1}^n |l_k^2(x)| = O(1), \quad x \in [-1,1], \tag{5.2.24}$$

$$\sum_{k=1}^n |x - x_k||h_k(x)| = O(1)\frac{|T_n(x)|}{\sqrt{n}}, \quad x \in [-1,1], \tag{5.2.25}$$

$$\sum_{k=1}^{n} |x - x_k| l_k^2(x) = O(1) \frac{|T_n(x)|}{\sqrt{n}}, \quad x \in [-1, 1], \tag{5.2.26}$$

and

$$\sum_{k=1}^{n} \frac{|\mu_k(x)|}{\sqrt{1 - x_k^2}} = O(1) \frac{|T_n(x)|}{\sqrt{n}}, \quad x \in [-1, 1]. \tag{5.2.27}$$

Proof. We prove only (5.2.23) because the others may be proved in exactly the same way. By (4.2.9) we have

$$\sum_{k=1}^{n} \frac{1}{x - x_k} = \frac{T_n'(x)}{T_n(x)} + \frac{R_n'(x)}{R_n(x)} = \frac{T_n'(x)}{T_n(x)} + \sum_{k=1}^{n} \frac{1}{x - a_k},$$

and

$$\sum_{k=1}^{n} \frac{1}{(x - x_k)^2} = \frac{(T_n'(x))^2 - T_n(x) T_n''(x)}{T_n^2(x)} + \sum_{k=1}^{n} \frac{1}{(x - a_k)^2}.$$

Assumption (A) implies that

$$\frac{1}{(x - a_k)^2} \le \frac{1}{(|a_k| - 1)^2} \le \frac{4\gamma^2}{(1 - \gamma)^4}.$$

Note that a Schur-type inequality (cf. Theorem 3.3.1)

$$\|P\|_{[-1,1]} \le \|B_n\|_{[-1,1]} \|\sqrt{1 - x^2} P(x)\|_{[-1,1]} \tag{5.2.28}$$

holds for $P \in \mathcal{P}_{n-}(a_1, \dots, a_n)$, we then deduce

$$\|U_n\|_{[-1,1]} \le \|B_n\|_{[-1,1]}.$$

Thus by a slightly longer calculation, and combining Lemma 3.4.2 and Lemma 5.2.4 as well as the above results, we have

$$\begin{aligned} &\sum_{k=1}^{n} (1 - x x_k) \left(\frac{T_n(x)}{B_n(x_k)(x - x_k)} \right)^2 \\ &= \sum_{k=1}^{n} (1 - x^2 + x(x - x_k)) \left(\frac{T_n(x)}{B_n(x_k)(x - x_k)} \right)^2 \\ &= \sum_{k=1}^{n} \frac{(1 - x^2) T_n^2(x)}{(B_n(x_k)(x - x_k))^2} + x T_n^2(x) \sum_{k=1}^{n} \frac{1}{B_n^2(x_k)(x - x_k)} \\ &\le d(\alpha), \end{aligned} \tag{5.2.29}$$

On recalling Lemma 3.4.2 and using the Cauchy's inequality we have

$$\begin{aligned}
&\sum_{k=1}^{n}(1-x_k^2)\left|\frac{B_n'(x_k)}{B_n^3(x_k)}\right|\frac{T_n^2(x)}{|x-x_k|} \\
&\le \frac{d(\alpha)}{n^2}\sum_{k=1}^{n}(1-x_k^2)\frac{T_n^2(x)}{|x-x_k|} \le d(\alpha)\frac{T_n^2(x)}{n^{3/2}}\left(\sum_{k=1}^{n}\frac{1-x_k^2}{(x-x_k)^2}\right)^{1/2} \\
&= d(\alpha)\frac{T_n^2(x)}{n^{3/2}}\left(\sum_{k=1}^{n}\frac{1-x^2+x^2-x_k^2}{(x-x_k)^2}\right)^{1/2} \\
&= d(\alpha)\frac{|T_n(x)|}{n^{3/2}}\left(\sum_{k=1}^{n}\frac{(T_n^2(x)(1-x^2)}{(x-x_k)^2}+2xT_n^2(x)\sum_{k=1}^{n}\frac{1}{x-x_k}-nT_n^2(x)\right)^{1/2} \\
&\le d(\alpha)\frac{|T_n(x)|}{\sqrt{n}}.
\end{aligned} \tag{5.2.30}$$

Now (5.2.23) follows from (5.2.29) and (5.2.30). ◻

Lemma 5.2.7 *Let* $\{a_k\}_{k=1}^{\infty} \subset \mathbb{C}\backslash[-1,1]$ *satisfy the Assumption* (A)*. Then*

$$\lim_{n\to\infty}\left\|\sum_{k=1}^{n}h_k(x)-1\right\|_{[-1,1]}=0, \tag{5.2.31}$$

and

$$\lim_{n\to\infty}\sqrt{1-x^2}\left|1-\sum_{k=1}^{n}l_k^2(x)\right|=0 \tag{5.2.32}$$

uniformly on $[-1,1]$.

Remark. By Lemma 5.2.1 we can see that $\sum_{k=1}^{n}h_k(x)\not\equiv 1$, Thus we cannot prove (5.2.31) in the usual way (cf. [69]).

Proof. By the partial fraction decomposition we know that there exist some constant A_0 and some $q_n \in \mathcal{P}_{n-1}(a_1,\dots,a_n)$ such that

$$T_n(x)=A_0-q_n(x).$$

Clearly, $A_0=\lim_{x\to\infty}T_n(x)$. By the definition of T_n and a simple calculation we have (cf. [6, Theorem 3.5.4])

$$A_0=\frac{(-1)^n}{2}\left((c_1\cdots c_n)^{-1}+c_1\cdots c_n\right).$$

Moreover, note that $\|T_n\|_{[-1,1]} = 1$ (cf. [8, Theorem 1.2]), so, if we take $p_n := q_n/A_0$, we have

$$\|1 - p_n\|_{[-1,1]} \leq \frac{1}{|A_0|} \leq 2\prod_{k=1}^{n} |c_k| \leq 2\gamma^n. \tag{5.2.33}$$

By the Bernstein-type inequality (3.2.2), Lemma 3.4.2 and (5.2.33) one can conclude that

$$\sqrt{1-x^2}|p_n'(x)| \leq B_n(x)\|1-p_n\|_{[-1,1]} \leq 2\gamma^n \left(\frac{1+\gamma}{1-\gamma}\right)^2 n \leq d(\gamma) \tag{5.2.34}$$

for $n \geq N(\gamma)$.

From Lemma 5.2.2 we can conclude that $H_n(p_n) \equiv p_n$. Thus, by (5.2.23), (5.2.27) and (5.2.34), we can show that

$$\begin{aligned}\left|\sum_{k=1}^{n} h_k(x) - 1\right| &= \left|\sum_{k=1}^{n} h_k(x) - p_n(x) + p_n(x) - 1\right| \\ &\leq \left|\sum_{k=1}^{n}(1-p_n(x_k))h_k(x) - \sum_{k=1}^{n} p_n'(x_k)\sigma_k(x)\right| + E_n^R(1) \\ &= O(1)\left(\frac{1}{n} + \frac{1}{\sqrt{n}}\right) \to 0, \quad n \to \infty.\end{aligned}$$

Since

$$\begin{aligned}\sqrt{1-x^2}\left|1 - \sum_{k=1}^{n} l_k^2(x)\right| &= \sqrt{1-x^2}\left|\left(1 - \sum_{k=1}^{n} h_k(x)\right) + \left(\sum_{k=1}^{n} h_k(x) - \sum_{k=1}^{n} l_k^2(x)\right)\right| \\ &\leq \sqrt{1-x^2}\left|1 - \sum_{k=1}^{n} h_k(x)\right| + \sqrt{1-x^2}\left|\sum_{k=1}^{n} h_k(x) - \sum_{k=1}^{n} l_k^2(x)\right| \\ &:= S_1(x) + S_2(x).\end{aligned}$$

(5.2.31) implies that $\lim_{n\to\infty} S_1(x) = 0$ uniformly on $[-1,1]$. By Lemma 5.2.4 we can show that

$$S_2(x) \leq \sum_{k=1}^{n} |x_k|\sqrt{1-x^2}\frac{T_n^2(x)}{B_n^2(x_k)|x-x_k|} + \sqrt{1-x^2}\sum_{k=1}^{n}\frac{|B_n'(x_k)|}{B_n^3(x_k)}(1-x_k^2)\frac{T_n^2(x)}{|x-x_k|}.$$

We may see that $S_2(x) \to 0$ uniformly on $[-1,1]$ as $n \to \infty$ by using similar techniques to those of Lemma 5.2.6. Therefore, we have completed the proof of Lemma 5.2.7. □

Proof of Theorem 5.2.3. We can now prove this in the usual way. One can easily show that

$$\begin{aligned}|\mathcal{H}_n(f,x) - f(x)| &\leq \sum_{k=1}^{n} |f(x_k) - f(x)||h_k(x)| + \|f\|_{[-1,1]}\left|\sum_{k=1}^{n} h_k(x) - 1\right| \\ &:= S_3(x) + S_4(x).\end{aligned} \tag{5.2.35}$$

Note that

$$\omega(f, \lambda\delta) \leq (1+\lambda)\omega(f, \delta), \quad (\lambda > 0, \quad \delta > 0), \tag{5.2.36}$$

where $\omega(f, .)$ is the modulus of continuity of f.
Hence, from (5.2.23) we have

$$\begin{aligned} S_3(x) &\leq \sum_{k=1}^{n} \omega(f, |x-x_k|)|h_k(x)| \leq \omega(f,\delta) \sum_{k=1}^{n} \left(1 + \frac{|x-x_k|}{\delta}\right) |h_k(x)| \\ &\leq \omega(f,\delta) \left(O(1) + 1/\delta \sum_{k=1}^{n} |x-x_k||h_k(x)| \right). \end{aligned}$$

Taking $\delta = \frac{|T_n(x)|}{\sqrt{n}}$ in the above inequality and using (5.2.25) we can conclude that

$$S_3(x) \to 0 \quad n \to \infty,$$

uniformly on $[-1, 1]$. Also, from (5.2.33) we conclude that

$$S_4(x) \to 0, \quad n \to \infty,$$

uniformly on $[-1, 1]$. Therefore we complete the proof of Theorem 5.2.3. □

Remark. Actually, we have also derived estimates of approximation for $f(x)$ by $H_n(f, x)$ on $[-1, 1]$.

5.3 Characterization on Grünwald Interpolation

If we drop the linear term $1 - 2\ell_k'(x_k)(x - x_k)$ in $h_k(x)$, $k = 1, \ldots, n$ (cf. (5.2.7)), then we obtain an simple positive operator:

$$G_n(f, x) := \sum_{k=1}^{n} f(x_k) l_k^2(x), \tag{5.3.37}$$

which was first studied by Grünwald for the classical polynomial case in 1940 (cf. [25]), hence, we here call it the generalized Grünwald interpolation. The studies of Grünwald interpolation and its applications can be found [42] [43] [49] [51] [64] and the references therein. It is also a close cousin of Nevai's G_n and F_n operators (cf. [57, (4.5.6), (4.10.35)] [55] [48]).

Theorem 5.3.1 *Let $f \in C[-1,1]$ and $\{a_k\}_{k=1}^{\infty} \subset \mathbb{C}\backslash[-1,1]$ satisfy Assumption (A). Then*

$$\lim_{n\to\infty} G_n(f,x) = f(x) \tag{5.3.38}$$

uniformly on the closed subset $[-\sigma,\sigma]$, where $0 < \sigma < 1$.

Theorem 5.3.2 characterizes the uniform approximation of Grünwald-type interpolation on the whole interval $[-1,1]$.

Theorem 5.3.2 *Let $f \in C[-1,1]$ and $\{a_k\}_{k=1}^{\infty} \subset \mathbb{C}\backslash[-1,1]$ satisfy Assumption (A). Then (5.3.29) holds uniformly on whole interval $[-1,1]$ if and only if*

$$f(1) = f(-1) = 0. \tag{5.3.39}$$

Therefore, Grünwald-type interpolation, in general, can not uniformly converge to all continuous functions on the whole interval $[-1,1]$. But by Theorem 5.3.2, one can easily show that mean convergence always holds for every continuous function on $[-1,1]$. That is,

Corollary 5.3.3 *Let $f \in C[-1,1]$, $\{a_k\}_{k=1}^{\infty} \subset \mathbb{C}\backslash[-1,1]$ satisfy Assumption (A). Then, for $0 < p < \infty$, we have*

$$\lim_{n\to\infty} \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} |G_n(f,x) - f(x)|^p \, dx = 0. \tag{5.3.40}$$

Lemma 5.3.4 gives a sufficient condition for the corresponding Grünwald interpolation to converge uniformly for the continuous function on $[-1,1]$.

Lemma 5.3.4 *Let $\{a_k\}_{k=1}^{\infty} \subset \mathbb{C}\backslash[-1,1]$ satisfy assumption (A). Then, for $f(x) = \sqrt{1-x^2}g(x)$, $g \in C[-1,1]$, we have*

$$\lim_{n\to\infty} \|G_n(f) - f\|_{[-1,1]} = 0.$$

Proof. Note that

$$\begin{aligned} G_n(f,x) - f(x) &= \sum_{k=1}^{n} (f(x_k) - f(x)) l_k^2(x) + f(x) \left(\sum_{k=1}^{n} l_k^2(x) - 1 \right) \\ &:= S_5(x) + S_6(x). \end{aligned}$$

By (5.2.24) (5.2.26) and using the same method in the proof of Theorem 5.2.3 we may deduce that

$$\lim_{n\to\infty} |S_5(x)| = 0, \tag{5.3.41}$$

uniformly on $[-1, 1]$.

Since

$$S_6(x) = g(x)\sqrt{1-x^2}\left(\sum_{k=1}^{n} l_k^2(x) - 1\right),$$

by (5.2.32) we can show that $S_6(x) \to 0$ uniformly on $[-1, 1]$ as $n \to \infty$. Therefore, Lemma 5.3.4 follows. □

Proof of Theorem 5.3.1. We can prove this in exactly the same method as in the proof of Theorem 5.2.3, so we omit it. □

Proof of Theorem 5.3.2. First we prove the *only if* part. By Hermite-Fejér interpolation and Lemma 4.2.6, as well as (5.2.20) (5.2.22), we can show that

$$\begin{aligned}
&(G_n(f,1) - f(1)) - \left(\sum_{k=1}^{n}(f(x_k) - f(1))l_k^2(1)\right) \\
&+ f(1)\left(1 - \sum_{k=1}^{n} h_k(1)\right) + f(1)\left(\sum_{k=1}^{n} \frac{B_n'(x_k)}{B_n(x_k)}(1 - x_k)l_k^2(1)\right) - f(1)\sum_{k=1}^{n}\frac{1}{B_n^2(x_k)} \\
&= -f(1)\sum_{k=1}^{n}\frac{x_k}{1-x_k^2}(1-x_k)l_k^2(1) - f(1)\sum_{k=1}^{n}\frac{1}{B_n^2(x_k)} \\
&= -f(1)\sum_{k=1}^{n}\frac{1}{1-x_k}\frac{1}{B_n^2(x_k)}.
\end{aligned} \tag{5.3.42}$$

Note that (5.3.41) yields

$$\lim_{n\to\infty}\sum_{k=1}^{n}(f(x_k) - f(1))l_k^2(1) = 0,$$

and (5.2.31) implies

$$\lim_{n\to\infty}\left(1 - \sum_{k=1}^{n} h_k(1)\right) = 0.$$

Combining Lemma 3.4.2 and (5.3.41) we have

$$\lim_{n\to\infty}\left|\sum_{k=1}^{n}\frac{B_n'(x_k)}{B_n(x_k)}(1-x_k)l_k^2(1)\right| = 0,$$

and

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{B_n^2(x_k)}=0.$$

Hence, by a simple calculation and [8, Theorem 2.3] we have

$$\sum_{k=1}^{n}\frac{1}{1-x_k}=\frac{T_n'(1)}{T_n(1)}+\frac{R_n'(1)}{R_n(1)}=\left(\sum_{k=1}^{n}\frac{1+c_k}{1-c_k}\right)^2+\sum_{k=1}^{n}\frac{1}{1-a_k}.$$

Since

$$\sum_{k=1}^{n}\frac{1-|c_k|}{1+|c_k|}\le\sum_{k=1}^{n}\frac{1+c_k}{1-c_k}\le\sum_{k=1}^{n}\frac{1+|c_k|}{1-|c_k|},$$

also, note that Assumption (A) is equivalent to Assumption (C), we may conclude that

$$\sum_{k=1}^{n}\frac{1}{1-x_k}\sim n^2.$$

Also, Lemma 3.4.2 asserts that

$$B_n(x)\sim n,\quad -1\le x\le 1.$$

Therefore, from (5.3.42) and the hypothesis of the uniform convergence we conclude that $f(1)=0$. Similarly, we can prove $f(-1)=0$.

Next we prove the *if* part. By using the Bernstein fundamental polynomials (cf. [40]) and a linear transformation, one can show that

$$\mathrm{span}\left\{(1+x)^k(1-x)^{n-k}\right\}_{k=1}^{n-1},\quad k=1,2,...,n-1,\ n=2,...$$

is dense in $C^*[-1,1]:=\{f\in C[-1,1], f(-1)=f(1)=0\}$. Therefore, we only need to check that

$$\lim_{n\to\infty}G_n(r_k,x)=r_k(x),\tag{5.3.43}$$

uniformly on $[-1,1]$, where $r_k(x):=(1+x)^k(1-x)^{m-k},\quad k=1,...,m-1,m=2,....$ Since $r_k(x)$ contains the factor $\sqrt{1-x^2}$, on applying Lemma 5.3.4, we complete the proof of the *if* part. □

Chapter 6

Bernstein-type Polynomials in Rational Systems

Overview

This chapter considers Bernstein-type polynomials for rational systems $\{p(x)/\prod_{i=1}^{n}(1+t_ix),\, p\in\mathcal{P}_n\}$ associated with $t_i>-1$, $i=1,\dots,n$ on the interval $[0,1]$. A Popoviciu-type theorem and asymptotic formula are established for these Bernstein-type polynomials. Some shape preserving properties of these Bernstein-type polynomials are presented. As an application of these Bernstein-type polynomials, we also consider the approximation problem in $\{p(x)/\prod_{i=1}^{n}(1+t_ix),\, p\in\mathcal{P}_n\}$ with $p(x)$ having integral coefficients.

6.1 Introduction

The Weierstrass Approximation Theorem states that every continuous function $f(x)$ on $0\le x\le 1$ can be uniformly approximated there by polynomials. In 1912, Bernstein gave an explicit method of constructing the approximation as follows:

$$B_n(f,x):=\sum_{k=0}^{n} f\left(\frac{k}{n}\right)p_k(x),\quad p_k(x)=\binom{n}{k}x^k(1-x)^{n-k}. \tag{6.1.1}$$

For a proof that $B_n(f,x)$ converges uniformly to continuous function $f(x)$ on $[0,1]$ as $n\to\infty$, see, for example, Lorentz [40], DeVore and Lorentz [15, Chapters 1 and 10] or Ditzian and

Totik [16]. Moreover, the Bernstein polynomials have some important shape preserving properties that play an important role in Computer Aided Geometric Design (CAGD) (cf. Farin [21]).

6.2 The Construction of Bernstein-type Polynomials

In 1979, Videnskii [81] introduced a set of generalized Bernstein polynomials for the rational system with prescribed poles

$$\mathcal{P}_n(t_1, \dots, t_n) := \left\{ \frac{p(x)}{\prod_{i=1}^n (1+t_i x)}, \quad p \in \mathcal{P}_n \right\}, \tag{6.2.2}$$

where $t_i \geq 0 \, (i = 1, \dots, n)$.

They are constructed as follows. For given $t_i \geq 0 \, (i = 1, \dots, n)$, let

$$\varphi_n(x) = \frac{x}{n} \sum_{i=1}^n \frac{1+t_i}{1+t_i x}. \tag{6.2.3}$$

Since $\varphi_n'(x) > 0$ on $[0, 1]$, $\varphi_n(x)$ is strictly increasing. The nodes $\{\tau_k\}_{k=0}^n$ are uniquely determined by the equations

$$\varphi_n(\tau_k) = \frac{k}{n}, \quad k = 0, 1, \dots, n. \tag{6.2.4}$$

Clearly, $\{\tau_k\}_{k=0}^n$ are functions of $\{t_i\}_{i=1}^n$ and $\tau_0 = 0, \tau_n = 1$. Note that $1 + t_i x = (1 - x) + (1 + t_i)x$, thus,

$$P_n(x) := \prod_{i=1}^n (1 + t_i x) = \sum_{k=0}^n a_k x^k (1-x)^{n-k}. \tag{6.2.5}$$

Obviously, $\{a_k\}_{k=0}^n (> 0)$ are functions of $\{t_i\}_{i=1}^n$, too. Then, for a given function $f(x)$ and $n \in \mathbb{Z}$, the *Bernstein-type polynomial* for the system (6.2.2) is defined as follows:

$$B_n(f, x) := \sum_{k=0}^n f(\tau_k) p_k(x), \tag{6.2.6}$$

where

$$p_k(x) := a_k x^k (1-x)^{n-k} / P_n(x) \geq 0, \quad k = 0, 1, \dots, n. \tag{6.2.7}$$

One can easily see that (6.2.6) is always defined for $t_i > -1\,(i = 1, \ldots, n)$, not only for the case $t_i \geq 0\,(i = 1, \ldots, n)$, the latter was treated by Videnskii [81]. Also, $B_n(f, x) \in \mathcal{P}_n(t_1, \ldots, t_n)$. We encounter a problem of terminology that our Bernstein "polynomial" here is actually a rational function with prescribed poles $-1/t_i$, $i = 1, \ldots, n$. For more information about the system (6.2.2) and related topics, see, for example, [1] [6, Chapter 7] [8] [9] [17] [25] [78] and references therein. It should be mentioned when all $t_i = 0$, (6.2.6) degenerates to classical Bernstein polynomials.

By introducing the index

$$s_n := \sum_{i=1}^{n} \frac{1}{1 + t_i}, \tag{6.2.8}$$

and assuming $s_n \to \infty, (n \to \infty)$, Vindeskii [81] proved that the operator (6.2.6) converges uniformly for continuous function on $[0, 1]$. More precisely, Videnskii [81] obtained the following

Theorem K. *Let $f \in C[0,1]$, and $t_i \geq 0\,(i = 1, \ldots, n)$. Then*

$$\|B_n(f) - f\|_{[0,1]} \leq 2\omega\left(f, 1/\sqrt{s_n}\right). \tag{6.2.9}$$

where $\omega(f, .)$ is the modulus of continuity of the function f on $[0, 1]$.

Therefore Videnskii [81] extended Popoviciu's estimate for classical Bernstein polynomials. He also established an asymptotic formula for these Bernstein polynomials under some conditions, which extended Voronovskaya's formula for classical Bernstein polynomials (cf. Theorem 6.4.1). As early as 1960, Videnskii and Shabozov (cf. [81] and references therein) tried to generalize classical Bernstein polynomials for the system (6.2.2), but they needed more restricted conditions and they constructed Bernstein-type operators of a somewhat different type from (6.2.6) (cf. [81] and references therein).

Since Videnskii [81] restricted $t_i \geq 0\,(i = 1, \ldots, n)$, he excluded the interesting case when the poles approach the end point 1. Moreover, we can easily see that the measure index (6.2.8) is not suitable in the case $t_i > -1\,(i = 1, \ldots, n)$. The restriction $t_i > -1\,(i = 1, \ldots, n)$ is necessary (otherwise, it is impossible for $B_n(f, x)$ to uniformly approximate continuous functions on $[0, 1]$).

In this paper, we shall study Bernstein-type polynomials (6.2.6) for the general case $t_i > -1\,(i = 1, \dots, n)$. By introducing another index

$$s_n := \sum_{-1<t_i<0} (1+t_i) + \sum_{t_i \geq 0} \frac{1}{1+t_i}, \tag{6.2.10}$$

which, indeed, is exactly (6.2.8) for the case of $t_i \geq 0\,(i = 1, \dots, n)$, we shall extend Videnskii's results. Moreover, we shall show that (6.2.6) still has some shape preserving properties. As an application of these Bernstein-type polynomials, we also consider the approximation problem by $\{p(x)/\prod_{i=1}^n (1+t_i x),\, p \in \mathcal{P}_n\}$ with $p(x)$ having integral coefficients.

6.3 Uniform Approximation

We first state our approximation theorem by Bernstein-type polynomials (6.2.6), which extends Theorem A.

Theorem 6.3.1. *Let $f \in C[0,1]$ and $t_i > -1\,(i = 1, \dots, n)$. Then*

$$\|B_n(f) - f\|_{[0,1]} \leq 2\omega\left(f, 1/\sqrt{s_n}\right). \tag{6.3.11}$$

Before we prove this, we make some observations.

Note that the distance between two consecutive interpolation nodes is $1/n$ for classical Bernstein polynomials, $1/n$ is exactly the square of the measure index of order of uniform approximation by classical Bernstein polynomials (cf. Lorentz [40]). Hence, we first estimate the distance between two consecutive nodes in order to find the measure index concerning the order of approximation by Bernstein-type polynomials (6.2.6). By the definition (6.2.4), we have

$$\frac{\tau_k}{n} \sum_{i=1}^n \frac{1+t_i}{1+t_i\tau_k} - \frac{\tau_{k-1}}{n} \sum_{i=1}^n \frac{1+t_i}{1+t_i\tau_{k-1}} = \frac{1}{n},$$

It immediately follows that

$$(\tau_k - \tau_{k-1}) \sum_{i=1}^n \frac{1+t_i}{(1+t_i\tau_k)(1+t_i\tau_{k-1})} = 1.$$

But

$$\sum_{-1<t_i<0} \frac{1+t_i}{(1+t_i\tau_k)(1+t_i\tau_{k-1})} \geq \sum_{-1<t_i<0} (1+t_i),$$

and

$$\sum_{t_i \geq 0} \frac{1+t_i}{(1+t_i\tau_k)(1+t_i\tau_{k-1})} \geq \sum_{t_i \geq 0} \frac{1}{1+t_i},$$

We thus have

<u>**Fact 1**</u>: *If $t_i > -1\,(i = 1, \dots, n)$, then*

$$0 < \tau_k - \tau_{k-1} \leq \left\{ \sum_{-1<t_i<0} (1+t_i) + \sum_{t_i \geq 0} \frac{1}{1+t_i} \right\}^{-1} = s_n^{-1}. \tag{6.3.12}$$

This is how we know to introduce the measure index s_n.

Due to Fact 1, we further revise a function $Q_n(x)$ given in Videnskii [81]:

$$Q_n(x) := \frac{1}{n} \left\{ \sum_{-1<t_i<0} \frac{1+t_i}{1+t_i x} + \sum_{t_i \geq 0} \frac{1}{1+t_i x} \right\}, \tag{6.3.13}$$

by a simple calculation, we then have

<u>**Fact 2**</u>: *If $t_i > -1\,(i = 1, \dots, n)$, then*

$$s_n/n \leq Q_n(x) \leq 1, \quad 0 \leq x \leq 1. \tag{6.3.14}$$

We introduce another function

$$\lambda_n(x) := \inf_{0 \leq y \leq 1} \frac{\varphi_n(y) - \varphi_n(x)}{y - x} \tag{6.3.15}$$

instead of $\varphi_n'(x)$ as in [81].

<u>**Fact 3**</u>: *If $t_i > -1\,(i = 1, \dots, n)$, then*

$$Q_n(x) \leq \lambda_n(x), \quad 0 \leq x \leq 1. \tag{6.3.16}$$

Proof. By a slightly longer calculation, we can show that

$$\frac{\varphi_n(y) - \varphi_n(x)}{y - x} - Q_n(x) \geq 0, \quad 0 \leq y \leq 1, \quad 0 \leq x \leq 1,$$

Fact 3 follows. ◻

We now let

$$\Phi_k(x) := \varphi_n(\tau_k) - \varphi_n(x), \quad k = 0, \dots, n. \tag{6.3.17}$$

__Fact 4__: *If* $t_i > -1\,(i = 1, \ldots, n)$, *then*

$$n\sum_{k=0}^{n}\Phi_k^2(x)p_k(x) \le Q_n(x), \quad 0 \le x \le 1. \tag{6.3.18}$$

Proof. Since $\sum_{k=0}^{n} p_k(x) \equiv 1$, it follows, $\sum_{k=0}^{n} p'_k(x) \equiv 0$. By a simple calculation we have

$$p'_k(x) = n\Phi_k(x)p_k(x)/x(1-x), \tag{6.3.19}$$

and it follows that

$$\sum_{k=0}^{n}\Phi_k(x)p_k(x) = 0. \tag{6.3.20}$$

Differentiating (6.3.20), we then have

$$\sum_{k=0}^{n}\Phi_k^2(x)p_k(x) = \frac{x(1-x)}{n}\varphi'_n(x) = \frac{x(1-x)}{n^2}\sum_{i=1}^{n}\frac{1+t_i}{(1+t_ix)^2}. \tag{6.3.21}$$

By a slightly longer calculation, we can check that

$$\frac{1}{1+t_ix} - \frac{x(1-x)(1+t_i)}{(1+t_ix)^2} \ge 0$$

for $t_i \ge 0$ and $0 \le x \le 1$, and

$$\frac{1+t_i}{1+t_ix} - \frac{x(1-x)(1+t_i)}{(1+t_ix)^2} \ge 0$$

for $-1 < t_i < 0$ and $0 \le x \le 1$. Therefore (6.3.18) follows. □

Proof of Theorem 6.3.1. We can now prove this in a usual way based on these facts. Noting that $\sum_{k=0}^{n} p_k(x) \equiv 1$ and applying Cauchy's inequality, we have,

$$\begin{aligned}|B_n(f,x) - f(x)| &\le \sum_{k=0}^{n}|f(\tau_k) - f(x)|p_k(x) \le \omega(f,\delta)\left\{1 + \delta^{-1}\sum_{k=0}^{n}|x-\tau_k|p_k(x)\right\}\\ &\le \omega(f,\delta)\left\{1 + \delta^{-1}\left(\sum_{k=0}^{n}(x-\tau_k)^2p_k(x)\right)^{1/2}\right\}, \quad 0 \le x \le 1.\end{aligned}$$

We denote that

$$D_n(x) := \sum_{k=0}^{n}(x-\tau_k)^2p_k(x). \tag{6.3.22}$$

By the definition of $\lambda_n(x)$, we can easily check that

$$\lambda_n^2(x)D_n(x) \le \sum_{k=0}^{n} \Phi_k^2(x)p_k(x).$$

Using Facts 2-4, we then have

$$\begin{aligned} D_n(x) &\le \lambda_n^{-2}(x)\sum_{k=0}^{n} \Phi_k^2(x)p_k(x) \le Q_n^{-2}(x)\sum_{k=0}^{n} \Phi_k^2(x)p_k(x) \\ &\le n^{-1}Q_n^{-1}(x) \le s_n^{-1}, \quad 0 \le x \le 1. \end{aligned} \tag{6.3.23}$$

We thus complete the proof by taking $\delta = s_n^{-1/2}$. □

6.4 Asymptotic Formula

We now let

$$\rho_n := \max\left\{\max_{t_i \ge 0} t_i,\ \max_{-1<t_i<0} \frac{-t_i}{1+t_i}\right\}, \tag{6.4.24}$$

and

$$W_n(x) := \frac{x(1-x)}{2n}\left[\frac{f'(x)}{\varphi_n'(x)}\right]'. \tag{6.4.25}$$

Next we shall establish a Voronvskaya-type theorem. Its special case, in which $t_i \ge 0\,(i = 1, \ldots, n)$, was proved by Videnskii [81]. Moreover, when $t_i = 0\,(i = 1, \ldots, n)$, it is exactly Voronvskaya's formula (cf. [15] [40]).

Theorem 6.4.1. *Let $f \in C^3[0,1]$, $t_i \ge -1/2\,(i = 1, \ldots, n)$, and ρ_n, $W_n(x)$ be defined by* (6.4.24) *and* (6.4.25)*, respectively. If*

$$\lim_{n\to\infty} n(\rho_n + 1)^2 s_n^{-3/2} = 0, \tag{6.4.26}$$

then

$$\lim_{n\to\infty} s_n \{B_n(f,x) - f(x) - W_n(x)\} = 0, \tag{6.4.27}$$

holds uniformly on $[0, 1]$.

Remark. Here $t_i \geq -1/2\,(i = 1, \ldots, n)$ is a technical assumption (cf. (6.4.33)), whether it can be dropped is still open.

Proof. Since

$$f(\tau_k) - f(x) = \int_x^{\tau_k} f'(t)\,dt = \int_{\tau_k}^{x} \frac{f'(t)}{\varphi_n'(t)}\Phi_k'(t)\,dt$$

integration by parts twice yields

$$f(\tau_k) - f(x) = \frac{f'(x)}{\varphi_n'(x)}\Phi_k(x) + \frac{1}{2\varphi_n'(x)}\left[\frac{f'(x)}{\varphi_n'(x)}\right]'\Phi_k^2(x) + R_k(x),$$

where

$$R_k(x) = \frac{1}{2}\int_x^{\tau_k}\left\{\frac{1}{\varphi_n'(t)}\left[\frac{f'(t)}{\varphi_n'(t)}\right]'\right\}'\Phi_k^2(t)\,dt. \tag{6.4.28}$$

Recalling (6.3.20) and (6.3.21), we then have

$$B_n(f, x) - f(x) = W_n(x) + \sum_{k=0}^{n} R_k(x)p_k(x). \tag{6.4.29}$$

Therefore, we need only to show that

$$s_n\left(\sum_{k=0}^{n} R_k(x)p_k(x)\right) \to 0, \quad (n \to \infty), \tag{6.4.30}$$

holds uniformly on $[0, 1]$ under the condition of (6.4.26).

By some simple calculations, we can show that

$$n\varphi_n'(x) \geq s_n, \quad 0 \leq x \leq 1, \tag{6.4.31}$$

and

$$|\varphi_n''(x)| \leq 2\rho_n\varphi_n'(x), \quad \varphi_n'''(x) \leq 6\rho_n^2\varphi_n'(x), \quad 0 \leq x \leq 1. \tag{6.4.32}$$

for $t_i > -1\,(i = 1, \ldots, n)$.
Moreover, if $t_i \geq -1/2\,(i = 1, \ldots, n)$, we then have

$$\left|\frac{t_i x}{1 + t_i x}\right| \leq 1, \quad 0 \leq x \leq 1.$$

Furthermore, by some slightly longer calculations, we can check that

$$x|\varphi_n''(x)| \le 2\varphi_n'(x), \quad x^2\varphi_n'''(x) \le \varphi_n'(x), \quad 0 \le x \le 1. \tag{6.4.33}$$

for $t_i \ge -1/2\,(i = 1, \dots, n)$.

We now denote

$$\Lambda_n := \sum_{k=0}^{n} \Phi_k^4(x) p_k(x).$$

Then we twice differentiate (6.3.21) to yield the following identity:

$$\begin{aligned}\Lambda_n =& 3\left(\frac{x(1-x)}{n}\right)^2 (\varphi_n'(x))^2 + \frac{1}{n^3}[(x(1-x))^3\varphi_n'''(x) + \\ & 3(x(1-x))^2(1-2x)\varphi_n''(x) + (1-6x(1-x))x(1-x)\varphi_n'(x)].\end{aligned} \tag{6.4.34}$$

Note that fact $|(1-x)(1-2x)| \le 1$ and $|1-6x(1-x)| \le 1$ for $x \in [0,1]$. Thus, if $t_i \ge -1/2\,(i = 1, \dots, n)$, by (6.4.31) and (6.4.33) we have

$$\sum_{k=0}^{n} \Phi_k^4(x) p_k(x) \le \frac{3(1-x)^2}{n^2} + \frac{8(1-x)}{n^3} \le \frac{11}{n^2}, \quad 0 \le x \le 1. \tag{6.4.35}$$

By the monotonicity of $\Phi_k(x)$, we can show that

$$\Phi_k^2(t) \le \Phi_k^2(x) \tag{6.4.36}$$

for t between x and τ_k. Recalling (6.4.31) – (6.4.33) and by a slightly longer calculation, we conclude that

$$\begin{aligned}|R_k(x)| &\le \frac{1}{2}(n/s_n)^2 \left(\|f'''\|_{[0,1]} + 6\rho_n\|f''\|_{[0,1]} + 18\rho_n^2\|f'\|_{[0,1]}\right)\Phi_k^2(x)|x-\tau_k| \\ &\le 9c_f\,(n/s_n)^2(\rho_n+1)^2\Phi_k^2(x)|x-\tau_k|, \quad 0 \le x \le 1,\end{aligned} \tag{6.4.37}$$

where $c_f := \max\{\|f'\|_{[0,1]}, \|f''\|_{[0,1]}, \|f'''\|_{[0,1]}\}$.

Hence, using (6.4.35) and (6.4.37), and applying Cauchy's inequality, we have

$$\begin{aligned}\sum_{k=0}^{n} |R_k(x)| p_k(x) &\le 9c_f\,(n/s_n)^2(\rho_n+1)^2 (D_n(x))^{1/2}\left(\sum_{k=0}^{n}\Phi_k^4(x)p_k(x)\right)^{1/2} \\ &\le 9\sqrt{11}c_f n(\rho_n+1)^2 s_n^{-5/2}, \quad 0 \le x \le 1.\end{aligned}$$

Now (6.4.30) follows, and we have completed the proof. □

We denote

$$q_k(x) := \frac{n\Phi_k^2(x)p_k(x)}{x(1-x)\varphi_n'(x)}, \quad k = 1, \ldots, n, \tag{6.4.38}$$

and $N_0 \in \mathbb{Z}$ such that $s_n \geq 1$ for $n \geq N_0$, respectively.

Using the same idea as in the estimate of $D_n(x)$, we can obtain that

$$\sum_{k=0}^{n}(x-\tau_k)^2 q_k(x) \leq \lambda_n^{-2}\sum_{k=0}^{n}\Phi_k^2(x)q_k(x) \leq 11/s_n \tag{6.4.39}$$

for $n \geq N_0$.

Combining Theorem 6.3.1 and (6.4.39), and using the exactly same method of the proof as in Theorem 3 in [81], we can show the following approximation result of $f'(x)$ by $B_n'(f,x)$.

Corollary 6.4.2. *Let $f \in C^1[0,1]$, $t_i \geq -1/2\,(i = 1, \ldots, n)$, and let ρ_n be defined by* (2.14). *Then*

$$\|B_n'(f) - f'\|_{[0,1]} \leq 8\left\{2\rho_n\omega(f, 1/\sqrt{s_n}) + \omega(f', 1/\sqrt{s_n})\right\} \tag{6.4.40}$$

for $n \geq N_0$.
In particular, if $\rho_n/\sqrt{s_n} \to 0(n \to \infty)$, then $B_n'(f,x)$ converges uniformly to $f'(x)$ on $[0,1]$.

6.5 Shape Preserving Properties

We now extend some basic shape preserving properties of classical Bernstein polynomials to the Bernstein-type polynomials (6.2.6).

Let $Z_{(0,1)}f$ and $S_{(0,1)}f$ denote the number of zeros and the number of sign changes of the function $f(x)$ on $[0,1]$, respectively, and let $Varf$ denote the total variation of the function $f(x)$ on $[0,1]$. We then have

Theorem 6.5.1. *Let $B_n(f,x)$ be defined by* (6.2.6) *and $t_i > -1\,(i = 1, \ldots, 1)$. Then*
(1) The polynomial $B_n(f,x)$ increases on $[0,1]$ if $f(x)$ is increasing on this interval.
(2) One has $Z_{(0,1)}B_n(f) \leq S_{(0,1)}f$.
(3) $VarB_n(f) \leq Varf$.

Proof. Note that

$$B'_n(f,x) = \sum_{k=0}^{n} f(\tau_k)p'_k(x)$$

and $\sum_{k=0}^{n} p'_k(x) \equiv 0$, and it follows that

$$B'_n(f,x) = \sum_{k=0}^{n} (f(\tau_k) - f(x))\, p'_k(x).$$

So, from (6.3.19) we have

$$B'_n(f,x) = \sum_{k=0}^{n} (f(\tau_k) - f(x))\,(\varphi_n(\tau_k) - \varphi_n(x))\,\frac{np_k(x)}{x(1-x)}. \tag{6.5.41}$$

We now suppose that $x \in [\tau_{k_0}, \tau_{k_0+1}]$, where $0 \le k_0 \le n-1$. Since $f(x)$ and $\varphi_n(x)$ are increasing on $[0,1]$, then $f(\tau_k) - f(x) \ge 0$ and $\varphi_n(\tau_k) - \varphi_n(x) > 0$ for $k \ge k_0 + 1$. Similarly, $f(\tau_k) - f(x) \le 0$ and $\varphi_n(\tau_k) - \varphi_n(x) < 0$ for $k \le k_0$. Therefore we conclude that

$$(f(\tau_k) - f(x))\,(\varphi_n(\tau_k) - \varphi_n(x)) \ge 0$$

for $k = 0, 1, \dots, n$. It follows that $B'_n(f,x) \ge 0$ for $0 \le x \le 1$ and we have shown the conclusion (1).

Recall that $a_k > 0\,(k = 0, 1, \dots, n)$ (cf. (6.2.5)). Thus, by a trivial modification of [15, p. 309] and applying Descartes rule (cf. Borwein and Erdélyi [6] or Pólya and Szegő [62]), one can easily show (2).

Since $B_n(f,x)$ preserves constants and $Z_{(0,1)}B_n(f) \le S_{(0,1)}f$, the conclusion (3) now follows by a nice observation of DeVore and Lorentz (cf. [15, Remark, p. 309]). □

6.6 An Application to Approximation by Rational Systems with Integer Coefficients

Let $\mathcal{P}_n^e$ denote the set of all polynomials of degree$\le n$ with integral coefficients. We further let

$$\mathcal{P}_n^e(t_1, \dots, t_n) := \left\{ \frac{p(x)}{\prod_{i=1}^{n}(1 + t_i x)}, \quad p \in \mathcal{P}_n^e \right\}, \tag{6.6.42}$$

and

$$E_n^e(f)_{[a,b]} := \inf_{q \in \mathcal{P}_n^e(t_1,\dots,t_n)} \|f(x) - q(x)\|_{[a,b]}. \tag{6.6.43}$$

As an application of the Bernstein-type polynomials (6.2.6), we treat an approximation problem by $\mathcal{P}_n^e(t_1, \dots, t_n)$ for continuous functions on $[0, 1]$.

In this section, we suppose $t_i \geq 0\,(i = 1, \dots, n)$. Clearly, $f(0)$ has to be integer if $f \in C[0,1]$ can be uniformly approximated by $\mathcal{P}_n^e(t_1, \dots, t_n)$. Using Bernstein-type polynomials (6.2.6), we can prove

Theorem 6.6.1. *Let $f \in C[0, 1]$, and let $f(0)$ be an integer. If $t_i (i = 1, \dots, n)$ are integers and $f(1)$ is also integer, then, when $s_n \to \infty$ $(n \to \infty)$, we have*

$$\lim_{n\to\infty} E_n^e(f)_{[0,1]} = 0, \tag{6.6.44}$$

Proof. We denote that

$$[B_n(f,x)] := \frac{\sum_{k=0}^n [f(\tau_k)a_k] p_k(x)}{P_n(x)}.$$

where the symbol $[\,]$ denotes the greatest integer function. Therefore, $[B_n(f,x)] \in \mathcal{P}_n^e(t_1, \dots, t_n)$. Recalling the definition of $\{a_k\}_{k=0}^n$ (cf. (6.2.5)), we have $a_0 = 1$ and $a_i = \prod_{i=1}^n (1 + t_i)$. Since $f(0)$ and $f(1)$ are integers, we then have from the given assumption

$$|B_n(f,x) - [B_n(f,x)]| \leq \frac{\sum_{k=1}^{n-1} x^k (1-x)^{n-k}}{\prod_{i=1}^n (1 + t_i x)} \leq \frac{1}{n},$$

and, by the usual method (cf. Ferguson [22]), one can easily prove this. □

Corollary 6.6.2. *Let $f \in C[a, b]$, where $0 < a < b < 1$, and let $t_i \geq 0\,(i = 1, \dots, n)$. If $s_n \to \infty (n \to \infty)$. Then*

$$\lim_{n\to\infty} E_n^e(f)_{[a,b]} = 0.$$

Remark. The assumption of $f(1)$ being integer is not necessary if $f \in C[0, 1]$ can be uniformly approximated by $\mathcal{P}_n^e(t_1, \dots, t_n)$. Indeed, we have

Theorem 6.6.3. *Let $f \in C[0, 1]$, and let $f(0)$ be integer. then, when $s_n \to \infty, \gamma_n \to \infty$ $(n \to \infty)$, we have*

$$\lim_{n\to\infty} E_n^e(f)_{[0,1]} = 0, \tag{6.6.45}$$

where

$$\gamma_n := \sum_{i=1}^{n} \frac{t_i}{1+t_i}.$$

Proof. In this case, we have

$$|B_n(f,x) - [B_n(f,x)]| \le \frac{\sum_{k=1}^{n-1} x^k (1-x)^{n-k}}{\prod_{i=1}^{n}(1+t_i x)} + \prod_{i=1}^{n} \frac{x}{1+t_i x}.$$

Since $\frac{x}{1+t_i x}$ is increasing function on $[0,1]$, we have

$$0 \le \prod_{i=1}^{n} \frac{x}{1+t_i x} \le \prod_{i=1}^{n} \frac{1}{1+t_i} = \prod_{i=1}^{n} \left(1 - \frac{t_i}{1+t_i}\right).$$

Note that

$$\prod_{i=1}^{n} \left(1 - \frac{t_i}{1+t_i}\right) = 0 \quad \Leftrightarrow \quad \sum_{i=1}^{n} \frac{t_i}{1+t_i} = \infty,$$

so the conclusion follows. □

For the case $t_i = 0\,(i = 1, \dots, n)$, Theorem 6.6.1 states that a continuous function $f(x)$ is uniformly approximated by classical polynomials with integral coefficients on $[0,1]$ if $f(0)$ and $f(1)$ are integers. This was treated by Kantorovic (cf. Ferguson [22] or Lorentz [40]).

6.7 Problems

1. It's well known that classical Bernstein polynomials preserve convexity. This means, it is convex if $f(x)$ is convex. But, whether Bernstein-type polynomials (6.2.6) preserve this property or not is still open.

2. The question of how to establish the necessary and sufficient condition for the uniform approximation in $\mathcal{P}_n^e(t_1, \dots, t_n)$ is certainly worth studying.

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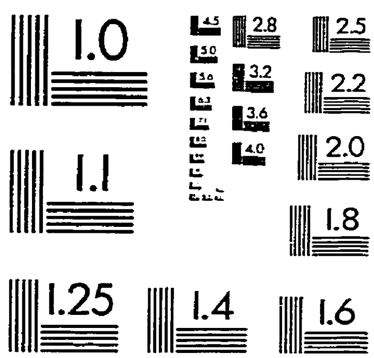

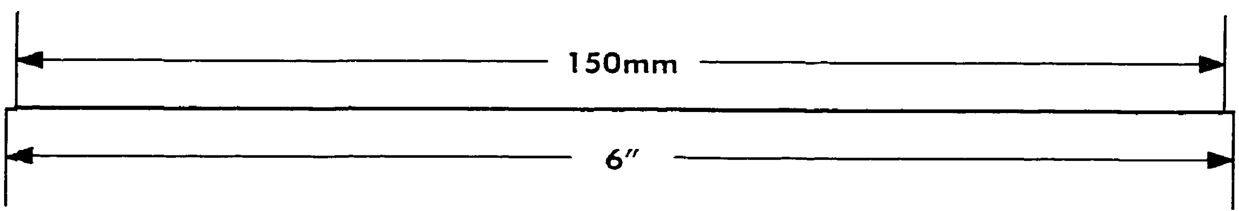

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